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UNIQUENESS OF MEROMORPHIC FUNCTIONS WHEN TWO DIFFERENTIAL POLYNOMIALS SHARE ONE VALUE

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Abstract. The paper deals with the uniqueness problems of meromorphic functions related to differential polynomials that share one value with finite weight. We prove three uniqueness results which significantly rectify, improve and supplement some recent results of [6] and [8].

1. Introduction, definitions and results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [12, 23, 25]. Let E denotes any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a non-constant meromorphic function f, we denote by $T(r,f)$ the Nevanlinna characteristic of f and by $S(r,f)$ any quantity satisfying $S(r,f) = o\{T(r,f)\}(r \to \infty, r \notin E)$. We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$, by $S(r)$ any quantity satisfying $S(r) = o\{T(r)\}$ $(r \to \infty, r \notin E)$.

Let f and g be two non-constant meromorphic functions. We say that f and g share the value a CM (counting multiplicities), if $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share the value a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. Throughout this paper, we need the following definition.

$$\Theta(a,f) = 1 - \limsup_{r \longrightarrow \infty} \frac{\overline{N}(r,a;f)}{T(r,f)},$$

where a is a value in the extended complex plane.

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In [13] following question was asked: What can be said if two non-linear differential polynomials generated by two meromorphic functions share 1 CM ?

During the last couple of years a significant number of authors worked on the uniqueness problem of meromorphic functions when the differential polynomials generated by them share certain values and continuous efforts are being put in to relax the hypothesis of the results. {cf. [2], [3], [5], [9], [10], [13], [16], [19], [20], [22]}. In the following we are recalling a few of them.

In 1997, Yang and Hua [22] proved the following result.

Theorem A. *Let f and g be two non-constant meromorphic functions, $n(\geq 11)$ an integer and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share the value a CM, then either $f = tg$ for some $(n+1)th$ root of unity 1 or $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1c_2)^{n+1}c^2 = -a^2$.*

In 2004 Lin and Yi [20] proved the following results.

Theorem B. *Let f and g be two non-constant meromorphic functions satisfying $\Theta(\infty, f) > 2/(n+1)$, $n \geq 12$ an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share the value 1 CM, then $f \equiv g$.*

Theorem C. *Let f and g be two non-constant meromorphic functions, $n \geq 13$ an integer. If $f^n(f-1)^2 f'$ and $g^n(g-1)^2 g'$ share the value 1 CM, then $f \equiv g$.*

A new trend in this direction is to consider the uniqueness of a meromorphic function concerning the value sharing of the k-th derivatives of a linear expression of a meromorphic function which ultimately produce a non-linear differential polynomial. Recently, in order to consider the uniqueness of meromorphic function related to the value sharing of two non-linear differential polynomials Dyavanal [6] also raised the same question. First of all we state the following results of Dyavanal in which the multiplicities of zeros and poles of f and g are taken into account for the value sharing of differential functions.

Theorem E (Theorem 1.1, [6])**.** *Let f and g be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let $n \geq 2$ be an integer satisfying $(n+1)s \geq 12$. If $f^n f'$ and $g^n g'$ share the value 1 CM, then either $f = dg$ for some $(n+1)$-th root of unity 1 or $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1c_2)^{n+1}c^2 = -1$.*

Theorem F (Theorem 1.2, [6])**.** *Let f and g be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least s, where* *s is a positive integer and* $\Theta(\infty, f) > 2/(n+1)$. *Let* $n \geq 4$ *be an integer satisfying* $(n+1)s \geq 12$. *If* $f^n(f-1)f'$ *and* $g^n(g-1)g'$ *share the value* 1 *CM, then* $f \equiv g$.

Theorem G (Theorem 1.3, [6]). *Let* f *and* g *be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least* s, *where* s *is a positive integer. Let* $n \geq 3$ *be an integer satisfying* $(n+1)s \geq 12$. *If* $f^n(f-1)^2f'$ *and* $g^n(g-1)^2g'$ *share the value* 1 *CM, then* $f \equiv g$.

Next we state the the following open questions posed by Dyavanal in [6].

Question 1.1. *Can the differential polynomials in Theorems E - G be replaced by differential polynomials of the form* $(f^n)^{(k)}$ *and* $[f^n(f-1)]^{(k)}$ *?*

Question 1.2. *Can CM shared value be replaced by an IM shared value in Theorems E - G ?*

The ideas resorted by the author in [6] are no doubt novel but there are some mistakes in the paper. For example in page 7, in the proof of Theorem 1.2 [6] there is a serious lacuna when a counting function is being elaborated and then restricted in terms of Nevanlinna's characteristic function.

Actually in Page 7, line 8 on-wards from bottom should be

$$\begin{aligned}\overline{N}\left(r, \frac{1}{F}\right) &= \overline{N}\left[r, \frac{1}{f^{n+1}(f - \frac{n+2}{n+1})}\right] \\ &\leq \frac{1}{s(n+1)} N\left(r, \frac{1}{f^{n+1}}\right) + \overline{N}\left(r, \frac{1}{f - \frac{n+2}{n+1}}\right) \not\leq \frac{1}{s(n+1)} N\left(r, \frac{1}{F}\right),\end{aligned}$$

since nowhere in the paper it has been assumed that the zeros of $f - \frac{n+2}{n+1}$ are of multiplicities $s(n+1)$. Since the above counting function just mentioned above has a vital contribution in the proofs of *Theorems 1.2, 1.3* and *1.5* in [6], the three theorems namely *Theorems 1.2, 1.3* and *1.5* in [6] are not correct.

However recently the author has rectified the paper {see [7], [8]} and *Theorems F* and *G* has been replaced by the following theorems.

Theorem H (Theorem 1.2, [8]). *Let* f *and* g *be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least* s, *where* s *is a positive integer. Let* n *be an integer satisfying* $(n-2)s \geq 10$. *If* $f^n(f-1)f'$ *and* $g^n(g-1)g'$ *share the value* 1 *CM, then* $g = \frac{(n+2)(1-h^{n+1})}{(n+1)(1-h^{n+2})}$, $f = \frac{(n+2)h(1-h^{n+1})}{(n+1)(1-h^{n+2})}$, *where* h *is a non-constant meromorphic function.*

Theorem I (Theorem 1.3, [8]). *Let* f *and* g *be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least* s, *where*

s is a positive integer. Let n be an integer satisfying $(n-3)s \geq 10$. If $f^n(f-1)^2f'$ and $g^n(g-1)^2g'$ share the value 1 CM, then $f \equiv g$.

Since in *Theorem H* one can easily check $h = \frac{f}{g}$, *Theorem H* is not very significant. In this paper though our main intention is to deal with the questions 1.1-1.2 made by the author in [6] but at the time of execution of the process we will rectify, improve and supplement all the theorems of [6] and [8]. To this end we are now introducing the notion of weighted sharing of values introduced by I. Lahiri in [15] which measure how close a shared value is to being shared IM or to being shared CM. The notion is explained in the following definition.

Definition 1.1. *Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and k+1 times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.*

This definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g *share* (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

We now state the main results of the paper.

Theorem 1.1. *Let f and g be two transcendental meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let $(f^n)^{(k)}$ and $(g^n)^{(k)}$ share (b, l) where $n(\geq 3)$, $k(\geq 1)$, $l(\geq 0)$ are integers, $b(\neq 0)$ is a constant and one of the following conditions holds:*
(a) $l \geq 2$ *and* $n > (3k+8)/s$;
(b) $l = 1$ *and* $n > (4k+9)/s$;
(c) $l = 0$ *and* $n > (9k+14)/s$.
Then either $(f^n)^{(k)}(g^n)^{(k)} \equiv b^2$ or $f = dg$ for some $(n+1)$-th root of unity 1.

If $k = 1$, then $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1c_2)^nc^2 = -\frac{b^2}{n^2}$.

Putting $n = t+1$ in the above theorem we obtain the following corollary.

Corollary 1.1. *Let f and g be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Suppose f^tf' and g^tg' share (b, l) where $l(\geq 0)$ is an integer and*

$b(\neq 0)$ is a constant. Then the conclusion of Theorem E holds provided one of the following holds:
(a) $l \geq 2$ *and* $t > 11/s - 1$;
(b) $l = 1$ *and* $t > 13/s - 1$;
(c) $l = 0$ *and* $t > 23/s - 1$.

Remark 1.1. Putting $s = 1$ in the above corollary we get the improved and rectified form of Theorem E.

Theorem 1.2. *Let f and g be two transcendental meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer and $\Theta(\infty, f) + \Theta(\infty, g) > 4/n$. Let $[f^n(a_1 f + a_2)]^{(k)}$ and $[g^n(a_1 g + a_2)]^{(k)}$ share $(b, 1)$ where $k(\geq 1)$, $l(\geq 0)$ are integers, a_1, a_2, b are non-zero constants and one of the following conditions holds:*
(a) $l \geq 2$ *and* $n > \max\{(3k + 8)/s + 1, 3 + 2/s\}$;
(b) $l = 1$ *and* $n > \max\{(4k + 9)/s + 3/2, 3 + 2/s\}$;
(c) $l = 0$ *and* $n > \max\{(9k + 14)/s + 4, 3 + 2/s\}$.
Then either $[f^n(a_1 f + a_2)]^{(k)}[g^n(a_1 g + a_2)]^{(k)} \equiv b^2$ or $f \equiv g$.

The possibility $[f^n(a_1 f + a_2)]^{(k)}[g^n(a_1 g + a_2)]^{(k)} \equiv b^2$ does not occur for $k = 1$.

Putting $n = t + 1$, $a_1 = \frac{1}{t+2}$, $a_2 = -\frac{1}{t+1}$ and $k = 1$ in the above theorem we can immediately deduce the following corollary.

Corollary 1.2. *Let f and g be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer and $\Theta(\infty, f) + \Theta(\infty, g) > 4/(t + 1)$. Suppose $f^t(f - 1)f'$ and $g^t(g - 1)g'$ share (b, l) where $l(\geq 0)$ is an integer and $b(\neq 0)$ is a constant. Then $f \equiv g$ provided one of the following holds:*
(a) $l \geq 2$ *and* $t > \max\{11/s, 2 + 2/s\}$;
(b) $l = 1$ *and* $t > \max\{13/s + 1/2, 2 + 2/s\}$;
(c) $l = 0$ *and* $t > \max\{23/s + 3, 2 + 2/s\}$.

Remark 1.2. Putting $s = 1$ in the above corollary we get the improved and rectified form of Theorem F.

Theorem 1.3. *Let f and g be two transcendental meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let $[f^n(a_1 f^2 + a_2 f + a_3)]^{(k)}$ and $[g^n(a_1 g^2 + a_2 g + a_3)]^{(k)}$ share $(b, 1)$ where $k(\geq 1)$, $l(\geq 0)$ are integers, a_1, a_2, a_3, b are non-zero constants and one of the following conditions holds:*
(a) $l \geq 2$ *and* $n > \max\{(3k + 8)/s + 2, 4 + 4/s\}$;
(b) $l = 1$ *and* $n > \max\{(4k + 9)/s + 3, 4 + 4/s\}$;
(c) $l = 0$ *and* $n > \max\{(9k + 14)/s + 8, 4 + 4/s\}$.

Then either $[f^n(a_1f^2+a_2f+a_3)]^{(k)}[g^n(a_1g^2+a_2g+a_3)]^{(k)} \equiv b^2$ *or* $f \equiv g$ *or* f, g *satisfy the algebraic equation* $R(f,g)=0$, *where*

$$R(x,y) = x^n(a_1x^2+a_2x+a_3) - y^n(a_1y^2+a_2y+a_3).$$

The possibility $[f^n(a_1f^2+a_2f+a_3)]^{(k)}[g^n(a_1g^2+a_2g+a_3)]^{(k)} \equiv b^2$ *does not occur for* $k=1$.

Putting $n=t+1$, $a_1=\frac{1}{t+3}$, $a_2=-\frac{2}{t+2}$, $a_3=\frac{1}{t+1}$ and $k=1$ in the above theorem we can immediately deduce the following corollary.

Corollary 1.3. *Let* f *and* g *be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least* s, *where* s *is a positive integer. Suppose* $f^t(f-1)^2f'$ *and* $g^t(g-1)^2g'$ *share* (b,l) *where* $l(\geq 0)$ *is an integer and* $b(\neq 0)$ *is a constant. Then* $f \equiv g$ *provided one of the following holds:*
(a) $l \geq 2$ *and* $t > \max\{11/s+1, 3+4/s\}$;
(b) $l = 1$ *and* $t > \max\{13/s+2, 3+4/s\}$;
(c) $l = 0$ *and* $t > \max\{23/s+7, 3+4/s\}$.

Remark 1.3. Putting $s=1$ in the above corollary we get the improved and rectified form of Theorem G. This corollary improves Theorem I as well.

We now explain the following definitions and notations which are used in the paper.

Definition 1.2. [14] *For* $a \in \mathbb{C}\cup\{\infty\}$ *we denote by* $N(r,a;f\mid=1)$ *the counting function of simple* a*-points of* f. *For a positive integer* p *we denote by* $N(r,a;f\mid\leq p)$ *the counting function of those* a*-points of* f *(counted with multiplicities) whose multiplicities are not greater than* p. *By* $\overline{N}(r,a;f\mid\leq p)$ *we denote the corresponding reduced counting function.*

In an analogous manner we define $N(r,a;f\mid\geq p)$ *and* $\overline{N}(r,a;f\mid\geq p)$.

Definition 1.3. [18] *Let* p *be a positive integer or infinity. We denote by* $N_p(r,b;f)$ *the counting function of* b*-points of* f, *where a* b*-point of multiplicity* m *is counted* m *times if* $m \leq p$ *and* p *times if* $m > p$. *Then*

$$N_p(r,b;f) = \overline{N}(r,b;f) + \overline{N}(r,b;f\mid\geq 2) + \ldots + \overline{N}(r,b;f\mid\geq p).$$

Clearly $N_1(r,b;f) = \overline{N}(r,b;f)$.

Definition 1.4. [4] *Let* $a,b \in \mathbb{C}\cup\{\infty\}$ *and* p *be a positive integer. Then we denote by* $\overline{N}(r,a;f\mid\geq p\mid g=b)$ $(\overline{N}(r,a;f\mid\geq p\mid g\neq b))$ *the reduced counting function of those* a*-points of* f *with multiplicities* $\geq p$, *which are* b*-points (not* b*-points) of* g.

Definition 1.5. [1, 2] *Let* f *and* g *be two non-constant meromorphic functions such that* f *and* g *share the value* 1 *IM. Let* z_0 *be a* 1*-point of* f

with multiplicity p and also a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r,1;f)$ the counting function of those 1-points of f and g, where $p>q$, by $N_E^{1)}(r,1;f)$ the counting function of those 1-points of f and g, where $p=q=1$ and by $\overline{N}_E^{(2}(r,1;f)$ the counting function of those 1-points of f and g, where $p=q\geq 2$, where each point in these counting functions is counted only once. In the same manner we can define $\overline{N}_L(r,1;g)$, $N_E^{1)}(r,1;g)$ and $\overline{N}_E^{(2}(r,1;g)$.

Definition 1.6. [1, 2] *Let f and g be two non-constant meromorphic functions such that f and g share the value 1 IM. Let z_0 be a 1-point of f with multiplicity p and also a 1-point of g with multiplicity q. For a positive integer k, $\overline{N}_{f>k}(r,1;g)$ denotes the reduced counting function of those 1-points of f and g such that $p>q=k$. In an analogous way we can define $\overline{N}_{g>k}(r,1;f)$.*

2. Lemmas

In this section we present some lemmas which will be needed to prove the theorem.

Lemma 2.1. [12] *Let f be a non-constant meromorphic function, k be a positive integer, and let c be a non-zero finite complex number. Then*

$$\begin{aligned}T(r,f) &\leq \overline{N}(r,\infty;f)+N(r,0;f)+N(r,c;f^{(k)})-N(r,0;f^{(k+1)})+S(r,f)\\ &\leq \overline{N}(r,\infty;f)+N_{k+1}(r,0;f)+\overline{N}(r,c;f^{(k)})-N_0(r,0;f^{(k+1)})+S(r,f),\end{aligned}$$

where $N_0(r,0;f^{(k+1)})$ denotes the counting function which only counts those points such that $f^{(k+1)}=0$ but $f(f^{(k)}-c)\neq 0$.

Lemma 2.2. [2] *Let f and g be two non-constant meromorphic functions that share $(1,0)$. Then*

$$\begin{aligned}\overline{N}_L(r,1;f)+2\overline{N}_L(r,1;g)+N_E^{(2}(r,1;f)-\overline{N}_{f>1}(r,1;g)-\overline{N}_{g>1}(r,1;f)&\\ \leq N(r,1;g)-\overline{N}(r,1;g).&\end{aligned}$$

Lemma 2.3. [24] *Let f and g be two non-constant meromorphic functions sharing $(1,0)$. Then*

$$\overline{N}_L(r,1;f)\leq \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+S(r,f).$$

Lemma 2.4. [2] *Let f and g be two non-constant meromorphic functions sharing $(1,0)$. Then*
(i) $\overline{N}_{f>1}(r,1;g)\leq \overline{N}(r,0;f)+\overline{N}(r,\infty;f)-N_{\oplus}(r,0;f')+S(r,f)$;
(ii) $\overline{N}_{g>1}(r,1;f)\leq \overline{N}(r,0;g)+\overline{N}(r,\infty;g)-N_{\oplus}(r,0;g')+S(r,g)$, *where*

$N_{\oplus}(r,0;f')$ $(N_{\oplus}(r,0;g'))$ *denotes the counting function of those zeros of* f' (g') *which are not zeros of* $f(f-1)$ $(g(g-1))$.

Lemma 2.5. [1] *Let* f *and* g *be two non-constant meromorphic functions that share* $(1,1)$. *Then*

$$2\overline{N}_L(r,1;f)+2\overline{N}_L(r,1;g)+N_E^{(2}(r,1;f)-\overline{N}_{f>2}(r,1;g) \le N(r,1;g)-\overline{N}(r,1;g).$$

Lemma 2.6. [2] *Let* f *and* g *be two non-constant meromorphic functions that share* $(1,1)$. *Then*

$$\overline{N}_{f>2}(r,1;g) \le \frac{1}{2}\overline{N}(r,0;f)+\frac{1}{2}\overline{N}(r,\infty;f)-\frac{1}{2}N_{\oplus}(r,0;f')+S(r,f),$$

where $N_{\oplus}(r,0;f')$ *is defined as in Lemma* 2.4.

Lemma 2.7. [21] *Let* f *be a non-constant meromorphic function and* $P(f)$ $=a_0+a_1f+a_2f^2+\cdots+a_nf^n$, *where* a_0, a_1, $a_2,\ldots,a_n$ *are constants and* $a_n \ne 0$. *Then*

$$T(r,P(f)) = nT(r,f)+S(r,f).$$

Lemma 2.8. [26] *Let* f *and* g *be two non-constant meromorphic functions, and let* p, k *be two positive integers. Then*

$$N_p(r,0;f^{(k)}) \le N_{p+k}(r,0;f)+k\overline{N}(r,\infty;f)+S(r,f).$$

Lemma 2.9. *Let* f *and* g *be two non-constant meromorphic functions whose zeros and poles are of multiplicities at least* s, *where* s *is a positive integer. If* $n(>3+2/s)$ *is an integer, then*

$$[f^n(a_1f+a_2)]'[g^n(a_1g+a_2)]' \not\equiv b^2,$$

where a_1, a_2 *and* b *are non-zero constants.*

Proof. We suppose that $[f^n(a_1f+a_2)]'[g^n(a_1g+a_2)]' \equiv b^2$. T hen

$$f^{n-1}(k_1f+k_2)f'g^{n-1}(k_1g+k_2)g' \equiv b^2, \tag{2.1}$$

where $k_1 = a_1(n+1)$, $k_2 = a_2n$. Let z_0 be a zero of f with multiplicity $p_0(\ge s)$. Then z_0 is a pole of g with multiplicity $q_0(\ge s)$, say. From (2.1) we obtain

$$np_0 - 1 = (n+1)q_0 + 1$$

and so

$$n(p_0-q_0) = q_0+2. \tag{2.2}$$

From (2.2) we get $q_0 \ge n-2$ and so again from (2.2) we obtain

$$p_0 \ge \frac{(n+1)q_0+2}{n} \ge n-1.$$

Let z_1 be a zero of $k_1 f + k_2$ with multiplicity $p_1(\geq 1)$. Then z_1 is a pole of g with multiplicity $q_1(\geq s)$, say. So from (2.1) we get

$$2p_1 - 1 \quad = \quad (n+1)q_1 + 1$$

which gives

$$p_1 \geq \frac{(n+1)s+2}{2}.$$

Since a pole of f is either a zero of $g^n(k_1 g + k_2)$ or a zero of g', we have

$$\begin{aligned}\overline{N}(r,\infty;f) &\leq \overline{N}(r,0;g) + \overline{N}(r,-\frac{k_2}{k_1};g) + \overline{N}_0(r,0;g') \\ &\leq \left(\frac{1}{n-1} + \frac{2}{(n+1)s+2}\right) T(r,g) + \overline{N}_0(r,0;g'),\end{aligned}$$

where $\overline{N}_0(r,0;g')$ denotes the reduced counting function of those zeros of g' which are not the zeros of $g(k_1 g + k_2)$.

Then by the second fundamental theorem of Nevanlinna we get

$$\begin{aligned}T(r,f) &\leq \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + \overline{N}(r,-\frac{k_2}{k_1};f) - \overline{N}_0(r,0;f') + S(r,f) \\ &\leq \left(\frac{1}{n-1} + \frac{2}{(n+1)s+2}\right)\{T(r,f)+T(r,g)\} + \overline{N}_0(r,0;g') \\ &\qquad - \overline{N}_0(r,0;f') + S(r,f). \qquad (2.3)\end{aligned}$$

Similarly we get

$$\begin{aligned}T(r,g) &\leq \left(\frac{1}{n-1} + \frac{2}{(n+1)s+2}\right)\{T(r,f)+T(r,g)\} + \overline{N}_0(r,0;f') \\ &\qquad - \overline{N}_0(r,0;g') + S(r,g). \qquad (2.4)\end{aligned}$$

Adding (2.3) and (2.4) we obtain

$$\left(1 - \frac{2}{n-1} - \frac{4}{(n+1)s+2}\right)\{T(r,f)+T(r,g)\} \leq S(r,f) + S(r,g),$$

which leads to a contradiction as $n > 3 + 2/s$. This proves the lemma. □

Lemma 2.10. *Let f and g be two non-constant entire functions and n be a positive integer. If $f^n f' g^n g' = b^2$, where b is a non-zero constant, then $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1 c_2)^{n+1} c^2 = -b^2$.*

Proof. We omit the proof since it can be proved in a manner similar to the proof of Theorem 3 in [22]. □

Lemma 2.11. *Let f and g be two non-constant meromorphic functions whose zeros and poles are of multiplicities at least s, where s is a positive integer. If $n(> 4 + 4/s)$ is an integer, then*

$$[f^n(a_1f^2 + a_2f + a_3)]'[g^n(a_1g^2 + a_2g + a_3)]' \not\equiv b^2,$$

where a_1, a_2, a_3 and b are non-zero constants.

Proof. we assume that $[f^n(a_1f^2 + a_2f + a_3)]'[g^n(a_1g^2 + a_2g + a_3)]' \equiv b^2$. Then

$$f^{n-1}(k_1f^2 + k_2f + k_3)f'g^{n-1}(k_1g^2 + k_2g + k_3)g' \equiv b^2, \tag{2.5}$$

where $k_1 = a_1(n + 2)$, $k_2 = a_2(n + 1)$ and $k_3 = a_3n$. We consider the following two cases:

Case 1. Let $k_2^2 - 4k_1k_3 \neq 0$.
Let z_0 be a zero of f with multiplicity $p_0(\geq s)$. Then from (2.5) we know z_0 is a pole of g with multiplicity $q_0(\geq s)$, such that

$$np_0 - 1 = (n + 2)q_0 + 1,$$

and so

$$2q_0 + 2 = n(p_0 - q_0). \tag{2.6}$$

From (2.6) we get $q_0 \geq \frac{n-2}{2}$ which implies

$$p_0 \geq \frac{1}{n}\left[\frac{(n + 2)(n - 2)}{2} + 2\right] = \frac{n}{2}.$$

Let z_1 be a zero of $k_1f^2 + k_2f + k_3$ with multiplicity $p_1(\geq 1)$. Then from (2.5) we know z_1 is a pole of g with multiplicity $q_1(\geq s)$, say for which

$$2p_1 - 1 = (n + 2)q_1 + 1,$$

i.e.,

$$p_1 \geq \frac{(n + 2)s + 2}{2}.$$

Since a pole of f is either a zero of $g(k_1g^2 + k_2g + k_3)$ or a zero of g', we have

$$\begin{aligned}\overline{N}(r,\infty; f) &\leq \overline{N}(r, 0; g) + \overline{N}(r, 0; k_1g^2 + k_2g + k_3)\\ &\qquad + \overline{N}_0(r, 0; g') + S(r, f) + S(r, g)\\ &\leq \left(\frac{2}{n} + \frac{4}{(n + 2)s + 2}\right)T(r, g) + \overline{N}_0(r, 0; g') + S(r, f) + S(r, g),\end{aligned}$$

where $\overline{N}_0(r, 0; g')$ denotes the reduced counting function of those zeros of g' which are not the zeros of $g(k_1g^2 + k_2g + k_3)$.

Let

$$k_1 f^2 + k_2 f + k_3 = k_1(f-\alpha_1)(f-\alpha_2),$$

where α_1, α_2 are distinct complex numbers. Then by the second fundamental theorem of Nevanlinna we get

$$\begin{aligned}2T(r,f) &\leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \sum_{j=1}^{2} \overline{N}(r,\alpha_j;f) - \overline{N}_0(r,0;f') + S(r,f)\\ &\leq \left(\frac{2}{n} + \frac{4}{(n+2)s+2}\right)\{T(r,f)+T(r,g)\} - \overline{N}_0(r,0;f') +\\ &\qquad \overline{N}_0(r,0;g') + S(r,f) + S(r,g). \quad (2.7)\end{aligned}$$

Similarly

$$\begin{aligned}2T(r,g) \leq &\left(\frac{2}{n} + \frac{4}{(n+2)s+2}\right)\{T(r,f)+T(r,g)\} + \overline{N}_0(r,0;f')\\ &- \overline{N}_0(r,0;g') + S(r,f) + S(r,g). \quad (2.8)\end{aligned}$$

Adding (2.7) and (2.8) we obtain

$$\left(1 - \frac{2}{n} - \frac{4}{(n+2)s+2}\right)\{T(r,f)+T(r,g)\} \leq S(r,f) + S(r,g),$$

which is a contradiction as $n > 2 + 2/s$.

Case 2. Let $k_2^2 - 4k_1k_3 = 0$. Then (2.5) can be written as

$$f^{n-1}(f-\alpha)^2 f' g^{n-1}(g-\alpha)^2 g' \equiv b^2.$$

Let z_2 be a zero of f with multiplicity $p_2(\geq s)$. Then z_2 is a pole of g with multiplicity $q_2(\geq s)$, say and so we obtain as in *Case 1* $p_2 \geq n/2$. Let z_3 be a zero of $f-\alpha$ with multiplicity $p_3(\geq 1)$. Then z_3 is a pole of g with multiplicity $q_3(\geq s)$, say. Then proceeding in the same way as done in the *Case 1* we obtain $p_3 \geq \frac{(n+2)s+2}{3}$. Since a pole of f is either a zero of $g(g-\alpha)^2$ or a zero of g', we have

$$\begin{aligned}\overline{N}(r,\infty;f) &\leq \overline{N}(r,0;g) + \overline{N}(r,\alpha;g) + \overline{N}_0(r,0;g') + S(r,f) + S(r,g)\\ &\leq \left(\frac{2}{n} + \frac{3}{(n+2)s+2}\right)T(r,g) + \overline{N}_0(r,0;g') + S(r,f) + S(r,g),\end{aligned}$$

where $\overline{N}_0(r,0;g')$ denotes the reduced counting function of those zeros of g' which are not the zeros of $g(g-\alpha)^2$. Then by the second fundamental theorem of Nevanlinna we get

$$T(r,f) \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}(r,\alpha;f) - \overline{N}_0(r,0;f') + S(r,f)$$

$$\leq \left(\frac{2}{n}+\frac{3}{(n+2)s+2}\right)\{T(r,f)+T(r,g)\}-\overline{N}_0(r,0;f')$$
$$+\overline{N}_0(r,0;g')+S(r,f)+S(r,g). \quad (2.9)$$

Similarly

$$T(r,g)\leq \left(\frac{2}{n}+\frac{3}{(n+2)s+2}\right)\{T(r,f)+T(r,g)\}+\overline{N}_0(r,0;f')$$
$$-\overline{N}_0(r,0;g')+S(r,f)+S(r,g). \quad (2.10)$$

Adding (2.9) and (2.10) we obtain

$$\left(1-\frac{4}{n}-\frac{6}{(n+2)s+2}\right)\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g),$$

which is a contradiction as $n>4+4/s$. This proves the lemma. □

Lemma 2.12. *Let f and g be two non-constant meromorphic functions such that*

$$\Theta(\infty,f)+\Theta(\infty,g)>\frac{4}{n},$$

where $n(\geq 3)$ is an integer. Then

$$f^n(af+b)\equiv g^n(ag+b)$$

implies $f\equiv g$, where a, b are two non-zero constants.

Proof. We omit the proof since it is similar to the proof of Lemma 6 [17]. □

Lemma 2.13. [11] *Let*

$$Q(w)=(n-1)^2(w^n-1)(w^{n-2}-1)-n(n-2)(w^{n-1}-1)^2,$$

then

$$Q(w)=(w-1)^4(w-\nu_1)(w-\nu_2)\ldots(w-\nu_{2n-6}),$$

where $\nu_j\in C\setminus\{0,1\}$ $(j=1,2,\ldots,2n-6)$, which are distinct respectively.

3. Proof of the theorem

Proof of Theorem 1.2. Let $F(z)=\frac{f^n(a_1f+a_2)}{b}$ and $G(z)=\frac{g^n(a_1g+a_2)}{b}$. Then $F^{(k)}$ and $G^{(k)}$ share $(1,l)$. Let

$$H=\left(\frac{F^{(k+2)}}{F^{(k+1)}}-\frac{2F^{(k+1)}}{F^{(k)}-1}\right)-\left(\frac{G^{(k+2)}}{G^{(k+1)}}-\frac{2G^{(k+1)}}{G^{(k)}-1}\right). \quad (3.1)$$

We assume that $H\not\equiv 0$. Let $l\geq 1$. Suppose that z_0 be a simple 1-point of $F^{(k)}$. Then z_0 is a simple 1-point of $G^{(k)}$. So by a simple computation on local expansions we see that z_0 is a zero of H. Thus

$$N(r,1;F^{(k)}\mid=1)\leq N(r,0;H)\leq T(r,H)+O(1)$$

$$\leq N(r,\infty;H)+S(r,F)+S(r,G). \qquad (3.2)$$

From (3.1) we know that poles of H possibly result from those zeros of $F^{(k+1)}$ and $G^{(k+1)}$ which are not common 1-points of $F^{(k)}$ and $G^{(k)}$, from poles of F and G, and from those common 1-points of $F^{(k)}$ and $G^{(k)}$ such that each such point has different multiplicity related to $F^{(k)}$ and $G^{(k)}$. Thus

$$\begin{aligned} N(r,\infty;H) &\leq \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}(r,0;F^{(k)} \mid\geq 2) \\ &\quad + \overline{N}(r,0;G^{(k)} \mid\geq 2)+\overline{N}_L(r,1;F^{(k)})+\overline{N}_L(r,1;G^{(k)}) \\ &\quad + \overline{N}_{\otimes}(r,0;F^{(k+1)})+\overline{N}_{\otimes}(r,0;G^{(k+1)}), \end{aligned} \qquad (3.3)$$

where $\overline{N}_{\otimes}(r,0;F^{(k+1)})$ denotes the reduced counting function of those zeros of $F^{(k+1)}$ which are not zeros of $F^{(k)}(F^{(k)}-1)$. Now we consider the following three cases.

Case 1. Let $l\geq 2$. By (3.2) and (3.3) we obtain

$$\begin{aligned} \overline{N}(r,1;F^{(k)}) &\leq N(r,1;F^{(k)} \mid= 1)+\overline{N}(r,1;F^{(k)} \mid\geq 2) \\ &\leq \overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}(r,0;F^{(k)} \mid\geq 2) \\ &\quad +\overline{N}(r,0;G^{(k)} \mid\geq 2)+\overline{N}_L(r,1;F^{(k)})+\overline{N}_L(r,1;G^{(k)}) \\ &\quad +\overline{N}(r,1;F^{(k)} \mid\geq 2)+\overline{N}_{\otimes}(r,0;F^{(k+1)}) \\ &\quad +\overline{N}_{\otimes}(r,0;G^{(k+1)})+S(r,F)+S(r,G). \end{aligned} \qquad (3.4)$$

From (3.4) and Lemma 2.1 we obtain

$$\begin{aligned} T(r,F)+T(r,G) &\leq 2\overline{N}(r,\infty;F)+2\overline{N}(r,\infty;G)+N_{k+1}(r,0;F) \\ &\quad +N_{k+1}(r,0;G)+\overline{N}(r,0;F^{(k)} \mid\geq 2)+\overline{N}(r,0;G^{(k)} \mid\geq 2) \\ &\quad +\overline{N}_L(r,1;F^{(k)})+\overline{N}_L(r,1;G^{(k)})+\overline{N}(r,1;F^{(k)} \mid\geq 2) \\ &\quad +\overline{N}(r,1;G^{(k)})+\overline{N}_{\otimes}(r,0;F^{(k+1)})+\overline{N}_{\otimes}(r,0;G^{(k+1)}) \\ &\quad -N_0(r,0;F^{(k+1)})-N_0(r,0;G^{(k+1)})+S(r,F)+S(r,G). \end{aligned} \qquad (3.5)$$

It is clear that

$$\begin{aligned} & N_{k+1}(r,0;F)+\overline{N}(r,0;F^{(k)} \mid\geq 2)+\overline{N}_{\otimes}(r,0;F^{(k+1)}) \\ &\leq N_{k+1}(r,0;F)+\overline{N}(r,0;F^{(k)} \mid\geq 2 \mid F=0) \\ &\quad +\overline{N}(r,0;F^{(k)} \mid\geq 2 \mid F\neq 0)+\overline{N}_{\otimes}(r,0;F^{(k+1)}) \\ &\leq N_{k+1}(r,0;F)+\overline{N}(r,0;F \mid\geq k+2)+N_0(r,0;F^{(k+1)}) \\ &\leq N_{k+2}(r,0;F)+N_0(r,0;F^{(k+1)}). \end{aligned} \qquad (3.6)$$

A similar result holds for G also. Again

$$\overline{N}(r,1;F^{(k)} \mid\geq 2)+\overline{N}_L(r,1;F^{(k)})+\overline{N}_L(r,1;G^{(k)})+\overline{N}(r,1;G^{(k)})$$

$$
\begin{aligned}
&\leq \overline{N}(r,1;G^{(k)} \mid= 2) + 2\overline{N}_L(r,1;F^{(k)}) + 2\overline{N}_L(r,1;G^{(k)}) \\
&\quad + \overline{N}_E^{(3}(r,1;G^{(k)}) + \overline{N}(r,1;G^{(k)}) \\
&\leq N(r,1;G^{(k)}) \\
&\leq T(r,G^{(k)}) + O(1) \\
&\leq T(r,G) + k\overline{N}(r,\infty;G) + S(r,G). \qquad (3.7)
\end{aligned}
$$

So from (3.5), (3.6) and (3.7) we obtain

$$
\begin{aligned}
T(r,F) &\leq N_{k+2}(r,0;F) + N_{k+2}(r,0;G) + 2\overline{N}(r,\infty;F) \\
&\quad + (k+2)\overline{N}(r,\infty;G) + S(r,F) + S(r,G) \\
&\leq N_{k+2}(r,0;f^n) + N_{k+2}(r,0;a_1f+a_2) + N_{k+2}(r,0;g^n) + N_{k+2}(r,0;a_1g+a_2) \\
&\quad + 2\overline{N}(r,\infty;f) + (k+2)\overline{N}(r,\infty;g) + S(r,f) + S(r,g).
\end{aligned}
$$

From this and using Lemma 2.7 we obtain

$$
\begin{aligned}
&(n+1)T(r,f) \leq \left(\frac{k+4}{s}+1\right)T(r,f) \\
&+\left(\frac{2k+4}{s}+1\right)T(r,g) + S(r,f) + S(r,g) \leq \left(\frac{3k+8}{s}+2\right)T(r) + S(r). \qquad (3.8)
\end{aligned}
$$

Similarly

$$
(n+1)T(r,g) \leq \left(\frac{3k+8}{s}+2\right)T(r) + S(r). \qquad (3.9)
$$

From (3.8) and (3.9) we get

$$
\left(n - \frac{3k+8}{s} - 1\right)T(r) \leq S(r),
$$

which is a contradiction as $n > \frac{3k+8}{s} + 1$.

Case 2. Let $l = 1$. In view of Lemmas 2.1, 2.5, 2.6, 2.8, (3.2) and (3.3) we obtain

$$
\begin{aligned}
&\overline{N}(r,1;F^{(k)}) + \overline{N}(r,1;G^{(k)}) \\
&\leq N(r,1;F^{(k)} \mid= 1) + \overline{N}_L(r,1;F^{(k)}) + \overline{N}_L(r,1;G^{(k)}) \\
&\quad + \overline{N}_E^{(2}(r,1;F^{(k)}) + \overline{N}(r,1;G^{(k)}) \\
&\leq N(r,1;F^{(k)} \mid= 1) + N(r,1;G^{(k)}) - \overline{N}_L(r,1;F^{(k)}) \\
&\quad - \overline{N}_L(r,1;G^{(k)}) + \overline{N}_{F^{(k)}>2}(r,1;G^{(k)}) \\
&\leq \overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + \overline{N}(r,0;F^{(k)} \mid\geq 2) + \overline{N}(r,0;G^{(k)} \mid\geq 2)
\end{aligned}
$$

$$\begin{aligned}
&+\frac{1}{2}\overline{N}(r,0;F^{(k)})+\frac{1}{2}\overline{N}(r,\infty;F^{(k)})+T(r,G^{(k)})+\overline{N}_{\otimes}(r,0;F^{(k+1)})\\
&+\overline{N}_{\otimes}(r,0;G^{(k+1)})+S(r,F)+S(r,G)\\
\leq\ &\left(\frac{k}{2}+\frac{3}{2}\right)\overline{N}(r,\infty;F)+(k+1)\overline{N}(r,\infty;G)+\overline{N}(r,0;F^{(k)}\mid\geq 2)\\
&+\overline{N}(r,0;G^{(k)}\mid\geq 2)+\frac{1}{2}N_{k+1}(r,0;F)+T(r,G)+\overline{N}_{\otimes}(r,0;F^{(k+1)})\\
&+\overline{N}_{\otimes}(r,0;G^{(k+1)})+S(r,F)+S(r,G). \qquad (3.10)
\end{aligned}$$

Using (3.6) and (3.10) we obtain from Lemma 2.1 that

$$\begin{aligned}
T(r,F)\leq\ &N_{k+2}(r,0;F)+N_{k+2}(r,0;G)+\left(\frac{k}{2}+\frac{5}{2}\right)\overline{N}(r,\infty;F)\\
&+(k+2)\overline{N}(r,\infty;G)+\frac{1}{2}N_{k+1}(r,0;F)+S(r,F)+S(r,G).
\end{aligned}$$

From this and using Lemma 2.7 we obtain

$$\begin{aligned}
(n+1)T(r,f)\leq\ &\left(\frac{2k+5}{s}+\frac{3}{2}\right)T(r,f)+\left(\frac{2k+4}{s}+1\right)T(r,g)\\
&+S(r,f)+S(r,g)\leq\left(\frac{4k+9}{s}+\frac{5}{2}\right)T(r)+S(r). \qquad (3.11)
\end{aligned}$$

Similarly

$$(n+1)T(r,g)\leq\left(\frac{4k+9}{s}+\frac{5}{2}\right)T(r)+S(r). \qquad (3.12)$$

From (3.11) and (3.12) we get

$$\left(n-\frac{4k+9}{s}-\frac{3}{2}\right)T(r)\leq S(r),$$

which is a contradiction as $n>\frac{4k+9}{s}+\frac{3}{2}$.

Case 3. Let $l=0$. In this case (3.2) becomes

$$\begin{aligned}
N_E^{1)}(r,1;F^{(k)})&\leq N(r,0;H)\leq T(r,H)+O(1)\\
&\leq N(r,\infty;H)+S(r,F)+S(r,G). \qquad (3.13)
\end{aligned}$$

Using Lemmas 2.2, 2.3, 2.4, 2.8, (3.3), (3.4) and (3.13) we obtain

$$\begin{aligned}
&\overline{N}(r,1;F^{(k)})+\overline{N}(r,1;G^{(k)})\\
\leq\ &N_E^{1)}(r,1;F^{(k)})+\overline{N}_L(r,1;F^{(k)})+\overline{N}_L(r,1;G^{(k)})\\
&+\overline{N}_E^{(2}(r,1;F^{(k)})+\overline{N}(r,1;G^{(k)})\\
\leq\ &\overline{N}(r,\infty;F)+\overline{N}(r,\infty;G)+\overline{N}(r,0;F^{(k)}\mid\geq 2)+\overline{N}(r,0;G^{(k)}\mid\geq 2)
\end{aligned}$$

$$\begin{aligned}
&+\overline{N}_L(r,1;F^{(k)})+T(r,G^{(k)})+\overline{N}_{F^{(k)}>1}(r,1;G^{(k)})+\overline{N}_{G^{(k)}>1}(r,1;F^{(k)})\\
&+\overline{N}_{\otimes}(r,0;F^{(k+1)})+\overline{N}_{\otimes}(r,0;G^{(k+1)})+S(r,F)+S(r,G)\\
\leq\ &(2k+3)\overline{N}(r,\infty;F)+(2k+2)\overline{N}(r,\infty;G)+\overline{N}(r,0;F^{(k)}\mid\geq 2)\\
&+\overline{N}(r,0;G^{(k)}\mid\geq 2)+2N_{k+1}(r,0;F)+N_{k+1}(r,0;G)+T(r,G)\\
&+\overline{N}_{\otimes}(r,0;F^{(k+1)})+\overline{N}_{\otimes}(r,0;G^{(k+1)})+S(r,F)+S(r,G)\\
\leq\ &(2k+3)\overline{N}(r,\infty;F)+(2k+2)\overline{N}(r,\infty;G)+N_{k+1}(r,0;F)\\
&+N_{k+2}(r,0;F)+N_{k+2}(r,0;G)+T(r,G)+N_0(r,0;F^{(k+1)})\\
&+N_0(r,0;G^{(k+1)})+S(r,F)+S(r,G). \qquad (3.14)
\end{aligned}$$

Using Lemma 2.1 we get

$$\begin{aligned}
&T(r,F)\leq(2k+4)\overline{N}(r,\infty;F)+(2k+3)\overline{N}(r,\infty;G)+2N_{k+1}(r,0;F)\\
&\quad+N_{k+2}(r,0;F)+N_{k+1}(r,0;G)+N_{k+2}(r,0;G)+S(r,F)+S(r,G).
\end{aligned}$$

In view of Lemma 2.7 we obtain

$$\begin{aligned}
(n+1)T(r,f)&\leq\left(\frac{5k+8}{s}+3\right)T(r,f)+\left(\frac{4k+6}{s}+2\right)T(r,g)\\
&\quad+S(r,f)+S(r,g)\leq\left(\frac{9k+14}{s}+5\right)T(r)+S(r). \qquad (3.15)
\end{aligned}$$

Similarly

$$(n+1)T(r,g)\leq\left(\frac{9k+14}{s}+5\right)T(r)+S(r). \qquad (3.16)$$

From (3.15) and (3.16) we get

$$\left(n-\frac{9k+14}{s}-4\right)T(r)\leq S(r),$$

which is a contradiction as $n>\frac{9k+14}{s}+4$.

We now assume that $H\equiv 0$. That is

$$\frac{F^{(k+2)}}{F^{(k+1)}}-\frac{2F^{(k+1)}}{F^{(k)}-1}\equiv\frac{G^{(k+2)}}{G^{(k+1)}}-\frac{2G^{(k+1)}}{G^{(k)}-1}.$$

Integrating two sides of the above equality twice we get

$$\frac{1}{F^{(k)}-1}\equiv\frac{BG^{(k)}+A-B}{G^{(k)}-1}, \qquad (3.17)$$

where $A(\neq 0)$ and B are constants. From (3.17) it is clear that $F^{(k)}$ and $G^{(k)}$ share 1 CM and hence $F^{(k)}$ and $G^{(k)}$ share $(1,2)$. Thus $n>\max\{(3k+8)/s+1, 3+2/s\}$. Now we consider the following three cases.

Case I. Let $B \neq 0$ and $A = B$.
If $B = -1$, from (3.17) we obtain $F^{(k)}G^{(k)} \equiv 1$. That is

$$[f^n(a_1 f + a_2)]^{(k)}[g^n(a_1 g + a_2)]^{(k)} \equiv b^2. \tag{3.18}$$

Also by Lemma 2.9, (3.18) does not occur for $k = 1$.
Let $B \neq -1$. Then from (3.17) we get

$$\frac{1}{F^{(k)}} \equiv \frac{BG^{(k)}}{(1+B)G^{(k)} - 1}.$$

This together with Lemma 2.8 gives

$$\overline{N}\left(r, \frac{1}{1+B}; G^{(k)}\right) \leq k\overline{N}(r, \infty; f) + N_{k+1}(r, 0; F). \tag{3.19}$$

Using Lemma 2.1 we obtain from (3.19) that

$$\begin{aligned}
T(r, G) &\leq \overline{N}(r, \infty; G) + N_{k+1}(r, 0; G) + \overline{N}\left(r, \frac{1}{1+B}; G^{(k)}\right) \\
&\qquad - N_0(r, 0; G^{(k+1)}) + S(r, G) \\
&\leq \overline{N}(r, \infty; g) + N_{k+1}(r, 0; G) + k\overline{N}(r, \infty; f) + N_{k+1}(r, 0; F) \\
&\qquad + S(r, F) + S(r, G) \\
&\leq \left(\frac{2k+1}{s} + 1\right) T(r, f) + \left(\frac{k+2}{s} + 1\right) T(r, g) + S(r, f) + S(r, g).
\end{aligned}$$

Thus by Lemma 2.7 we obtain

$$\left(n - \frac{3k+3}{s} - 1\right) T(r, g) \leq S(r, g),$$

a contradiction.

Case II. Let $B \neq 0$ and $A \neq B$.
If $B = -1$, we have from (3.17) that

$$F^{(k)} = \frac{A}{-G^{(k)} + A + 1}.$$

Therefore

$$\overline{N}\left(r, \infty; \frac{A}{-G^{(k)} + A + 1}\right) = \overline{N}(r, \infty; f).$$

By Lemma 2.1 and using the same argument as in case I, we arrive at a contradiction.
If $B \neq -1$, we have from (3.17)

$$F^{(k)} - \left(1 + \frac{1}{B}\right) \equiv \frac{-A}{B^2\left(G^{(k)} + \frac{A-B}{B}\right)}.$$

Therefore

$$\overline{N}\left(r, 0; G^{(k)} + \frac{A-B}{B}\right) = \overline{N}(r, \infty; f).$$

Again by Lemma 2.1 and proceeding as above we reach a contradiction.

Case III. Now we assume that $B = 0$. Then from (3.17) we obtain

$$F^{(k)} = \frac{G^{(k)} + A - 1}{A}. \tag{3.20}$$

If $A \neq 1$ then from (3.20) we obtain

$$\overline{N}\left(r, 1-A; G^{(k)}\right) = \overline{N}\left(r, 0; F^{(k)}\right).$$

We can similarly deduce a contradiction as in Subcase I. So $A = 1$ and from (3.20) we obtain

$$F = G + \varphi(z), \tag{3.21}$$

where $\varphi(z)$ is a polynomial with degree at most $k-1$. We claim that $\varphi(z) \equiv 0$. Otherwise noting that f is transcendental when $k \geq 2$, in view of Lemma 2.7, by the second fundamental theorem of Nevanlinna we get

$$\begin{aligned}
(n+1)T(r,f) + O(1) &= T(r,F) \\
&\leq \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,\varphi(z);F) \\
&\quad + S(r,F) \\
&\leq \overline{N}(r,\infty;f) + \overline{N}(r,0;F) + \overline{N}(r,0;G) + S(r,F) \\
&\leq (1+2/s)T(r,f) + (1+1/s)T(r,g) \\
&\quad + S(r,f).
\end{aligned} \tag{3.22}$$

It is clear from (3.21) that

$$T(r,f) = T(r,g) + S(r,f).$$

This together with (3.22) gives

$$(n-1-3/s)T(r,f) \leq S(r,f),$$

which is impossible. Hence $\varphi(z) \equiv 0$. Hence (3.21) becomes $F \equiv G$. That is

$$f^n(a_1 f + a_2) \equiv g^n(a_1 g + a_2). \tag{3.23}$$

This by Lemma 2.12 gives $f \equiv g$. This completes the proof of the Theorem. □

Proof of Theorem 1.1. Let $F(z) = \frac{f^n}{b}$ and $G(z) = \frac{g^n}{b}$. Then $F^{(k)}$ and $G^{(k)}$ share $(1, l)$. Proceeding in the same way as the proof of Theorem 1.2 we obtain either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$. If $F^{(k)}G^{(k)} \equiv 1$, then for $k = 1$ we obtain

$$f^{n-1}f'g^{n-1}g' = b_1^2. \tag{3.24}$$

Let z_0 be a zero of f with multiplicity $p(\geq s)$. Then z_0 is a pole of g with multiplicity $q(\geq s)$, say. From (3.24) we obtain $np - 1 = nq + 1$ and so $n(p - q) = 2$, which is impossible since $n \geq 3$ and p, q are integers. Therefore, we conclude that f and g are entire functions. So by Lemma 2.10 we obtain $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1c_2)^nc^2 = -b_1^2$.

If $F \equiv G$, then we obtain $f = dg$ for some $(n+1)$-th root of unity 1. This completes the proof of the Theorem. □

Proof of Theorem 1.3. Let $F(z) = \frac{f^n(a_1f^2+a_2f+a_3)}{b}$ and $G(z) = \frac{g^n(a_1g^2+a_2g+a_3)}{b}$. Then $F^{(k)}$ and $G^{(k)}$ share $(1, l)$. Proceeding similarly as in the proof of Theorem 1.2 we obtain either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$. Also by Lemma 2.11 the case $F^{(k)}G^{(k)} \equiv 1$ does not arise for $k = 1$. Let $F \equiv G$, That is

$$f^n(a_1f^2 + a_2f + a_3) \equiv g^n(a_1g^2 + a_2g + a_3). \tag{3.25}$$

Let $h = f/g$. If h is a constant, then substituting $f = gh$ in (3.25) we obtain

$$a_1g^{n+2}(h^{n+2} - 1) + a_2g^{n+1}(h^{n+1} - 1) + a_3g^n(h^n - 1) = 0,$$

which imply $h = 1$. Hence $f \equiv g$.

If h is not a constant, by (3.25) we can say that f and g satisfy the algebraic equation $R(f, g) = 0$, where

$$R(x, y) = x^n(a_1x^2 + a_2x + a_3) - y^n(a_1y^2 + a_2y + a_3).$$

This completes the proof of the Theorem. □

Proof of Corollary 1.3. Proceeding in a manner as in the proof of Theorem 1.3 and using Case 2 of Lemma 2.11 we obtain (3.25). Putting $n = t + 1$, $a_1 = \frac{1}{t+3}$, $a_2 = -\frac{2}{t+2}$, $a_3 = \frac{1}{t+1}$ and $h = \frac{f}{g}$ in (3.25) we obtain

$$\begin{aligned}(t+2)(t+1)g^2(h^{t+3} - 1) - 2(t+3)(t+1)g(h^{t+2} - 1)\\ +(t+2)(t+3)(h^{t+1} - 1) = 0.\end{aligned} \tag{3.26}$$

By (3.26) and by Lemma 2.13, we can conclude that

$$\{(t+1)(t+2)(h^{t+3} - 1)g - (t+3)(t+1)(h^{t+2} - 1)\}^2 = -(t+3)(t+1)Q(h),$$

where $Q(h) = (h - 1)^4(h - \nu_1)(h - \nu_2) \cdots (h - \nu_{2t})$, where $\nu_j \in C \setminus \{0, 1\}$ $(j = 1, 2, \dots, 2t)$, which are pairwise distinct.

If h is not a constant, this implies that every zero of $h - \nu_j$ $(j = 1, 2, \dots, 2t)$, has a multiplicity of at least 2. By the second fundamental theorem of Nevanlinna we obtain that $t \leq 2$, which is again a contradiction. Therefore,

h is a constant. We have from (3.26) that $h^{t+1} - 1 = 0$ and $h^{t+2} - 1 = 0$, which imply $h = 1$, and hence $f \equiv g$. □

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