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NONLINEAR THREE POINT BOUNDARY VALUE PROBLEM

A. GUEZANE-LAKOUD AND A. FRIOUI

ABSTRACT. In this work, we establish sufficient conditions for the existence of solutions for a three point boundary value problem generated by a third order differential equation. We give sufficient conditions that allow us to obtain the existence of a nontrivial solution. Then by using the Leray Schauder nonlinear alternative we prove the existence of at least one solution of the posed problem. As an application, we give two examples to illustrate our results.

1. INTRODUCTION

Three-point boundary value problems have received great attention and have become an important subject of research This is due to the fact that many physical phenomena can be modeled by ordinary differential equations with nonlocal conditions. It is found that problems with nonlocal conditions have many applications in many problems such as population dynamics, the process of heat conduction, control theory, etc.. In particular, the introduction of non-local conditions can improve the qualitative and quantitative characteristics of the problem which lead to good results concerning existence and uniqueness of the solution. Non local conditions come up when values of the function on the boundary are connected to values inside the domain or when direct measurements on the boundary are not possible. In this paper, we will discuss the existence of non trivial solutions for the third-order three-point boundary value problem (BVP):

$$u''' + f(t,u) = 0, \quad 0 < t < 1, \tag{1}$$

$$u(0) = \alpha u(1), \quad u'(1) = \beta u'(\eta), \quad u'(0) = 0, \tag{2}$$

where $\eta \in (0,1)$, $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$, the parameters α and β are such that $(1-\alpha)(\beta\eta - 1) \neq 0$. We do not assume any monotonicity condition

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on the nonlinearity f, we assume that $f(t,0) \neq 0$ and that there exist two nonnegative functions $k, h \in L^1[0,1]$ such that

$$|f(t,x)| \leq k(t)|x|^p + h(t),$$

where $p > 0$, $(t,x) \in [0,1] \times \mathbb{R}$. The third order equations (1) are used to model various phenomena in physics, chemistry and epidemiology. The nonlinearities that refer to source terms represent specific physical laws in chemistry. For example, they can represent the Arrheninus law for chemistry reactions, see [1].

Many of the results involving nonlocal boundary value problems are studied and the main tools are the method of upper and lower solutions, topological degree, fixed point theorem in cones, continuation methods based on a priori bounds on solutions, nonlinear alternative and so on [2,4-24]. In [10] Graef, Kong and Yang studied a third order equation with the nonlocal conditions $u(0) = u(p) - u(1) = u''(1) = 0$. Guezane-Lakoud and Khaldi in [12] considered the boundary conditions $u(0) = \alpha u'(0)$, $u(1) = \beta u'(\eta)$, $u'(1) = 0$, and established the existence of solution for $p = 1$. Our aim in the present work is to give new conditions on the nonlinearity f, then, using the Leray Schauder nonlinear alternative, to establish the existence of at least one solution for problem (1)-(2). As an application, some examples to illustrate our results are given.

This paper is organized as follows. First, we list some preliminary material to be used later and give the solution of an auxiliary problem. Then, in Section 3, we present and prove our main results which consist in existence theorems, we establish some existence criteria of at least one solution by using the Leray Schauder nonlinear alternative. We end our work with two illustrating examples.

2. Preliminary lemmas

Let $E = C[0,1]$, with the norm $||y|| = \max_{t\in[0,1]} |y(t)|$, $\forall\, y \in E$. Firstly we state two preliminary results.

Lemma 1. *Let $y \in E$. If $\zeta = (1-\alpha)(\beta\eta - 1) \neq 0$, then the three point BVP*

$$\begin{cases} u''' + y(t) = 0, & 0 < t < 1, \\ u(0) = \alpha u(1), \quad u'(1) = \beta u'(\eta), & u'(0) = 0, \end{cases} \tag{3}$$

has a unique solution

$$u(t) = -\frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds - \frac{\beta}{2\zeta}\left(t^2(1-\alpha)+\alpha\right)\int_0^\eta (\eta - s)\,y(s)\,ds$$
$$+\frac{1}{2\zeta}\int_0^1 (1-s)\left(t^2(1-\alpha) + \alpha\beta\eta(1-s) + \alpha s\right) y(s)\,ds.$$

Proof. The proof is easy, so we omit it. □

We define the integral operator $T : E \to E$, by

$$\begin{aligned} Tu(t) = -\frac{1}{2}\int_0^t (t-s)^2 f(s,u(s))\, ds \\ - \frac{\beta}{2\zeta}\left(t^2(1-\alpha)+\alpha\right)\int_0^\eta (\eta - s) f(s,u(s))\, ds \\ + \frac{1}{2\zeta}\int_0^1 (1-s)\left(t^2(1-\alpha)+\alpha\beta\eta(1-s)+\alpha s\right) f(s,u(s))\, ds. \end{aligned} \tag{4}$$

By Lemma 1, the BVP (1)-(2) has a solution if and only if the operator T has a fixed point in E. By Ascoli Arzela Theorem we prove that T is completely continuous operator. Now we cite the Leray–Schauder nonlinear alternative.

Lemma 2. [3] *Let F be a Banach space and Ω a bounded open subset of F, $0 \in \Omega$. Let $T : \overline{\Omega} \to F$ be a completely continuous operator. Then, either there exists $x \in \partial\Omega$, $\lambda > 1$ such that $T(x) = \lambda x$, or there exists a fixed point $x^* \in \overline{\Omega}$ of T.*

3. Main results

In this section, we present and prove our main results. Let us introduce the following notations

$$\begin{aligned} M = \left(1+\eta\frac{|\alpha\beta|}{2|\zeta|}\right)\int_0^1 (1-s)^2 k(s)\, ds + \frac{|\beta|(1+2|\alpha|)}{2|\zeta|}\int_0^\eta (\eta - s)k(s)\, ds \\ + \frac{(1+2|\alpha|)}{2|\zeta|}\int_0^1 (1-s)\, k(s)\, ds \end{aligned}$$

and

$$\begin{aligned} N = \left(1+\eta\frac{|\alpha\beta|}{2|\zeta|}\right)\int_0^1 (1-s)^2 h(s)\, ds + \frac{|\beta|(1+2|\alpha|)}{2|\zeta|}\int_0^\eta (\eta - s)h(s)\, ds \\ + \frac{(1+2|\alpha|)}{2|\zeta|}\int_0^1 (1-s)\, h(s)\, ds. \end{aligned}$$

Theorem 3. *Assume that $f(t,0) \neq 0$, $\zeta \neq 0$, $0 < p < 1$, and that there exist two nonnegative functions $k, h \in L^1[0,1]$, $k(t) \neq 0$ almost everywhere on $[0,1]$ and such that*

$$|f(t,x)| \leq k(t)\,|x|^p + h(t). \tag{5}$$

Then the BVP (1)-(2) has at least one nontrivial solution $u^* \in E$.

Proof. Since k is nonnegative and $k(t) \neq 0$, *a.e.* $t \in [0,1]$, then $M \neq 0$. Let $m = (M+N)^{\frac{1}{1-p}}$, $r = \max(1,m)$, $B_r = \{u \in E : \|u\| < r+1\}$, $u \in \partial B_r$ and $0 < \lambda < 1$ such that $u = \lambda T u$. We have

$$||u|| = \lambda \, ||Tu|| \leq M \, \|u\|^p + N.$$

If $\|u\| > 1$, it yields

$$\|u\|^{1-p} \leq M + N \, \|u\|^p \leq M + N,$$

hence

$$\|u\| \leq (M+N)^{\frac{1}{1-p}} = m. \tag{6}$$

Otherwise $\|u\| < 1$, consequently $\|u\| \leq \max(1,m) = r$, and from here we see that (6) contradicts the fact that $u \in B_r$. By Lemma 2 we conclude that operator T has a fixed point $u^* \in \overline{B_r}$ and then the BVP (1)-(2) has a nontrivial solution $u^* \in C[0,1]$. □

Theorem 4. *Assume that* $f(t,0) \neq 0$, $\zeta \neq 0$, $p = 1$ *and that there exist nonnegative functions* $k, h \in L^1[0,1]$ *such that*

$$|f(t,x)| \leq k(t)\,|x| + h(t),$$

$$M < 1. \tag{7}$$

Then the BVP (1)-(2) *has at least one nontrivial solution* $u^* \in C[0,1]$.

Proof. Since the function f is continuous and $f(t,0) \neq 0$, there exists an interval $[\sigma,\tau] \subset [0,1]$ such that $\min\limits_{\sigma \leq t \leq \tau} |f(t,0)| > 0$ and as $h(t) \geq |f(t,0)|$, *a.e.* $t \in [0,1]$ then $N > 0$. Let $r = \dfrac{N}{1-M}$, so $r \neq 0$. Let $B_r = \{u \in E : \|u\| < r\}$, $u \in \partial B_r$ and $\lambda > 1$ such $Tu = \lambda u$. Then

$$\lambda r = \lambda \, ||u|| = ||Tu|| = \max_{0 \leq t \leq 1} |(Tu)(t)| \leq M \, \|u\| + N.$$

Therefore, we have $\lambda \leq M + \frac{N}{r} = 1$. This contradicts the fact that $\lambda > 1$. By Lemma 2 we conclude that the operator T has a fixed point $u^* \in \overline{B_r}$ and then the BVP (1)-(2) has a nontrivial solution $u^* \in C[0,1]$. □

Theorem 5. *Assume that* $f(t,0) \neq 0$, $\zeta \neq 0$, $p > 1$,*and that there exist nonnegative functions* $k, h \in L^1[0,1]$, $k(t) \neq 0$, *a.e. on* $[0,1]$ *and such that*

$$|f(t,x)| \leq k(t)\,|x|^p + h(t),$$

$$M + N < 1. \tag{8}$$

Then the BVP (1)-(2) *has at least one nontrivial solution* $u^* \in C[0,1]$.

Proof. Let $B_1 = \{u \in E : \|u\| < 1\}$, $u \in \partial B_1$ and $\lambda > 1$ such $Tu = \lambda u$. Then

$$\lambda \, ||u|| = ||Tu|| \leq M \, \|u\|^p + N.$$

Consequently $\lambda \leq M + N$. Condition (8) gives $\lambda < 1$, and this contradicts the fact that $\lambda > 1$. Hence T has a fixed point $u^* \in \overline{B_1}$ that is a nontrivial solution for the BVP (1)-(2). This finishes the proof. $\square$

In order to illustrate our results, we give two examples.

Example 6. Consider the three point

$$\text{(BVP2)} \begin{cases} u''' + \frac{u^{\frac{1}{3}}}{2} \arcsin t + 2\cos t + 4\sin t = 0, \quad 0 < t < 1, \\ u\,(0) = -\frac{1}{5} u\,(1)\,, \quad u'\,(1) = -\frac{1}{2} u'\left(\frac{1}{3}\right), \quad u'\,(0) = 0. \end{cases}$$

We have $f\,(t, x) = \left(\frac{x^{\frac{1}{3}}}{2}\right) \arcsin t + 2\cos t + 4\sin t$. So $|f\,(t, x)| \leq \left(\frac{1}{2} \arcsin t\right)$ $|x|^{\frac{1}{3}} + 2\cos t + 4\sin t = k(t)\,|x|^{\frac{1}{3}} + h(t)$, $p = \frac{1}{3} < 1$, $\zeta = \frac{7}{5} \neq 0$, $f(t, 0) = 2\cos t + 4\sin t \neq 0$. From Theorem 3, we conclude that the (BVP2) has at least one nontrivial solution u^* in E.

Example 7. Consider the three point

$$\text{(BVP3)} \begin{cases} u''' + \frac{1}{(1+t)^5} u^{\frac{3}{2}} + \frac{t}{1+t^2} = 0, \quad 0 < t < 1, \\ u\,(0) = \frac{1}{6} u\,(1)\,, \quad u'\,(1) = 8u'\left(\frac{1}{4}\right), \quad u'\,(0) = 0. \end{cases}$$

We have $f\,(t, x) = \frac{1}{(1+t)^5} x^{\frac{3}{2}} + \frac{t}{1+t^2}$. So $|f\,(t, x)| \leq \frac{1}{(1+t)^5} |x|^{\frac{3}{2}} + \frac{t}{1+t^2} = k(t)\,|x|^{\frac{3}{2}} + h(t)$, $f\,(t, 0) \neq 0$, $p = \frac{3}{2} > 1$, $\zeta = \frac{5}{6} \neq 0$. Applying Theorem 5 we get

$$M + N = 0.4564 + 0.20690 = 0.6633 < 1.$$

Then the (BVP3) has at least one nontrivial solution u^* in E.

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Laboratory of advanced materials
University Badji Mokhtar – Annaba
B.P. 12, 23000, Annaba, Algeria