SARAJEVO JOURNAL OF MATHEMATICS
Vol.8 (20) (2012), 91–100

FURTHER RESULTS ON THE LOGARITHMIC INTEGRAL

BRIAN FISHER, BILJANA JOLEVSKA-TUNESKA AND ARPAD TAKACI

Abstract. The logaritmic integral $\mathrm{li}(x^r)$ and its associated functions $\mathrm{li}_+(x^r)$ and $\mathrm{li}_-(x^r)$ are defined as locally summable functions on the real line. Some convolutions and neutrix convolutions of these functions and other functions are then found.

1. Introduction

The *logarithmic integral* $\mathrm{li}(x)$, see Abramowitz and Stegun [1] is defined by

$$\begin{aligned}\mathrm{li}(x) &= \begin{cases} \displaystyle\int_0^x \frac{dt}{\ln|t|}, & \text{for } |x|<1,\\ \displaystyle\mathrm{PV}\int_0^x \frac{dt}{\ln t}, & \text{for } x>1,\\ \displaystyle\mathrm{PV}\int_0^x \frac{dt}{\ln|t|}, & \text{for } x<-1 \end{cases}\\ &= \begin{cases} \displaystyle\int_0^x \frac{dt}{\ln|t|}, & \text{for } |x|<1,\\ \displaystyle\lim_{\epsilon\to0^+}\left[\int_0^{1-\epsilon}\frac{dt}{\ln t}+\int_{1+\epsilon}^x\frac{dt}{\ln t}\right], & \text{for } x>1,\\ \displaystyle\lim_{\epsilon\to0^+}\left[\int_0^{-1+\epsilon}\frac{dt}{\ln|t|}+\int_{-1-\epsilon}^x\frac{dt}{\ln|t|}\right], & \text{for } x<-1 \end{cases}\end{aligned}$$

where PV denotes the Cauchy principal value of the integral. We will therefore write

$$\mathrm{li}(x)=\mathrm{PV}\int_0^x\frac{dt}{\ln|t|}$$

for all values of x.

2000 *Mathematics Subject Classification.* 33B10, 46F10.
Key words and phrases. Logarithmic integral, distribution, convolution, neutrix, neutrix convolution.

More generally, we have

$$\operatorname{li}(x^r) = \mathrm{PV}\int_0^{x^r} \frac{dt}{\ln|t|}$$

and the associated functions $\operatorname{li}_+(x^r)$ and $\operatorname{li}_-(x^r)$ are now defined by

$$\operatorname{li}_+(x^r) = H(x)\operatorname{li}(x^r), \quad \operatorname{li}_-(x^r) = H(-x)\operatorname{li}(x^r),$$

where $H(x)$ denotes Heaviside's function.

It follows that

$$\operatorname{li}(x^r) = \mathrm{PV}\int_0^{x} \frac{t^{r-1}dt}{\ln|t|}, \tag{1}$$

see [4], the distribution $x^{r-1}\ln^{-1}|x|$ is then defined by

$$x^{r-1}\ln^{-1}|x| = [\operatorname{li}(x^r)]'$$

and its associated distributions $x_+^{r-1}\ln^{-1}x_+$ and $x_-^{r-1}\ln^{-1}x_-$ are defined by

$$x_+^{r-1}\ln^{-1}x_+ = H(x)x^{r-1}\ln^{-1}|x| = [\operatorname{li}_+(x^r)]',$$
$$x_-^{r-1}\ln^{-1}x_- = H(-x)x^{r-1}\ln^{-1}|x| = [\operatorname{li}_-(x^r)]',$$

for $r = 1, 2, \ldots$.

The classical definition of the convolution of two functions f and g is as follows:

Definition 1. *Let f and g be functions. Then the convolution $f*g$ is defined by*

$$(f*g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t)\,dt$$

for all points x for which the integral exist.

It follows easily from the definition 1 that if $f*g$ exists then $g*f$ exists and

$$f*g = g*f \tag{2}$$

and if $(f*g)'$ and $f*g'$ (or $f'*g$) exists, then

$$(f*g)' = f*g' \quad (\text{or } f'*g). \tag{3}$$

Definition 1 can be extended to define the convolution $f*g$ of two distributions f and g in $\mathcal{D}'$ with the following definition, see Gel'fand and Shilov [7].

Definition 2. *Let f and g be distributions in $\mathcal{D}'$. Then the convolution $f*g$ is defined by the equation*

$$\langle (f*g)(x), \varphi(x)\rangle = \langle f(y), \langle g(x), \varphi(x+y)\rangle\rangle$$

for arbitrary φ in $\mathcal{D}$, provided f and g satisfy either of the conditions

(a) *either f or g has bounded support,*
(b) *the supports of f and g are bounded on the same side.*

It follows that if the convolution $f * g$ exists by this definition, then equations (2) and (3) are satisfied.

The above definition of the convolution is rather restrictive and so a neutrix convolution was defined in [3]. In order to define the neutrix convolution, we first of all let τ be a function in $\mathcal{D}$, see [8], satisfying the following properties:

(i) $\tau(x) = \tau(-x)$,
(ii) $0 \le \tau(x) \le 1$,
(iii) $\tau(x) = 1$ for $|x| \le \frac{1}{2}$,
(iv) $\tau(x) = 0$ for $|x| \ge 1$.

The function τ_n is now defined by

$$\tau_n(x) = \begin{cases} 1, & |x| \le n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n, \end{cases}$$

for $n = 1, 2, \ldots$.

The following definition of the non-commutative neutrix convolution was given in [3].

Definition 3. *Let f and g be distributions in $\mathcal{D}'$ and let $f_n = f\tau_n$ for $n = 1, 2, \ldots$. Then the non-commutative neutrix convolution $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_n * g\}_{n\in\mathbf{N}}$, provided the limit h exists in the sense that*

$$\mathrm{N}\!-\!\lim_{n\to\infty} \langle f_n * g, \varphi\rangle = \langle h, \varphi\rangle$$

for all φ in $\mathcal{D}$, where N is the neutrix, see van der Corput [2], having domain N' the positive reals and range N'' the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n : \ \lambda > 0, \ r = 1, 2, \ldots$$

and all functions which converge to zero in the normal sense as n tends to infinity.

It is easily seen that any results proved with the original definition of the convolution hold with the new definition of the neutrix convolution. The following results proved in [3] hold, first showing that the neutrix convolution is a generalization of the convolution.

Theorem 1. *Let f and g be distributions in $\mathcal{D}'$, satisfying either condition* (a) *or condition* (b) *of Gel'fand and Shilov's definition. Then the neutrix* *convolution* $f \circledast g$ *exists and*

$$f \circledast g = f * g.$$

Theorem 2. *Let* f *and* g *be distributions in* $\mathcal{D}'$ *and suppose that the neutrix convolution* $f \circledast g$ *exists. Then the neutrix convolution* $f \circledast g'$ *exists and*

$$(f \circledast g)' = f \circledast g'.$$

If $\operatorname*{N-lim}_{n\to\infty} \langle (f\tau_n') * g, \varphi\rangle$ *exists and equals* $\langle h, \varphi\rangle$ *for all* φ *in* $\mathcal{D}$, *then* $f' \circledast g$ *exists and*

$$(f \circledast g)' = f' \circledast g + h.$$

In the following, we need to extend our set of negligible functions to include finite linear sums of the functions $n^s \operatorname{li}(n^r)$ and $n^s \ln^{-r} n$, $(n > 1)$ for $s = 0, 1, 2, \dots$ and $r = 1, 2, \dots$.

2. Main Results

The following lemmas were proved in [4].

Lemma 1.

$$\lim_{n\to\infty} \int_n^{n+n^{-n}} \tau_n(t) \operatorname{li}(t)(x-t)^r \, dt = 0 \tag{4}$$

for $r = 1, 2, \dots$.

Lemma 2.

$$\operatorname*{N-lim}_{n\to\infty} \operatorname{li}[(x+n)^r] = 0, \tag{5}$$

$$\operatorname*{N-lim}_{n\to\infty} n^r \operatorname{li}[(x+n)] = 0 \tag{6}$$

for $r = 1, 2, \dots$.

We now prove a number of results involving the convolution. First of all we have

Theorem 3. *The convolutions* $\operatorname{li}_+(x^s) * x_+^r$ *and* $x_+^{s-1} \ln^{-1} x_+ * x_+^r$ *exist and*

$$\operatorname{li}_+(x^s) * x_+^r = \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^{r-i+1} x^i \operatorname{li}_+(x^{r+s-i+1}), \tag{7}$$

$$x_+^{s-1} \ln^{-1} x_+ * x_+^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} x^i \operatorname{li}_+(x^{r+s-i}) \tag{8}$$

for $r = 0, 1, 2, \dots$ *and* $s = 1, 2, \dots$.

Proof. It is obvious that $\mathrm{li}_+(x^s) * x_+^r = 0$ if $x < 0$.
When $x > 0$, we have

$$\begin{aligned}
\mathrm{li}_+(x^s) * x_+^r &= PV \int_0^x (x-t)^r \int_0^t \frac{u^{s-1}du}{\ln u}\,dt \\
&= PV \int_0^x \frac{u^{s-1}}{\ln u} \int_u^x (x-t)^r\,dt\,du \\
&= PV \frac{1}{r+1} \sum_{i=0}^{r+1} (-1)^{r-i+1} x^i \binom{r+1}{i} \int_0^x \frac{u^{r+s-i}}{\ln u}\,du \\
&= \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^{r-i+1} x^i\,\mathrm{li}(x^{r+s-i+1}),
\end{aligned}$$

on using equation (1) and equation (7) is proved.
Now, using equation (3) and (7), we get

$$\begin{aligned}
x_+^{s-1} \ln^{-1} x_+ * x_+^r &= r\,\mathrm{li}_+(x^s) * x_+^{r-1} \\
&= \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} x^i\,\mathrm{li}_+(x^{r+s-i}),
\end{aligned}$$

proving equation (8). □

Corollary 1. *The convolutions* $\mathrm{li}_-(x^s) * x_-^r$ *and* $x_-^{s-1} \ln^{-1} x_- * x_-^r$ *exist and*

$$\mathrm{li}_-(x^s) * x_-^r = \frac{1}{r+1} \sum_{i=0}^{r+1} \binom{r+1}{i} (-1)^{r-i+2} x^i\,\mathrm{li}_-(x^{r+s-i+1}), \tag{9}$$

$$x_-^{s-1} \ln^{-1} x_- * x_-^r = \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i+1} x^i\,\mathrm{li}_-(x^{r+s-i}) \tag{10}$$

for $r = 0, 1, 2, \ldots,$ *and* $s = 1, 2, \ldots.$

Proof. Equations (9) and (10) follow on replacing x by $-x$ in equation (7) and (8). □

Theorem 4. *The neutrix convolutions* $\mathrm{li}_+(x^s) \circledast x^r$ *and* $x^{s-1} \ln^{-1} x_+ \circledast x^r$ *exist and*

$$\mathrm{li}_+(x^s) \circledast x^r = 0, \tag{11}$$

$$x^{s-1} \ln^{-1} x_+ \circledast x^r = 0 \tag{12}$$

for $r = 0, 1, 2, \ldots,$ *and* $s = 1, 2, \ldots.$

Proof. We put $[\mathrm{li}_+(x^s)]_n = \mathrm{li}_+(x^s)\tau_n(x)$. Then the convolution $[\mathrm{li}_+(x^s)]_n * x^r$ exists by definition 1 and

$$[\mathrm{li}_+(x^s)]_n * x^r = \int_0^n \mathrm{li}(t^s)(x-t)^r\,dt + \int_n^{n+n^{-n}} \tau_n(t)\,\mathrm{li}(t^s)(x-t)^r\,dt, \quad (13)$$

where

$$\begin{aligned}
\int_0^n \mathrm{li}(t^s)(x-t)^r\,dt &= PV\int_0^n (x-t)^r \int_0^t \frac{u^{s-1}du}{\ln u}\,dt \\
&= PV\int_0^n \frac{u^{s-1}}{\ln u}\int_u^n (x-t)^r\,dt\,du \\
&= PV\frac{1}{r+1}\sum_{i=0}^{r+1}(-1)^{r-i+1}x^i\binom{r+1}{i}\int_0^n \frac{u^{r+s-i}-n^{r+s-i}}{\ln u}\,du \\
&= \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^{r-i+1}x^i[\mathrm{li}(n^{r+s-i+1}) - n^{r+s-i+1}\,\mathrm{li}(n)].
\end{aligned}$$

Thus from Lemma 1 we have

$$\operatorname*{N-lim}_{n\to\infty}\int_0^n \mathrm{li}(t^s)(x-t)^r\,dt = 0. \quad (14)$$

Equation (11) now follows using Lemma 1, equations (13) and (14).

Differentiating equation (11) and using Theorem 2 we get

$$x^{s-1}\ln^{-1}x_+ \circledast x^r = \operatorname*{N-lim}_{n\to\infty}[\mathrm{li}_+(x^s)\tau_n'(x)] * x^r \quad (15)$$

where, on integration by parts we have

$$\begin{aligned}
[\mathrm{li}_+(x^s)\tau_n'(x)] * x^r &= \int_n^{n+n^{-n}} \tau_n'(t)\,\mathrm{li}(t^s)(x-t)^r\,dt \\
&= -\,\mathrm{li}(n^s)(x-n)^r - \int_n^{n+n^{-n}} t^{s-1}\ln^{-1}(t^s)(x-t)^r\tau_n(t)\,dt \\
&\quad + r\int_n^{n+n^{-n}} \mathrm{li}(t^s)(x-t)^{r-1}\tau_n(t)\,dt. \quad (16)
\end{aligned}$$

It is clear that

$$\lim_{n\to\infty}\int_n^{n+n^{-n}} t^{s-1}\ln^{-1}(t)(x-t)^r\tau_n(t)\,dt = 0 \quad (17)$$

and now equation (12) follows from lemma 1 and equations (15), (16) and (17). □

Corollary 2. *The neutrix convolutions* $\mathrm{li}_-(x^s) \circledast x^r$ *and* $x_-^{s-1}\ln^{-1} x_- \circledast x^r$ *exist and*

$$\mathrm{li}_-(x^s) \circledast x^r = 0, \tag{18}$$

$$x_-^{s-1}\ln^{-1} x_- \circledast x^r = 0 \tag{19}$$

for $r = 0, 1, 2, \dots,$ *and* $s = 1, 2, \dots.$

Proof. Equations (18) and (19) on replacing x by $-x$ in equation(11) and (12). □

Corollary 3. *The neutrix convolutions* $\mathrm{li}(x^s) \circledast x^r$ *and* $x^{s-1}\ln^{-1}|x| \circledast x^r$ *exist and*

$$\mathrm{li}(x^s) \circledast x^r = 0, \tag{20}$$

$$x^{s-1}\ln^{-1}|x| \circledast x^r = 0 \tag{21}$$

for $r = 0, 1, 2, \dots,$ *and* $s = 1, 2, \dots.$

Proof. Equation (20) follows on adding equation (18) and (11) and equation (21) follows on adding equations (12) and (19). □

Corollary 4. *The neutrix convolutions* $\mathrm{li}_+(x^s) \circledast x_-^r$, $\mathrm{li}_-(x^s) \circledast x_+^r$, $x_+^{s-1}\ln^{-1} x_+ \circledast x_-^r$ *and* $x_-^{s-1}\ln^{-1} x_- \circledast x_+^r$ *exist and*

$$\mathrm{li}_+(x^s) \circledast x_-^r = \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^i x^i \,\mathrm{li}_+(x^{r+s-i+1}), \tag{22}$$

$$\mathrm{li}_-(x^s) \circledast x_+^r = \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^{i+1} x^i \,\mathrm{li}_-(x^{r+s-i+1}), \tag{23}$$

$$x_+^{s-1}\ln^{-1} x_+ \circledast x_-^r = \sum_{i=0}^{r}\binom{r}{i}(-1)^i x^i \,\mathrm{li}_+(x^{r+s-i}), \tag{24}$$

$$x_-^{s-1}\ln^{-1} x_- \circledast x_+^r = \sum_{i=0}^{r}\binom{r}{i}(-1)^{i+1} x^i \,\mathrm{li}_-(x^{r+s-i}), \tag{25}$$

for $r = 0, 1, 2, \dots,$ *and* $s = 1, 2, \dots.$

Proof. Noting that $x^r = x_+^r + (-1)^r x_-^r$ and the fact that the neutrix convolution product is distributive with respect to addition, we have

$$\mathrm{li}_+(x^s) \circledast x^r = \mathrm{li}_+(x^s) * x_+^r + (-1)^r \,\mathrm{li}_+(x^s) \circledast x_-^r.$$

Equation (22) follows from equations (7) and (11). Equation (23) follows on replacing x by $-x$ in equation (22).

Equation (24) follows from equations (8) and (12) and equation (25) follows on replacing x by $-x$ in equation (24). □

Theorem 5. *The neutrix convolutions* $x^r \circledast \mathrm{li}_+(x^s)$ *and* $x^r \circledast x_+^{s-1}\ln^{-1} x_+$ *exist and*

$$x^r \circledast \mathrm{li}_+(x^s) = 0, \tag{26}$$

$$x^r \circledast x_+^{s-1}\ln^{-1} x_+ = 0 \tag{27}$$

for $r = 0, 1, 2, \dots,$ *and* $s = 1, 2, \dots$.

Proof. We put $(x^r)_n = x^r\tau_n(x)$ for $r = 0, 1, 2, \dots$. Then the convolution $(x^r)_n * \mathrm{li}_+(x^s)$ exists by Definition 1 and

$$(x^r)_n * \mathrm{li}_+(x^s) = \int_0^{x+n} \mathrm{li}(t^s)(x-t)^r\,dt + \int_{x+n}^{x+n+n^{-n}} \tau_n(x-t)\,\mathrm{li}(t^s)(x-t)^r\,dt, \tag{28}$$

where

$$\begin{aligned}
\int_0^{x+n} \mathrm{li}(t^s)(x-t)^r\,dt &= PV\int_0^{x+n}(x-t)^r\int_0^t \frac{u^{s-1}du}{\ln u}\,dt \\
&= PV\int_0^{x+n}\frac{u^{s-1}}{\ln u}\int_u^{x+n}(x-t)^r\,dt\,du \\
&= PV\frac{1}{r+1}\sum_{i=0}^{r+1}(-1)^{r-i+1}x^i\binom{r+1}{i}\int_0^{x+n}\frac{u^{r+s-i}}{\ln u}\,du \\
&\qquad - PV\frac{(-n)^{r+1}}{r+1}\int_0^{x+n}\frac{u^{s-1}du}{\ln u} \\
&= \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^{r-i+1}x^i\,\mathrm{li}[(x+n)^{r+s-i+1}] - \frac{(-n)^{r+1}}{r+1}\,\mathrm{li}\left((x+n)^s\right).
\end{aligned}$$

Thus, on using Lemma 2, we have

$$\mathrm{N-}\lim_{n\to\infty}\int_0^{x+n}\mathrm{li}(t^s)(x-t)^r\,dt = 0. \tag{29}$$

Further, using lemma 2 it is easily seen that

$$\lim_{n\to\infty}\int_{x+n}^{x+n+n^{-n}}\tau_n(x-t)\,\mathrm{li}(t^s)(x-t)^r\,dt = 0 \tag{30}$$

and equation (26) follows from equations (28), (29) and (30).

Differentiating equation (26) gives equation (27). □

Corollary 5. *The neutrix convolutions* $x^r \circledast \mathrm{li}_-(x^s)$ *and* $x^r \circledast x_-^{s-1}\ln^{-1} x_-$ *exist and*

$$x^r \circledast \mathrm{li}_-(x^s) = 0, \tag{31}$$

$$x^r \circledast x_-^{s-1}\ln^{-1} x_- = 0 \tag{32}$$

for $r = 0, 1, 2, \dots,$ *and* $s = 1, 2, \dots.$

Proof. Equations (31) and (32) follow on replacing x by $-x$ in equations (26) and (27). □

Corollary 6. *The neutrix convolutions* $x^r \circledast \mathrm{li}(x^s)$ *and* $x^r \circledast x^{s-1}\ln^{-1}|x|$ *exist and*

$$x^r \circledast \mathrm{li}(x^s) = 0, \tag{33}$$

$$x^r \circledast x^{s-1}\ln^{-1}|x| = 0 \tag{34}$$

for $r = 0, 1, 2, \dots,$ *and* $s = 1, 2, \dots.$

Proof. Equation (33) follows on adding equation (31) and (26) and equation (34) follows on adding equations (27) and (32). □

Corollary 7. *The neutrix convolutions* $x_-^r \circledast \mathrm{li}_+(x^s)$, $x_+^r \circledast \mathrm{li}_-(x^s)$, $x_-^r \circledast x_+^{s-1}\ln^{-1}x_+$ *and* $x_+^r \circledast x_-^{s-1}\ln^{-1}x_-$ *exist and*

$$x_-^r \circledast \mathrm{li}_+(x^s) = \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^i x^i\,\mathrm{li}_+(x^{r+s-i+1}), \tag{35}$$

$$x_+^r \circledast \mathrm{li}_-(x^s) = \frac{1}{r+1}\sum_{i=0}^{r+1}\binom{r+1}{i}(-1)^{i+1} x^i\,\mathrm{li}_-(x^{r+s-i+1}), \tag{36}$$

$$x_-^r \circledast x_+^{s-1}\ln^{-1}x_+ = \frac{1}{r+1}\sum_{i=0}^{r}\binom{r}{i}(-1)^i x^i\,\mathrm{li}_+(x^{r+s-i}), \tag{37}$$

$$x_+^r \circledast x_-^{s-1}\ln^{-1}x_- = \frac{1}{r+1}\sum_{i=0}^{r}\binom{r}{i}(-1)^{i+1} x^i\,\mathrm{li}_-(x^{r+s-i}) \tag{38}$$

for $r = 0, 1, 2, \dots,$ *and* $s = 1, 2, \dots.$

Proof. Equation (35) follows from equations (7) and (26) on noting that

$$x^r \circledast \mathrm{li}_+(x^s) = x_+^r * \mathrm{li}_+(x^s) + (-1)^r x_-^r \circledast \mathrm{li}_+(x^s).$$

Equation (36) follows on replacing x by $-x$ in equation (35). Equation (37) follows from equations (8) and (27) and equation (38) follows on replacing x by $-x$ in equation (37). □

For further results involving the convolution see [5] and [6].

References

[1] M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with formulas, Graphs and Mathematical Tables*, 9th printing. New York: Dover, p. 879, 1972.

[2] J. G. van der Corput, *Introduction to the neutrix calculus*, J. Anal. Math., 7 (1959-60), 291-398.

[3] B. Fisher, *Neutrices and the convolution of distributions*, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat., 17 (1987), 119-135.

[4] B. Fisher and B. Jolevska-Tuneska, *On the logarithmic integral*, Hacet. J. Math. Stat., 39 (3) (2010), 393-401.

[5] B. Fisher, B. Jolevska-Tuneska and A.Takaci, *On convolutions and neutrix convolutions involving the incomplete gamma function*, Integral Transforms Spec. Funct., 15 (5) (2004), 404-415.

[6] B. Jolevska-Tuneska and A. Takaci, *Results on the commutative neutrix convolution of distributions*, Hacet. J. Math. Stat., 37 (2)(2008), 135-141.

[7] I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, Vol. I, Academic Press Chap. 1, 1964.

[8] D.S. Jones, *The convolution of generalized functions*, Quart. J. Math. Oxford (2), 24 (1973), 145-163.

(Received: March 21, 2011)

B. Fisher
Department of Mathematics
University of Leicester, Leicester
LE1 7RH, England
E–mail: fbr@le.ac.uk

B. Jolevska-Tuneska
Faculty of Electrical Engineering and
Informational Technologies
Karpos II bb, Skopje, Republic of Macedonia
E–mail: biljanaj@feit.ukim.edu.mk

A. Takači
Faculty of Natural sciences
Tgr Dositeja Obradovića 4
21000 Novi Sad, Serbia
E–mail: takaci@im.ns.ac.yu