SARAJEVO JOURNAL OF MATHEMATICS
Vol.8 (20) (2012), 53–68

ON JENSEN'S INEQUALITY INVOLVING AVERAGES OF CONVEX FUNCTIONS

J. JAKŠETIĆ, J. PEČARIĆ AND G. ROQIA

Abstract. Using Wulbert result from [18] we deduce a method for constructing exponentially convex functions. To this date there is no known operative criteria for recognizing exponentially convex functions, so our method is of a special interest since there is a lack of examples of this class of functions. Constructed exponentially convex functions are then used for the construction of two and three parameters Cauchy means, that have the very nice property: they have the monotonicity property in their defining parameters. Particularly, we generalize recent results on Cauchy means from [2] and [4].

1. **Introduction and preliminary results**

Let $(\Omega, \mathbf{A}, \mu)$ be a probability space. The integral power means of order $r \in \mathbb{R}$ of positive $\alpha \in L^1(\mu)$ are defined as follows:

$$M_r(\alpha,\mu) = \begin{cases} \left(\int\limits_{\Omega} (\alpha(u))^r d\mu(u)\right)^{\frac{1}{r}}, & r \neq 0; \\ \exp\left(\int\limits_{\Omega} \log(\alpha(u)) d\mu(u)\right), & r = 0. \end{cases} \tag{1}$$

M. R. Lipanović, J. Pečarić and H.N. Srivastava in [11] defined new means of Cauchy type as:

$$M^s_{r,t}(\alpha,\mu) = \left(\frac{t(t-s)}{r(r-s)} \frac{M^r_r(\alpha,\mu) - M^r_s(\alpha,\mu)}{M^t_t(\alpha,\mu) - M^r_t(\alpha,\mu)}\right)^{\frac{1}{r-t}}, \ t \neq r \neq s, \ t, r \neq 0. \tag{2}$$

It is interesting that for these means the following inequality is valid

$$M^s_{r,t}(\alpha,\mu) \leq M^s_{u,v}(\alpha,\mu), \tag{3}$$

2000 *Mathematics Subject Classification.* Primary 26D15, Secondary 26D99.
Key words and phrases. Jensen's inequality, log-convex function, exponential convexity.

for $r, t, u, v, s \in \mathbb{R}$ such that $r \leq u$, $t \leq v$. All limiting cases of (2) and the proof of (3) can be found in [2]. Moreover, some related results of this type are previously obtained in [8] and [12]. It is interesting that one year later in the paper [16] the case $s = 0$ of this mean as an extension of Stolarsky means is considered without mentioning reference [2] (see (2.2) in [16]). Also, in [16] there is no mention of any results from [8, 11, 12] (see Theorem 4.1 in [16] and compare it with results from [8, 11, 12]).

D. E. Wulbert in, [18] proved that for a convex function $\psi : I \to \mathbb{R}$,

$$F(a,b) = \begin{cases} \frac{1}{b-a}\int\limits_a^b \psi(x)dx, & a \neq b; \\ \\ \psi(a), & a = b. \end{cases} \tag{4}$$

is convex on $I \times I$.

In [4], D. E. Wulbert's result is given in the following form:

Theorem 1.1. *Let ψ be a continuous function on an interval I with a nonempty interior. If ψ is convex on I, then the integral arithmetic mean F defined in* (4) *is convex on I^2.*

Furthermore, for $a_i, b_i \in I$, $i = 1, \ldots, n$ and non-negative real weights $w_i, i = 1, \ldots, n$ such that $\sum_{i=1}^n w_i = 1$, the following holds

$$\frac{1}{\bar{b}-\bar{a}}\int_{\bar{a}}^{\bar{b}} \psi(x)dx \leq \sum_{i=1}^n w_i \frac{1}{b_i - a_i}\int_{a_i}^{b_i} \psi(x)dx \tag{5}$$

and

$$\psi\left(\frac{\bar{a}+\bar{b}}{2}\right) \leq \frac{1}{\bar{b}-\bar{a}}\int\limits_{\bar{a}}^{\bar{b}} \psi(t)dt \leq \sum_{i=1}^n w_i \frac{1}{b_i - a_i}\int\limits_{a_i}^{b_i} \psi(t)dt \leq \sum_{i=1}^n w_i \frac{\psi(a_i)+\psi(b_i)}{2} \tag{6}$$

where $\bar{a} = \sum_{i=1}^n w_i a_i$, $\bar{b} = \sum_{i=1}^n w_i b_i$.

Using Theorem 1.1, we gave a discrete extension of means (2) and we get new means of Cauchy type defined by :

$$M^s_{r,t}(w_i, a_i^s, b_i^s; n) = \left[\frac{t^3 - ts^2}{r^3 - rs^2} \frac{\sum\limits_{i=1}^n w_i \frac{a_i^{r+s} - b_i^{r+s}}{a_i^s - b_i^s} - \frac{\left(M^s_{w_i}(a_i)\right)^{r+s} - \left(M^s_{w_i}(b_i)\right)^{r+s}}{\left(M^s_{w_i}(a_i)\right)^{s} - \left(M^s_{w_i}(b_i)\right)^{s}}}{\sum\limits_{i=1}^n w_i \frac{a_i^{t+s} - b_i^{t+s}}{a_i^s - b_i^s} - \frac{\left(M^s_{w_i}(a_i)\right)^{t+s} - \left(M^s_{w_i}(b_i)\right)^{t+s}}{\left(M^s_{w_i}(a_i)\right)^{s} - \left(M^s_{w_i}(b_i)\right)^{s}}}\right]^{\frac{1}{r-t}}$$

where

$$M_{w_i}^S(x_i) = \begin{cases} \left(\sum\limits_{i=1}^n w_i x_i^S \right)^{\frac{1}{S}}, & S \neq 0, \\ \prod\limits_{i=1}^n x_i^{w_i}, & S = 0, \end{cases}$$

$r \neq t \neq s$, $r \neq s, -s$, $s \neq 0$.

Limiting cases are also derived and we have shown that these means are monotonic, that is for $r, t, u, v, s \in \mathbb{R}$ such that $r \leq u$, $t \leq v$

$$M_{r,t}^s(w_i; a_i, b_i; n) \leq M_{u,v}^s(w_i; a_i, b_i; n). \tag{7}$$

If we let $a_i \to b_i$, $i = 1, \ldots, n$, we have discrete version of means in (2). Let us note that the case $s = 0$ is also considered in [16] (see (1.14) and (1.15) in [16]).

In this paper, we will give further extension of results from [10]. To begin with, let us note that the following integral version of (5) is valid:

Theorem 1.2. *Let $\alpha, \beta : \Omega \to I$ (I, an interval with non-empty interior) be functions from $L^1(\mu)$ and let $\psi : I \to \mathbb{R}$ be a continuous convex function. Then the following inequality is valid:*

$$\frac{1}{\overline{\beta} - \overline{\alpha}} \int_{\overline{\alpha}}^{\overline{\beta}} \psi(x)dx \leq \int_\Omega \left(\frac{1}{\beta(u) - \alpha(u)} \int_{\alpha(u)}^{\beta(u)} \psi(x)dx \right) d\mu(u), \tag{8}$$

where $\overline{\alpha} = \int\limits_\Omega \alpha(u)d\mu(u)$, $\overline{\beta} = \int\limits_\Omega \beta(u)d\mu(u)$.

Proof. Follows from integral version of convexity on $I \times I$ (see [9], p. 51), and the fact that the function $F : I \times I \to \mathbb{R}$ defined with (4) is convex. $\square$

In the later text J stands for an open interval in $\mathbb{R}$; with $\mathbb{R}_+$ we denote positive reals.

Definition 1. *A function $f : J \to \mathbb{R}_+$ is a log-convex in Jensen's sense if*

$$f^2\left(\frac{x+y}{2}\right) \leq f(x)f(y),$$

for all $x, y \in J$.

The following simple result gives us necessary and sufficient condition to recognize *log*-convex function in Jensen's sense.

Lemma 1.3. *A function $f : J \to \mathbb{R}_+$ is a log-convex in Jensen's sense if and only if*

$$u_1^2 f(x) + 2u_1u_2 f\left(\frac{x+y}{2}\right) + u_2^2 f(y)) \geq 0, \tag{9}$$

for all $u_1, u_2 \in \mathbb{R}$ and $x, y \in J$.

Now we define exponentially convex functions (see [1]).

Definition 2. *A function $f : J \to \mathbb{R}$ is exponentially convex on J if it is continuous and*

$$\sum_{i,j=1}^{n} \xi_i \xi_j f\left(x_i + x_j\right) \geq 0$$

for all $n \in \mathbb{N}$ and all choices $\xi_i \in \mathbb{R},\ i = 1, \ldots, n$ such that $x_i + x_j \in J,\ 1 \leq i, j \leq n$.

For our purposes we will need the following simple characterization of exponential convexity.

Proposition 1.4. *Let $f : J \to \mathbb{R}$. The following propositions are equivalent.*

(i) *f is exponentially convex.*

(ii) *f is continuous and*

$$\sum_{i,j=1}^{n} \xi_i \xi_j f\left(\frac{x_i + x_j}{2}\right) \geq 0,$$

for all $n \in \mathbb{N}$, $\xi_i \in \mathbb{R}$ and $x_i \in J$, $i = 1, \ldots, n$.

The following corollary gives the inclusion relation between two very important subclasses of the class of convex functions.

Corollary 1.5. *A positive exponentially convex function $f : J \to \mathbb{R}_+$ is log-convex function i.e. $\log f$ is convex function on J.*

Exponentially convex functions have strong analytical properties: they are differentiable of all orders, and even more, they are analytical (see [3, 5]). Previous results follow from the integral representation of exponentially convex functions:

Theorem 1.6. *The function $f : J \to \mathbb{R}$ is exponentially convex on J if and only if*

$$f(x) = \int_{-\infty}^{\infty} e^{tx} d\sigma(t), \quad x \in J \tag{10}$$

for some non-decreasing function $\sigma : \mathbb{R} \to \mathbb{R}$.

Proof. See [1], p. 211. □

Remark 1.7. Theorem 1.6 enables us, in particular, to conclude that any real function that is a Laplace transform of some positive function is an exponentially convex function on its open domain.

The next theorem from [7] will be very helpful in the sequel.

Theorem 1.8. *Let $f : J \to \mathbb{R}_+$ be log-convex, derivable function.*
Let $M : J \times J \to \mathbb{R}_+$ be defined by

$$M(x,y) = \begin{cases} \left(\frac{f(x)}{f(y)}\right)^{\frac{1}{x-y}}, & x \neq y; \\ \exp\left(\frac{f'(x)}{f(x)}\right), & x = y. \end{cases} \tag{11}$$

If $x_1 \leq x_2,\ y_1 \leq y_2$ then

$$M(x_1, y_1) \leq M(x_2, y_2). \tag{12}$$

It is well known that a real valued convex function ψ defined on I is characterized by the second order divided difference at mutually distinct points $x_0, x_1, x_2 \in I$:

$$[x_0, x_1, x_2; \psi] = \frac{\psi(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{\psi(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{\psi(x_2)}{(x_2 - x_0)(x_2 - x_1)} \geq 0,$$

If ψ is a differentiable function, we have the support line convexity criteria for $x_0, x_1 \in I$

$$[x_0, x_0, x_1; \psi] = \frac{\psi(x_1) - \psi(x_0) - (x_1 - x_0)\psi'(x_0)}{(x_1 - x_0)^2} \geq 0.$$

Also, if ψ is twice differentiable on I, we have a very useful result about convexity for $x_0 \in I$

$$[x_0, x_0, x_0; \psi] = \frac{\psi''(x_0)}{2} \geq 0.$$

2. Main results

Let I be an interval with nonempty interior and J stand for an open interval in $\mathbb{R}$. Let $\alpha, \beta : \Omega \to I$ be functions from $L^1(\mu)$ as in introduction and let $\psi : I \to \mathbb{R}$ be a continuous function.

Denote

$$F(\alpha, \beta; \psi; \mu) = \int_{\Omega} \left(\frac{1}{\beta(u) - \alpha(u)} \int_{\alpha(u)}^{\beta(u)} \psi(x)dx \right) d\mu(u) - \frac{1}{\overline{\beta} - \overline{\alpha}} \int_{\overline{\alpha}}^{\overline{\beta}} \psi(x)dx, \tag{13}$$

where $\overline{\alpha} = \int_{\Omega} \alpha(u)d\mu(u)$, $\overline{\beta} = \int_{\Omega} \beta(u)d\mu(u)$. From Theorem 1.2 it follows that if ψ is convex function, then $F(\alpha, \beta; \psi; \mu) \geq 0$.
Further, we introduce the following families of functions:

- $X_1 = \{\psi_t : I \to \mathbb{R},\ t \in J\}$, a family of functions from $C(I)$ such that $t \to [x_0, x_1, x_2; \psi_t]$ is log-convex in Jensen's sense on J for every choice of distinct points $x_0, x_1, x_2 \in I$;
- $X_2 = \{\psi_t : I \to \mathbb{R},\ t \in J\}$, a family of differentiable functions on I such that $t \to [x_0, x_0, x_1; \psi_t]$ is log-convex in Jensen's sense on J for every choice of distinct points $x_0, x_1 \in I$;
- $X_3 = \{\psi_t : I \to \mathbb{R},\ t \in J\}$, a family of twice differentiable functions from I such that $t \to [x_0, x_0, x_0; \psi_t]$ is log-convex in Jensen's sense on J for every choice of $x_0 \in I$.

Theorem 2.1. *Let $(\Omega, \boldsymbol{A}, \mu)$ be a probability space, let $\alpha, \beta : \Omega \to I$ be functions from $L^1(\mu)$. Let $F : J \to \mathbb{R}$ be function defined by*

$$F(t) = F(\alpha, \beta; \psi_t; \mu) \tag{14}$$

$\psi_t \in X_k,\ t \in J;\ k = 1, 2, 3$. Then the following is valid

(i) *the function F is log-convex in Jensen's sense on J;*

(ii) *if the function F is continuous, it is log-convex and for $r, s, t \in J$ such that $r < s < t$ we have,*

$$(F(s))^{t-r} \le (F(r))^{t-s} (F(t))^{s-r}; \tag{15}$$

(iii) *if the function F is derivable, then for every $t, r, u, v \in J$ such that $r \le u,\ t \le v$, we have*

$$M_{r,t}(\alpha, \beta, \mu) \le M_{u,v}(\alpha, \beta, \mu), \tag{16}$$

where

$$M_{r,t}(\alpha, \beta, \mu) = \begin{cases} \left(\frac{F(r)}{F(t)}\right)^{\frac{1}{r-t}}, & t \ne r; \\ \exp\left(\frac{d}{dr} \log F(r)\right), & t = r. \end{cases} \tag{17}$$

Proof.

(i) Let $u_1, u_2 \in \mathbb{R}$ and $t, r \in J$ be arbitrary. Define function h on I by

$$h(x) = u_1^2 \psi_t(x) + 2u_1 u_2 \psi_{\frac{t+r}{2}}(x) + u_2^2 \psi_r(x).$$

Then

$$[x_0, x_1, x_2; h] = u_1^2 [x_0, x_1, x_2; \psi_t] + 2u_1 u_2 \left[x_0, x_1, x_2; \psi_{\frac{t+r}{2}}\right] + u_2^2 [x_0, x_1, x_2; \psi_r]$$

Since $t \to [x_0, x_1, x_2; \psi_t]$ is log-convex in Jensen's sense on J

$$[x_0, x_1, x_2; h] \ge 0$$

we conclude that h is convex function on I. Hence (8) gives

$$u_1^2 F(t) + 2u_1u_2 F\left(\frac{t+r}{2}\right) + u_2^2 F(r) \geq 0$$

F is log-convex in the Jensen's sense.

(ii) Since F is continuous on J it is a log-convex function.

(iii) This is a simple consequence of Theorem 1.8.

□

Remark 2.2. The case X_3 of the Theorem 2.1 is a generalization of Theorem 4.2 in [16]. It is interesting to note here that the author S. Simić has proved exactly the same theorem twice: once as Theorem 4.2 in [16] and the second time as Theorem 1 in [17], but there is no reference to either.

We now go a step further as we prove analogue of Theorem 2.1 for exponentially convex functions. We first introduce the following families of functions:

- $\overline{X}_1 = \{\psi_t : I \to \mathbb{R},\ t \in J\}$, a family of functions from $C(I)$ such that $t \to [x_0, x_1, x_2; \psi_t]$ is an exponentially convex function on J for every choice of 3 distinct points $x_0, x_1, x_2 \in I$;
- $\overline{X}_2 = \{\psi_t : I \to \mathbb{R},\ t \in J\}$, a family of differentiable functions on I such that $t \to [x_0, x_0, x_1; \psi_t]$ is an exponentially convex function on J for every choice of 2 distinct points $x_0, x_1 \in I$;
- $\overline{X}_3 = \{\psi_t : I \to \mathbb{R},\ t \in J\}$, a family of twice differentiable functions from I such that $t \to [x_0, x_0, x_0; \psi_t]$ is an exponentially convex function on J for every choice of $x_0 \in I$.

Theorem 2.3. *Let $(\Omega, \boldsymbol{A}, \mu)$ be a probability space, let I be an interval in $\mathbb{R}$ with a nonempty interior and let $\alpha,\ \beta : \Omega \to I$ be functions from $L^1(\mu)$. Let $\overline{F} : J \to \mathbb{R}$ be function defined by*

$$\overline{F}(t) = F(\alpha, \beta; \psi_t; \mu), \tag{18}$$

$\psi_t \in \overline{X}_k,\ t \in J;\ k = 1, 2, 3.$ *Then the following hold*

(i) *the function $\overline{F}$ is exponentially convex on J;*

(ii) *for each $n \in \mathbb{N}$ and $t_1, \ldots, t_n \in J$ the matrix $\left[\overline{F}\left(\frac{t_i+t_j}{2}\right)\right]_{i,j=1}^n$ is positive semi-definite. Particularly,*

$$det\left[\overline{F}\left(\frac{t_i+t_j}{2}\right)\right]_{i,j=1}^n \geq 0;$$

(iii) *for every $t,\ r,\ u,\ v \in J$ such that $r \leq u,\ t \leq v$ we have*

$$M_{r,t}(\alpha, \beta, \mu) \leq M_{u,v}(\alpha, \beta, \mu), \tag{19}$$

where

$$M_{r,t}(\alpha,\beta,\mu) = \begin{cases} \left(\frac{\overline{F}(r)}{\overline{F}(t)}\right)^{\frac{1}{r-t}}, & r \neq t; \\ \exp\left(\frac{d}{dr}\log \overline{F}(r)\right), & t = r. \end{cases} \tag{20}$$

Proof. (i) Let $u_i \in \mathbb{R}$, $t_i \in J$, $i = 1, \ldots, n$, for arbitrary fixed $n \in \mathbb{N}$. Define a function ν on I by

$$\nu(x) = \sum_{i,j=1}^{n} u_i u_j \psi_{\frac{t_i+t_j}{2}}(x). \tag{21}$$

By assumption

$$[x_0, x_1, x_2; \nu] = \sum_{i,j=1}^{n} \left[x_0, x_1, x_2; \psi_{\frac{t_i+t_j}{2}}\right] \geq 0,$$

hence ν is convex function on I. Now we can substitute $\psi = \nu$ in (8), and we get

$$\sum_{i,j=1}^{n} u_i u_j \overline{F}\left(\frac{t_i + t_j}{2}\right) \geq 0$$

concluding exponential convexity of $\overline{F}$ on J.
(ii) and (iii)-parts are simple consequences of (i).

□

Now we need to built mean value theorems in order to generate Cauchy type means.

Theorem 2.4. *Let $(\Omega, \boldsymbol{A}, \mu)$ be a probability space, let I be a compact interval in $\mathbb{R}$, and let α, $\beta : \Omega \to I$ be functions from $L^1(\mu)$ and let $f \in C^2(I)$. Then there exists $\xi \in I$ such that*

$$F(\alpha, \beta; f; \mu) = \frac{f''(\xi)}{2} F(\alpha, \beta; e_2; \mu) \tag{22}$$

where F is defined with (13) *and $e_2(x) = x^2$.*

Proof. Analogous to the proof of Theorem 2.4 in [10]. □

Following the steps of the proof of Theorem 2.5 in [10] , we can prove the following mean value theorem.

Theorem 2.5. *Let $(\Omega, \boldsymbol{A}, \mu)$ be a probability space, I be a compact interval in $\mathbb{R}$, let α, $\beta : \Omega \to I$ be functions from $L^1(\mu)$ and let f_1, $f_2 \in C^2(I)$. Then there exits $\xi \in I$ such that*

$$\frac{f_1''(\xi)}{f_2''(\xi)} = \frac{F(\alpha, \beta; f_1, \mu)}{F(\alpha, \beta; f_2; \mu)}, \tag{23}$$

assuming both denominators are non-zero.

Remark 2.6. The previous Theorem is a generalization of results in [8, 11, 12].

Remark 2.7. Theorem (2.5) enables us to define various types of means, because if the function $\frac{f_1''}{f_2''}$ has inverse from (23) we have

$$\xi = \left(\frac{f_1''}{f_2''}\right)^{-1}\left(\frac{F(\alpha,\beta;f_1,\mu)}{F(\alpha,\beta;f_2;\mu)}\right).$$

The number $\xi \in I$ we call Cauchy mean on I.

3. **Examples**

Example 3.1. Suppose that I is any compact interval in $\mathbb{R}_+$, $J = \mathbb{R}_+$ and that $\{\psi_t : I \to \mathbb{R}, t \in J\}$ is the family of functions from $C^2(I)$ defined by

$$\psi_t(x) = \frac{e^{-x\sqrt{t}}}{t}.$$

Since $t \mapsto \frac{d^2\psi_t}{dx^2}(x) = e^{-x\sqrt{t}}$, is exponentially convex on J by Example 3 in [7] (see [15] p. 214), by Theorem 2.3 we conclude the exponential convexity of function $\overline{F}$ on J, where

$$\overline{F}(t) = \frac{-1}{(\sqrt{t})^3}\left(\int_\Omega \frac{e^{-\sqrt{t}\beta(u)} - e^{-\sqrt{t}\alpha(u)}}{\beta(u)-\alpha(u)}d\mu(u) - \frac{e^{-\overline{\beta}\sqrt{t}} - e^{-\overline{\alpha}\sqrt{t}}}{\overline{\beta}-\overline{\alpha}}\right). \tag{24}$$

If we apply (20) on $\overline{F}$ we get expressions $M^1_{r,t}(\alpha,\beta;\mu)$:

$$M^1_{r,t}(\alpha,\beta;\mu) = \begin{cases} \left(\frac{F(\alpha,\beta;\psi_r;\mu)}{F(\alpha,\beta;\psi_t;\mu))}\right)^{\frac{1}{r-t}}, & r \neq t; \\ \\ \exp\left(-\frac{1}{r} - \frac{F(\alpha,\beta;id\cdot\psi_r;\mu)}{2\sqrt{r}F(\alpha,\beta;\psi_r;\mu)}\right), & r = t, \end{cases}$$

and we have the monotonicity property (19): for $(r,t),(u,v) \in J \times J$ such that $r \leq u,\ t \leq v$

$$M^1_{r,t}(\alpha,\beta;\mu) \leq M^1_{u,v}(\alpha,\beta;\mu). \tag{25}$$

Example 3.2. Suppose that I is a compact interval in $\mathbb{R}_+$ and $J = \mathbb{R}_+$. Define $\psi_t \in \overline{X}_3$ by

$$\psi_t(x) = \begin{cases} \frac{t^{-x}}{(\log t)^2}, & t \neq 1; \\ \frac{x^2}{2}, & t = 1. \end{cases}$$

Exponential convexity of $t \mapsto \frac{d^2}{dx^2}\psi_t(x) = t^{-x}$ on J for $x \in \mathbb{R}_+$ is given by Example 2 in [7] (see also [15], p. 210). For this family the exponentially convex function is:

$$\overline{F}(t) = \begin{cases} \frac{-1}{\log^3 t}\left[\int\limits_{\Omega} \frac{t^{-\beta(u)}-t^{-\alpha(u)}}{\beta(u)-\alpha(u)}d\mu(u) - \frac{t^{-\overline{\beta}}-t^{-\overline{\alpha}}}{\overline{\beta}-\overline{\alpha}}\right], & t \neq 1; \\ \frac{1}{6}\left[\int\limits_{\Omega} \frac{\beta^3(u)-\alpha^3(u)}{\beta(u)-\alpha(u)}d\mu(u) - \frac{\overline{\beta}^3-\overline{\alpha}^3}{\overline{\beta}-\overline{\alpha}}\right], & t = 1. \end{cases} \tag{26}$$

Then, as corresponding to (20), we define

$$M^2_{r,t}(\alpha,\beta;\mu) = \begin{cases} \left[\frac{F(\alpha,\beta;\psi_r;\mu)}{F(\alpha,\beta;\psi_t;\mu)}\right]^{\frac{1}{r-t}}, & r \neq t; \\ \exp\left[-\frac{F(\alpha,\beta;id\psi_r;\mu)}{r\,F(\alpha,\beta;\psi_r;\mu)} - \frac{2}{r\log r}\right], & r = t,\ r \neq 1; \\ \exp\left[-\frac{F(\alpha,\beta;id\cdot\psi_1;\mu)}{3F(\alpha,\beta;\psi_1;\mu)}\right], & r = t,\ t = 1. \end{cases}$$

Again, we have the monotonicity property (19):
If $(r,t),(u,v) \in J \times J$ such that $r \leq u,\ t \leq v$, then

$$M^2_{r,t}(\alpha,\beta;\mu) \leq M^2_{u,v}(\alpha,\beta;\mu). \tag{27}$$

Example 3.3. Define a family of functions $\{\psi_t : t \in J\}$, on $I \subseteq \mathbb{R}$, $I = [a,b],\ J = \mathbb{R}$ by

$$\psi_t(x) = \begin{cases} \frac{1}{t^2}e^{tx}, & t \neq 0; \\ \frac{1}{2}x^2, & t = 0, \end{cases} \tag{28}$$

Then $t \mapsto \frac{d^2}{dx^2}(\psi_t(x)) = e^{tx}$ which is a basic example of exponentially convex function on $\mathbb{R}$ for each $x \in [a,b]$. Using Theorem 2.3, for this family we can construct a new exponentially convex function defined by

$$\overline{F}(t) = \begin{cases} \frac{1}{t^3}\left[\int\limits_{\Omega} \frac{e^{t\beta(u)}-e^{t\alpha(u)}}{\beta(u)-\alpha(u)}d\mu(u) - \frac{e^{t\overline{\beta}}-e^{t\overline{\alpha}}}{\overline{\beta}-\overline{\alpha}}\right], & t \neq 0; \\ \frac{1}{6}\left[\int\limits_{\Omega} \frac{\beta^3(u)-\alpha^3(u)}{\beta(u)-\alpha(u)}d\mu(u) - \frac{\overline{\beta}^3-\overline{\alpha}^3}{\overline{\beta}-\overline{\alpha}}\right], & t = 0. \end{cases} \tag{29}$$

Using (20) we now construct $M^3_{r,t}(\alpha,\beta;\mu)$ on $\mathbb{R}\times\mathbb{R}$

$$M^3_{r,t}(\alpha,\beta;\mu) = \begin{cases} \left[\frac{F(\alpha,\beta;\psi_r;\mu)}{F(\alpha,\beta;\psi_t;\mu)}\right]^{\frac{1}{r-t}}, & r \neq t; \\ \exp\left[\frac{F(\alpha,\beta;id\cdot\psi_r;\mu)}{F(\alpha,\beta;\psi_r;\mu)} - \frac{2}{r}\right], & r = t \neq 0; \\ \exp\left[\frac{F(\alpha,\beta;id\cdot\psi_0;\mu)}{3F(\alpha,\beta;\psi_0);\mu}\right], & r = t = 0. \end{cases} \tag{30}$$

Again we have monotonicity

$$M^3_{r,t}(\alpha,\beta;\mu) \leq M^3_{u,v}(\alpha,\beta;\mu),\ r \leq u,\ \ t \leq v. \tag{31}$$

Observe here, that, according Theorem 2.5, we have here Cauchy means:

$$e^{\xi} = \left[\frac{F(\alpha,\beta;\psi_r;\mu)}{F(\alpha,\beta;\psi_t;\mu)}\right]^{\frac{1}{r-t}}, \tag{32}$$

for some(unique) $\xi \in I = [a,b]$. So we conclude that $M^3_{r,t}(\alpha,\beta;\mu)$ represents a Cauchy mean on $[e^a, e^b]$.

Remark 3.4.

(i) If we replace $\alpha \to \log\alpha$ and $\beta \to \log\beta$ in (30) we get the continuous version of the 3-parameters mean for $s = 0$ obtained in [4].
(ii) The case $\alpha = \beta$ on Ω, gives us the limiting cases of the new Cauchy means for $s = 0$ in [2], that are means from [16].

Example 3.5. Define a family of functions $\Phi = \{\phi_t : t \in J\}$, on $I \subseteq \mathbb{R}_+$, $I = [a,b]$, $J = \mathbb{R}$, by

$$\phi_t(x) = \begin{cases} \frac{x^t}{t(t-1)}, & t \neq 1, 0; \\ -\log x, & t = 0; \\ x\log x, & t = 1. \end{cases} \tag{33}$$

Then $t \mapsto \frac{d^2}{dx^2}(\phi_t(x)) = x^{t-2}$ is again an exponentially convex function on J since $x^{t-2} = e^{(t-2)\log x}$. Using Theorem 2.3, for this family construct a new exponentially convex function defined by

$$\overline{F}(t)=\begin{cases}\frac{1}{t^3-t}\left[\int\limits_{\Omega}\frac{\beta^{t+1}(u)-\alpha^{t+1}(u)}{\beta(u)-\alpha(u)}d\mu(u)-\frac{\overline{\beta}^{t+1}-\overline{\alpha}^{t+1}}{\overline{\beta}-\overline{\alpha}}\right], & t\neq -1,0,1;\\ \frac{1}{2}\left[\int\limits_{\Omega}\frac{\log(\beta(u))-\log(\alpha(u))}{\beta(u)-\alpha(u)}d\mu(u)-\frac{\log(\overline{\beta})-\log(\overline{\alpha})}{\overline{\beta}-\overline{\alpha}}\right], & t=-1;\\ \frac{\overline{\beta}\log\overline{\beta}-\overline{\alpha}\log\overline{\alpha}}{\overline{\beta}-\overline{\alpha}}-\int\limits_{\Omega}\frac{\beta(u)\log\beta(u)-\alpha(u)\log\alpha(u)}{\beta(u)-\alpha(u)}d\mu(u), & t=0;\\ \int\limits_{\Omega}\frac{\beta(u)^2\log\beta(u)-\alpha(u)^2\log\alpha(u)}{\beta(u)-\alpha(u)}d\mu(u)-\frac{\overline{\beta}^2\log\overline{\beta}-\overline{\alpha}^2\log\overline{\alpha}}{\overline{\beta}-\overline{\alpha}}, & t=1\end{cases}\tag{34}$$

Using (20), for family Φ we now construct $M^4_{r,t}(\alpha,\beta;\mu;\Phi)$ on $\mathbb{R}\times\mathbb{R}$:

$$M^4_{r,t}(\alpha,\beta;\mu;\Phi)=\begin{cases}\left[\frac{F(\alpha,\beta;\phi_r;\mu)}{F(\alpha,\beta;\phi_t;\mu)}\right]^{\frac{1}{r-t}}, & r\neq t,\ t\neq 0,1;\\ \exp\left[\frac{1-2r}{r^2-r}-\frac{F(\alpha,\beta;\phi_0\phi_r;\mu)}{F(\alpha,\beta;\phi_r;\mu)}\right], & r=t\neq 0,1;\\ \exp\left[1-\frac{F(\alpha,\beta;\phi_0^2;\mu)}{2F(\alpha,\beta;\phi_0;\mu)}\right], & r=t=0;\\ \exp\left[-1-\frac{F(\alpha,\beta;\phi_0\phi_1;\mu)}{2F(\alpha,\beta;\phi_1;\mu)}\right], & r=t=1.\end{cases}\tag{35}$$

$M^4_{r,t}(\alpha,\beta;\mu;\Phi)$ are Cauchy means since, for $r\neq t$,

$$\frac{\frac{d^2}{dx^2}(\phi_r(x))}{\frac{d^2}{dx^2}(\phi_t(x))}=x^{r-t}$$

is an invertible function. Particularly,

$$a\leq\left[\frac{F(\alpha,\beta;\phi_r;\mu)}{F(\alpha,\beta;\phi_t;\mu)}\right]^{\frac{1}{r-t}}\leq b.\tag{36}$$

We now impose one additional parameter $s\in\mathbb{R}$; after substitutions $\alpha\to\alpha^s$, $\beta\to\beta^s$, $r\to\frac{r}{s}$ and $t\to\frac{t}{s}$ in (36), we have

$$\min\{a^s,b^s\}\leq\left[\frac{F(\alpha^s,\beta^s;\psi_{\frac{r}{s}};\mu)}{F(\alpha^s,\beta^s;\psi_{\frac{t}{s}};\mu)}\right]^{\frac{s}{r-t}}\leq\max\{a^s,b^s\},\ r\neq t.\tag{37}$$

Hence, we get the (generalized) Cauchy means on $[a,b]$ since

$$a\leq\left[\frac{F(\alpha^s,\beta^s;\psi_{\frac{r}{s}};\mu)}{F(\alpha^s,\beta^s;\psi_{\frac{t}{s}};\mu)}\right]^{\frac{1}{r-t}}\leq b.$$

We define this new generalized Cauchy means $M^4_{r,t;s}(\alpha,\beta;\mu)$ as

$$M^4_{r,t;s}(\alpha,\beta;\mu)=\begin{cases}\left[M^4_{\frac{r}{s},\frac{t}{s}}(\alpha^s,\beta^s;\mu;\Phi)\right]^{\frac{1}{s}}, & s\neq 0;\\ M^4_{r,t}(\log(\alpha),\log(\beta);\mu;\Psi), & s=0,\end{cases}\tag{38}$$

where $\Psi=\{\phi_t : t\in J\}$ is the family of functions defined with (28).

Using Theorem 1.8 have monotonicity of these generalized Cauchy means; for $r,t,u,v,s\in\mathbb{R}$ such that $r\le u$ and $t\le v$

$$M^4_{r,t;s}(\alpha,\beta;\mu)\le M^4_{u,v;s}(\alpha,\beta;\mu).\tag{39}$$

Remark 3.6.

(i) Means $M^4_{r,t;s}(\alpha,\beta;\mu)$ are continuous version of the 3-parameters means obtained in [4].

(ii) In the special case $\alpha=\beta$ on Ω, we will get Cauchy means from [2] (see relation (2.4) in [2]).

4. Applications

4.1. **Tobey and Stolarsky-Tobey means.** Let Δ_{n-1} represents the $n-1$ -dimensional Euclidean simplex given by

$$\Delta_{n-1}=\left\{(u_1,u_2,\ldots,u_{n-1}) : u_i\ge 0,\ i=1,\ldots,n-1,\sum_{k=1}^{n-1}u_k\le 1\right\}.$$

Denote $\mathbf{u}=(u_1,\ldots,u_n)$ and $d\mathbf{u}=du_1\cdots du_n$, where $u_n=1-\sum_{i=1}^{n-1}u_n$. Let μ be a probability measure on Δ_{n-1}. The power mean of order $r\in\mathbb{R}$ of the positive n-tuple $\mathbf{x}=(x_1,\ldots,x_n)$, with the weights $u=(u_1,\ldots,u_n)$, is defined by

$$m_p(\mathbf{x},\mathbf{u})=\begin{cases}\left(\sum\limits_{i=1}^n u_i x_i^p\right)^{\frac{1}{p}}, & p\neq 0;\\ \prod\limits_{i=1}^n x_i^{u_i}, & p=0.\end{cases}\tag{40}$$

We note that $m_p(\mathbf{x},u)$ represents a mean because we have $\min\{x_i\}\le m_p(\mathbf{x},u)\le\max\{x_i\}$. Now setting $\alpha(u)=m_p(\mathbf{x},u)$ in (1) we get the *Tobey mean* $L_{p,r}(\mathbf{x};\mu)$:

$$L_{p,r}(\mathbf{x};\mu)=M_r(m_p(\mathbf{x},u);\mu).\tag{41}$$

If we replace $r\to r-p$ in (41) we get *Stolarsky-Tobey means* (see [14]):

$$E_{p,r}(\mathbf{x},\mu)=L_{p,r-p}(\mathbf{x},\mu)=M_{r-p}(m_p(\mathbf{x},u);\mu).\tag{42}$$

We now go a step further. First, take $\alpha(u)=m_p(\mathbf{x},u)$ and $\beta(u)=m_p(\mathbf{y},u)$ in means $M^4_{r,t;s}(\alpha,\beta;\mu)$ defined in (38). Then

$$\Upsilon^s_{p,r,t}(\mathbf{x},\mathbf{y};\mu):=M^4_{r,t;s}(m_p(\mathbf{x},\cdot),m_p(\mathbf{y},\cdot);\mu)\tag{43}$$

represents the Cauchy means. This fact generalizes the result from [2] since if we put $\mathbf{x} = \mathbf{y}$ in (43) we get Cauchy means defined with relation (4.7) from [2].

4.2. **The complete symmetric mean values.** The r-th complete symmetric polynomial mean the complete symmetric mean) of the positive real n-tuple $\mathbf{x}$ is defined by see [2, 11, 13].

$$Q_n^{[r]}(\mathbf{x}) = \left({q_n}^{[r]}(\mathbf{x})\right)^{\frac{1}{r}} = \left(\frac{{c_n}^{[r]}(x)}{\binom{n+r-1}{r}}\right)^{\frac{1}{r}}, \tag{44}$$

where $c_n^{[0]}(\mathbf{x}) = 1$ and $c_n^{[r]}(\mathbf{x}) = \sum_{(i_1,\ldots,i_n)}(\prod_{i=1}^n x_i^{i_j})$ and the sum is taken over all $\binom{n+r-1}{r}$ nonnegative integer n-tuples $(i_1,\ldots,i_n)$ with $\sum_{j=1}^n i_j = r, (r \neq 0)$. It is known that complete symmetric polynomial mean can expressed via integral (see [2]):

$$Q_n^{[r]}(\mathbf{x}) = \left(\int_{\Delta_{n-1}}\left(\sum_{j=1}^n x_i u_i\right)^r d\mu(\mathbf{u})\right)^{\frac{1}{r}}, \tag{45}$$

where $r \in \mathbb{N}\cup\{0\}$, μ represents a probability measure such that $d\mu(\mathbf{u}) = (n-1)!du_1,\ldots,du_{n-1}$. So (45) is in fact power mean (1) with $\alpha(u) = \sum_{j=1}^n x_i u_i$.

If we put $\alpha(u) = \sum_{j=1}^n x_i u_i$ and $\beta(u) = \sum_{j=1}^n y_i u_i$ in means $M_{r,t;s}^4(\alpha,\beta;\mu)$ defined in (38) we get a new Cauchy means:

$$\Theta_{n,r,t}^s(\mathbf{x},\mathbf{y};\mu) = M_{r,t;s}^4(\alpha,\beta;\mu). \tag{46}$$

If we put $\mathbf{x} = \mathbf{y}$ in (46) we get Cauchy means defined with (4.11) in [2].

4.3. **Whiteley means.** Let $\mathbf{x}$ be a positive real n-tuple, $s \in \mathbb{R}(s \neq 0)$ and $r \in \mathbb{N}\cup\{0\}$. Then, the $s-$th function of degree r, $t_n^{[r,s]}(\mathbf{x})$, is defined by :

$$1 + \sum_{r=0}^{\infty} t_n^{[r,s]}(\mathbf{x})t^r = \begin{cases} \prod\limits_{i=1}^n (1+x_i t)^s, & s > 0; \\ \prod\limits_{i=1}^n (1-x_i t)^s, & s < 0. \end{cases} \tag{47}$$

The *Whiteley mean* is now defined by

$$\mathcal{W}_n{}^{[r,s]}(\mathbf{x}) = \left(\omega_n{}^{[r,s]}(\mathbf{x})\right)^{\frac{1}{r}} = \begin{cases} \left(\frac{t_n^{[r,s]}(\mathbf{x})}{\binom{nr}{s}}\right)^{\frac{1}{r}}, & s > 0; \\ \left(\frac{t_n^{[r,s]}(\mathbf{x})}{(-1)^r\binom{nr}{s}}\right)^{\frac{1}{r}}, & s < 0. \end{cases} \tag{48}$$

For $s < 0$, the Whiteley means can be further generalized if we slightly change the definition of ${t_n}^{[r,s]}(\mathbf{x})$ and define ${h_n}^{[r,\sigma]}$ as follows:

$$\sum_{r=0}^{\infty} h_n^{[r,s]}(\mathbf{x})t^r = \prod_{i=1}^{n} \frac{1}{(1+x_i t)^{\sigma_i}},$$

where $\sigma = (\sigma_1, \ldots, \sigma_n)$; $\sigma_i \in \mathbb{R}_+,\ i = 1, \ldots, n$. The following generalization of the Whiteley mean for $s < 0$ is defined by (see , Lemma 2.3 [13])

$$\mathcal{H}_n{}^{[r,\sigma]}(\mathbf{x}) = \left(\frac{h_n^{[r,\sigma]}(\mathbf{x})}{\binom{\sum_{i=1}^{n} \sigma_i + r - 1}{r}} \right). \tag{49}$$

If we denote a measure on the simplex Δ_{n-1} by μ with

$$d\mu(\mathbf{u}) = \frac{\Gamma\left(\sum\limits_{i=1}^{n} \sigma_i\right)}{\prod\limits_{i=1}^{n} \Gamma(\sigma_i)} \prod_{i=1}^{n} u_i^{\sigma_i - 1} du_1, \ldots, du_{n-1}, \tag{50}$$

then we obtain a probability measure and we can also write the mean $\mathcal{H}_n{}^{[r,\sigma]}$ in integral form as follows:

$$\mathcal{H}_n{}^{[r,\sigma]}(\mathbf{x}) = \left(\int_{\Delta_{n-1}} \left(\sum_{i=1}^{n} x_i u_i \right)^r d\mu(\mathbf{u}) \right)^{\frac{1}{r}}. \tag{51}$$

Similar to above examples, for measure μ, and $\alpha(u) = \sum_{j=1}^{n} x_i u_i,\ \beta(u) = \sum_{j=1}^{n} y_i u_i$ and $\Omega = \Delta_{n-1}$, we define new Cauchy means:

$$\mathfrak{H}_{n,t,r}^{s}(\mathbf{x}, \mathbf{y}; \mu) = M_{r,t;s}^{4}(\alpha, \beta; \mu). \tag{52}$$

These means are generalizations of Cauchy means defined with (4.19) in [2] and again we have the monotonicity property:

for $t, r, u, v, s \in \mathbb{R},\ n \in \mathbb{N}$ such that, $t \leq v,\ r \leq u$, we have

$$\mathfrak{H}_{n,t,r}^{s}(\mathbf{x}, \mathbf{y}; \mu) \leq \mathfrak{H}_{n,v,u}^{s}(\mathbf{x}, \mathbf{y}; \mu). \tag{53}$$

References

[1] N. I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis*, Oliver and Boyd, Edinburgh, 1965.

[2] M. Anwar and J. Pečarić *New means of Cauchy's type*, J. Inequal. Appl., Art. ID 163202, 10 pages, (2008).

[3] T. M. Bisgaard and Z. Sasvári, *Characteristic Functions and Moment Sequences, Positive Definiteness in Probability,* Huntington, NY: Nova Science Publishers, 2000.

[4] V. Čuljak, I. Franjić, J. Pečarić and G. Roqia, *Schur-convexity of averges of convex functions* , J. Inequal. Appl., Art. ID 581918, 25 pages, (2011).

[5] W. Ehm, M. G. Genton and T. Gneiting, *Stationary covariances associated with exponentially convex functions*, Bernoulli, 9 (4) (2003), 607–615.

[6] E. Issacson and H. B. Keller, *Analysis of Numerical methods,* Dover Publications, Inc. New York, 1966.

[7] J. Jakšetić and J. Pečarić, *Exponential Convexity Method,* (submitted).

[8] A. McD. Mercer, *Some new inequalities involving elementary mean values,* J. Math. Anal. Appl., 229 1999), 677–681.

[9] J. E. Pečarić, F. Proschan, and Y. L.Tong, *Convex functions, partial orderings, and statistical applications,* Academic Press, Inc, 1992.

[10] J. Pečarić and G. Roqia, *Generalization of Stolarsky type means,* J. Inequal. Appl., Art. ID 720615, 15 pages, (2010).

[11] J. Pečarić, M. R. Lipanović and H. N. Srivastava, *Some mean value theorems of Cauchy type*, Fract. Calc. Appl. Anal., 9 (2) (2006), 143–158.

[12] J. Pečarić, I. Perić and H. N. Srivastava, *A family of Cauchy type mean value theorems,* J. Math. Anal. Appl.,36 (2005), 730–739.

[13] J. Pečarić, I. Perić and M. R. Lipanović, *Integral representations of generalized Whiteley means and related inequalities,* Math. Inequal. Appl.,12 (2) (2009), 295–309.

[14] J. Pečarić and V. Šimić, *Stolarsky-Tobey mean in n variables,* Math. Inequal. Appl., 2 (3) (1999), 325–341.

[15] J. L. Schiff, *The Laplace transform. Theory and applications.* Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1999.

[16] S. Simić, *An extension of Storalsky means to the multivariable case,* Int. J. Math. Mat. Sci., Art. ID 432857, 14 pages, (2009).

[17] S. Simić, *On certain new inequalities in information theory,* Acta Math. Hungar., 124 (4) (2009), 353–361.

[18] D. E. Wulbert, *Favard's Inequality on average values of convex functions,* Math. Comp. Mod., 30 (3) (2000), 853–856.

(Received: May 31, 2011)

J. Jakšetić
Faculty of Mechanical Engineering and
Naval Architecture
University of Zagreb
10000 Zagreb, Crotia
E–mail: julije@math.hr

J. Pečarić
Faculty of Textile Technology
University of Zagreb
Pierottijeva 6, 10 000 Zagreb, Croatia
E–mail: pecaric@hazu.hr

G. Roqia
Abdus Salam School of
Mathematical Sciences
68-B, New Muslim Town
Lahore 54000, Pakistan
E–mai: rukiyya@gmail.com