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OSCILLATORY BEHAVIOR OF SECOND ORDER INTEGRO-DYNAMIC EQUATIONS WITH MAXIMA AND SUPERLINEAR OR SUBLINEAR NEUTRAL TERMS

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ABSTRACT. In this work, we present some new criteria for the oscillatory and the asymptotic behavior of the solutions of the following second-order mixed nonlinear integro-dynamic equations with maxima and superlinear or sublinear neutral terms on time scales

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta}+\int_{0}^{t}a(t,s)f(s,x(s))\Delta s+\sum_{i=1}^{n}q_i(t)\max_{s\in[\tau_i(t),\xi_i(t)]}x^{\alpha}(s)=0,$$

where

$$z(t)=x(t)+p_1(t)x^{\lambda_1}(\eta_1(t))+p_2(t)x^{\lambda_2}(\eta_2(t)),t\in[0,+\infty)_{\mathbb{T}}.$$

The obtained results are new for both the discrete and continuous cases. Furthermore, our results extend known ones in the literature. An example is presented to illustrate the relevance of the results.

1. INTRODUCTION

This paper deals with the oscillatory and the asymptotic behavior of the solutions of the integro-dynamic equations with superlinear or sublinear neutral terms and with maxima of the form:

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta}+\int_{0}^{t}a(t,s)f(s,x(s))\Delta s+\sum_{i=1}^{n}q_i(t)\max_{s\in[\tau_i(t),\xi_i(t)]}x^{\alpha}(s)=0, \tag{1.1}$$

with

$$z(t)=x(t)+p_1(t)x^{\lambda_1}(\eta_1(t))+p_2(t)x^{\lambda_2}(\eta_2(t)),t\in[0,+\infty)_{\mathbb{T}}. \tag{1.2}$$

We take $\mathbb{T}\subseteq\mathbb{R}$ to be an arbitrary time scale with $0\in\mathbb{T}$ and $sup\mathbb{T}=+\infty$. By $t\geq s$ we mean as usual $t\in[s,\infty)\cap\mathbb{T}$.

We assume the following conditions:

(H_1) $\eta_i,\tau_i,\xi_i:\mathbb{T}\to\mathbb{T}$ are rd-continuous functions such that $\eta_1(t)\leq t\leq\eta_2(t)$, $\tau_i(t)\leq t\leq\xi_i(t)$, $i=1,2,...,n$ and $lim_{t\to+\infty}\eta_1(t)=+\infty=lim_{t\to+\infty}\tau_i(t)$.

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(H_2) p_1, p_2, q_i and r are non-negative rd-continuous functions on an arbitrary time scale $\mathbb{T}$ such that $r(t) > 0$, $i = 1, 2, ..., n$ and moreover, either

$$lim_{t\to+\infty}L(t,t_0) := \int_{t_0}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)} = +\infty, \tag{1.3}$$

or

$$lim_{t\to+\infty}L(t,t_0) < +\infty. \tag{1.4}$$

(H_3) $a(t,s) : \mathbb{T}\times\mathbb{R} \to \mathbb{R}$ is a rd-continuous function such that

$$a(t,s) \geq 0,\ a^{\Delta_t}(t,s) < 0 \text{ for } t > s.$$

(H_4) $f \in C(\mathbb{T}\times\mathbb{R},\mathbb{R})$ such that, $x(t)f(t,x(t)) \geq m(t)|x(t)|^{\beta+1} > 0$, $x \neq 0$, where $m(t) : \mathbb{T} \to (0,+\infty)$ is a positive increasing rd-continuous function and β is a quotient of odd positive integers.

(H_5) α and γ are quotients of odd positive integers.

By a solution of (1.1), we mean a nontrivial real valued Δ- differentiable function $x(t)$ satisfying (1.1) for $t \in \mathbb{T}$.

Definition 1.1. *A solution $x(t)$ of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative, otherwise, it is called nonoscillatory. Eq. (1.1) is said to be oscillatory if all of its solutions are oscillatory.*

The qualitative theory of differential equations with "maxima" has received very little attention, even though such equations are often used in the problem of automatic regulation and automatic control of various systems. These equations arise when the law of regulation depends on maximum values of some regulated state parameters over certain time intervals, see [5] and reference cited therein. In addition, the theory of integro-dynamic equations is limited due to the lack of available techniques.

In [8] and [4] the authors established oscillation results for the following second order neutral delay differential equations with "maxima"

$$(r(t)(x(t)+p(t)x(\tau(t)))')' + q(t) \max_{s\in[\sigma(t),t]} x^{\alpha}(s) = 0$$

and

$$(r(t)(x(t)+p(t)x(\tau(t))')^{\gamma})' + q(t) \max_{s\in[\sigma(t),t]} x^{\alpha}(s) = 0, \tag{1.5}$$

respectively. They obtained sufficient conditions for the solutions of these equations to oscillate or tend to zero as $t \to \infty$.

In [9], some oscillation criteria were introduced for (1.5) when taking $\gamma = \alpha$. Also, in [7], Grace et al., studied the asymptotic behavior of non-oscillatory solutions of the following second order integro-dynamic equation

$$(r(t)x^{\Delta}(t))^{\Delta} + \int_{0}^{t} a(t,s)f(s,x(s))\Delta s = 0. \tag{1.6}$$

In 2014, Agarwal et al. [2], studied the oscillatory and asymptotic behavior of the following second order integro-dynamic equation

$$(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta}+\int_{0}^{t}a(t,s)f(s,x(s))\Delta s=0. \tag{1.7}$$

Note that Eqs.(1.6) and (1.7) are special cases of our Eq. (1.1) when taking $q_i(t)=0=p_1(t)=p_2(t)$. Thus, the results of [2] and [9] can't be applied to Eq. (1.1). Note also that (1.5) is a special case of (1.1) when taking $p_2(t)\equiv 0\equiv a(t,s)$, $\lambda_1=1$ and $q_i(t)\equiv 0$, $i=2,3,...n$.
In 2019, Agwa et al. [3], studied the oscillatory behavior of Eq. (1.1) when taking $\lambda_1=\lambda_2=1$.
In this work we generalize the previous work in [3] by studying the same equation with superlinear and sublinear terms, considering the following cases
1) $0<\lambda_1<1$ and $\lambda_2>1$ (neutral equation with superlinear and sublinear neutral terms)
2) $0<\lambda_2<1$ and $\lambda_1>1$ (neutral equation with superlinear and sublinear neutral terms)
3) $0<\lambda_1<1$ and $0<\lambda_2<1$ (neutral equation with sublinear neutral terms)
4) $\lambda_1>1$ and $\lambda_2>1$ (neutral equation with superlinear neutral terms).

Therefore the obtained results in this paper extend and complement the results obtained in [2, 3, 4, 8, 9].

2. Basic lemmas

In this section, we state some lemmas that are essential to establishing our results.

Lemma 2.1. *[1] If X and Y are nonnegative real numbers, then*

$$X^{\lambda}+(\lambda-1)Y^{\lambda}-\lambda XY^{\lambda-1}\geq 0 \text{ for } \lambda>1$$

and

$$X^{\lambda}-(1-\lambda)Y^{\lambda}-\lambda XY^{\lambda-1}\leq 0 \text{ for } \lambda<1,$$

with equality holding if and only if $X=Y$ or $\lambda=1$.

Lemma 2.2. *If $f(s)$ and $a(u,s)$ are rd-continuous functions, then*

$$\int_{t_0}^{t}\int_{t_0}^{u}a(u,s)f(s)\Delta s\Delta u=\int_{t_0}^{t}(ta(t,s)-\sigma(s)a(\sigma(s),s))f(s)\Delta s$$
$$-\int_{t_0}^{t}\sigma(u)\int_{t_0}^{u}a^{\Delta_u}(u,s)f(s)\Delta s\Delta u. \tag{2.1}$$

Proof. Let $F(u):=\int_{t_0}^{u}a(u,s)f(s)\Delta s$, and $g(u):=u$, then Theorem 5.37 in [**?**], leads to

$$F^{\Delta}(u)=a(\sigma(u),u)f(u)+\int_{t_0}^{u}a^{\Delta_u}(u,s)f(s)\Delta s.$$

Now by using, $Fg^{\Delta}=[Fg]^{\Delta}-F^{\Delta}g^{\sigma}$, we see that (2.1) holds. □

Lemma 2.3. *[6] (Grönwall's Inequality) Let $p \in \Re^+$. Also, assume that y and $f \in C_{rd}$. If*

$$y^{\Delta}(t) \leq p(t)y(t) + f(t) \;\; \textit{for all} \;\; t \in \mathbb{T},$$

then

$$y(t) \leq y(t_0)e_p(t,t_0) + \int_{t_0}^{t} e_p(t,\sigma(\tau))f(\tau)\Delta\tau \;\; \textit{for all} \;\; t,t_0 \in \mathbb{T}.$$

Lemma 2.4. *Assume (1.3) and conditions H_1-H_5 hold. Let x be a non-oscillatory solution of Eq. (1.1) on $[t_0,\infty)_{\mathbb{T}}$ and z be the function defined by Eq. (1.2). Then there exists $t_4 > t_0$ sufficiently large such that*

$$z(t) > 0, \; z^{\Delta}(t) > 0 \; \textit{and} \; (r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} < 0 \;\; \textit{for} \;\; t \in [t_4,+\infty)_{\mathbb{T}}. \tag{2.2}$$

Moreover, there exist suitable constants $b_1 > 0$ and $b_2 := \frac{z(t_3)}{L(t_4,t_3)} + r^{\frac{1}{\gamma}}(t_3)z^{\Delta}(t_3) \geq 0$, such that

$$b_1 \leq z(t) \leq b_2 L(t,t_3). \tag{2.3}$$

Proof. Let $x(t)$ be a non-oscillatory solution of Eq. (1.1). Then, we may assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$ for all $t \geq t_1$ and $t_2 \geq t_1 + \max\{\eta_1, \tau_i, i = 1,2,\ldots,n\}$, such that $x(\eta_1(t)) > 0$ and $x(\tau_i(t)) > 0$ for all $t \geq t_2$. Now from (1.1) and (1.2), we have $z(t) > 0$ and

$$\begin{aligned}(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} &= -\int_0^t a(t,s)f(s,x(s))\Delta s - \sum_{i=1}^{n} q_i(t) \max_{s\in[\tau_i(t),\xi_i(t)]} x^{\alpha}(s)\\ &= -\int_0^{t_2} a(t,s)f(s,x(s))\Delta s - \int_{t_2}^{t} a(t,s)f(s,x(s))\Delta s\\ &\quad - \sum_{i=1}^{n} q_i(t) \max_{s\in[\tau_i(t),\xi_i(t)]} x^{\alpha}(s),\end{aligned} \tag{2.4}$$

for $t \in [t_2,\infty)_{\mathbb{T}}$. Choosing $t_3 > t_2$ sufficiently large, then from H_4, we can find $k \geq 0$ such that

$$k := \int_0^{t_2} a(t,s)f(s,x(s))\Delta s + \int_{t_2}^{t_3} a(t,s)m(s)x^{\beta}(s)\Delta s.$$

So (2.4) can be written as

$$\begin{aligned}(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} \leq &-\int_0^{t_2} a(t,s)f(s,x(s))\Delta s - \int_{t_2}^{t_3} a(t,s)m(s)x^{\beta}(s)\Delta s\\ &- \int_{t_3}^{t} a(t,s)f(s,x(s))\Delta s - \sum_{i=1}^{n} q_i(t) \max_{s\in[\tau_i(t),\xi_i(t)]} x^{\alpha}(s)\end{aligned}$$

$$= -k - \int_{t_3}^{t} a(t,s) f(s,x(s)) \Delta s - \sum_{i=1}^{n} q_i(t) \max_{s \in [\tau_i(t), \xi_i(t)]} x^{\alpha}(s)$$

$$< - \int_{t_3}^{t} a(t,s) f(s,x(s)) \Delta s - \sum_{i=1}^{n} q_i(t) \max_{s \in [\tau_i(t), \xi_i(t)]} x^{\alpha}(s) < 0. \quad (2.5)$$

Then, $r(t)(z^{\Delta}(t))^{\gamma}$ is strictly decreasing on $[t_3, +\infty)_{\mathbb{T}}$. Now we claim that $r(t)(z^{\Delta}(t))^{\gamma} > 0$ on $[t_3, +\infty)_{\mathbb{T}}$. For the sake of contradiction, assume that this is not true. Then there is $t_3^* \in [t_3, +\infty)_{\mathbb{T}}$, such that $G_1 := r(t_3^*)(z^{\Delta}(t_3^*))^{\gamma} < 0$. Using the fact that $r(t)(z^{\Delta}(t))^{\gamma}$ is decreasing, we have

$$z^{\Delta}(t) \leq \frac{G_1^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(t)}.$$

Integrating from t_3^* to t and using condition (1.3), we get

$$z(t) \leq z(t_3^*) + G_1^{\frac{1}{\gamma}} \int_{t_3^*}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)} \to -\infty \text{ as } t \to \infty.$$

Hence, $z(t)$ is eventually negative. This is a contradiction. Thus (2.2) holds. Using the fact that $z(t)$ is increasing, we have

$$z(t) > z(t_3) := b_1. \quad (2.6)$$

Now integrating $z^{\Delta}(t)$ from t_3 to t and using (2.2), we obtain

$$z(t) = z(t_3) + \int_{t_3}^{t} \frac{[r(s)(z^{\Delta}(s))^{\gamma}]^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)} \Delta s \leq z(t_3) + r^{\frac{1}{\gamma}}(t_3) z^{\Delta}(t_3) L(t,t_3),$$

where $L(t,t_3) := \int_{t_3}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}$, hence $L(t,t_3)$ is a positive increasing function. Choosing $t_4 \geq t_3$ sufficiently large, we can write

$$z(t) \leq b_2 L(t,t_3) \text{ for all } t \in [t_4, +\infty)_{\mathbb{T}}, \quad (2.7)$$

where $b_2 := \frac{z(t_3)}{L(t_4,t_3)} + r^{\frac{1}{\gamma}}(t_3) z^{\Delta}(t_3)$. By combining the previous inequality and (2.6), we get

$$b_1 \leq z(t) \leq b_2 L(t,t_3) \qquad \text{for } t \in [t_4, \infty)_{\mathbb{T}}.$$

If $x(t)$ is an eventually negative solution of Eq. (1.1), then we can see that using the transformation $y(t) = -x(t), y(t) > 0$, Eq. (1.1) becomes

$$(r(t)(v^{\Delta}(t))^{\gamma})^{\Delta} - \int_{0}^{t} a(t,s) f(s,-y(s)) \Delta s + \sum_{i=1}^{n} q_i(t) \min_{s \in [\tau_i(t), \xi_i(t)]} y^{\alpha}(s) = 0,$$

where

$$v(t) = y(t) + p_1(t) y^{\lambda_1}(\eta_1(t)) + p_2(t) y^{\lambda_2}(\eta_2(t)).$$

Thus,

$$
\begin{aligned}
(r(t)(v^{\Delta}(t))^{\gamma})^{\Delta} &= \int_{0}^{t} a(t,s)f(s,-y(s))\Delta s - \sum_{i=1}^{n} q_i(t) \min_{s\in[\tau_i(t),\xi_i(t)]} y^{\alpha}(s) \\
&= \int_{0}^{t_2} a(t,s)f(s,x(s))\Delta s + \int_{t_2}^{t} a(t,s)f(s,x(s))\Delta s \\
&\quad - \sum_{i=1}^{n} q_i(t) \min_{s\in[\tau_i(t),\xi_i(t)]} y^{\alpha}(s).
\end{aligned} \tag{2.8}
$$

Choosing $t_4 > t_2$ sufficiently large, and in view of H_4, we can find $k_1 \le 0$ such that

$$
k_1 := \int_{0}^{t_2} a(t,s)f(s,x(s))\Delta s - \int_{t_2}^{t_4} a(t,s)m(s)y^{\beta}(s)\Delta s.
$$

Thus, (2.8) can be written as

$$
\begin{aligned}
(r(t)(v^{\Delta}(t))^{\gamma})^{\Delta} \le& \int_{0}^{t_2} a(t,s)f(s,x(s))\Delta s - \int_{t_2}^{t_4} a(t,s)m(s)y^{\beta}(s)\Delta s \\
&- \int_{t_4}^{t} a(t,s)f(s,x(s))\Delta s - \sum_{i=1}^{n} q_i(t) \min_{s\in[\tau_i(t),\xi_i(t)]} y^{\alpha}(s) \\
=& k_1 + \int_{t_4}^{t} a(t,s)f(s,x(s))\Delta s - \sum_{i=1}^{n} q_i(t) \min_{s\in[\tau_i(t),\xi_i(t)]} y^{\alpha}(s) \\
<& - \int_{t_4}^{t} a(t,s)m(s)y^{\beta}(s)\Delta s - \sum_{i=1}^{n} q_i(t) \min_{s\in[\tau_i(t),\xi_i(t)]} y^{\alpha}(s) < 0.
\end{aligned}
$$

It follows in a similar manner that (2.3) holds for $v(t)$. This completes the proof.□

Lemma 2.5. *Assume that (1.3) and conditions H_1-H_5 hold. Let x be a non-oscillatory solution of (1.1) on $[t_0,\infty)_{\mathbb{T}}$ and z be the function defined by Eq. (1.2). Then, there exists $t_4 \ge t_0$ (t_4 is as given in Lemma 2.4), such that*

$$
\frac{z(\eta_2(t))}{z(t)} \le \psi(t,t_4) \quad \textit{for all} \quad t \ge t_4. \tag{2.9}
$$

Proof. Let $x(t)$ be a non-oscillatory solution of Eq. (1.1). Then, we may assume that there exists $t_4 \ge t_0$ sufficiently large such that $x(t) > 0$, $x(\eta_1(t)) > 0$ and $x(\tau_i(t)) > 0$ for all $t \ge t_4$, where t_4 is as specified in Lemma 2.4. Integrating $z^{\Delta}(t)$ from t_4 to t and using the fact that $r(t)(z^{\Delta}(t))^{\gamma}$ is decreasing, we have

$$
z(t) \ge z(t) - z(t_4) = \int_{t_4}^{t} \frac{(r(s)(z^{\Delta}(s))^{\gamma})^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)} \Delta s \ge r^{\frac{1}{\gamma}}(t) z^{\Delta}(t) \int_{t_4}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}.
$$

Thus

$$\frac{r^{\frac{1}{\gamma}}(t)z^{\Delta}(t)}{z(t)} \leq \Big(\int_{t_4}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}\Big)^{-1}. \tag{2.10}$$

Integrating $z^{\Delta}(t)$ from t to $\eta_2(t)$, we get

$$z(\eta_2(t)) - z(t) = \int_{t}^{\eta_2(t)} \frac{(r(s)(z^{\Delta}(s))^{\gamma})^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)}\Delta s \leq r^{\frac{1}{\gamma}}(t)z^{\Delta}(t) \int_{t}^{\eta_2(t)} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)},$$

i.e.,

$$\frac{z(\eta_2(t))}{z(t)} \leq 1 + \frac{r^{\frac{1}{\gamma}}(t)z^{\Delta}(t)}{z(t)}\Big(\int_{t_4}^{\eta_2(t)} \frac{\Delta s}{r^{\frac{1}{\gamma}}(t)} - \int_{t_4}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(t)}\Big). \tag{2.11}$$

Therefore, (2.10) and (2.11) imply

$$\frac{z(\eta_2(t))}{z(t)} \leq \Big(\int_{t_4}^{\eta_2(t)} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}\Big)\Big(\int_{t_4}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}\Big)^{-1} := \psi(t,t_4) \qquad \text{for all } \ t \geq t_4,$$

which is the desired inequality. □

3. Main results

Theorem 3.1. *Assume that (1.3) and conditions $H_1 - H_5$ hold with $\beta \geq 1, \gamma \geq 1$, $0 < \lambda_1 < 1$ and $\lambda_2 > 1$. Furthermore, suppose that there exist positive rd- continuous functions $N(t)$ and $h_1(t)$ such that for all t_4 sufficiently large such that $t_4 \geq t_3 > t_0$, we have*

$$\limsup_{t\to+\infty} \int_{t_4}^{t} \Big(\frac{1}{r(u)} \int_{t_4}^{u} g^{*}(s)\Delta s\Big)^{\frac{1}{\gamma}}\Delta u] < +\infty, \tag{3.1}$$

then every nonoscillatory solution $x(t)$ of Eq. (1.1) satisfies

$$\mid x(t) \mid = O[A_1 e_{B(t)}(t,t_4) + \int_{t_4}^{t} e_{B(t)}(t,\sigma(v))f(v)\Delta v],$$

where

$$B(t) := \frac{1}{\gamma r(t)} \int_{t_4}^{t} \sigma(s)a(\sigma(s),s)Q^{\beta}(s)N(s)\Delta s,$$

$$f(t) := \frac{c_4}{r(t)^{\frac{1}{\gamma}}} + (1 - \frac{1}{\gamma}),$$

$$g_1(t) := \frac{1-\lambda_1}{\lambda_1^{\frac{\lambda_1}{\lambda_1-1}}} h_1^{\frac{\lambda_1}{\lambda_1-1}}(t)p_1^{\frac{1}{1-\lambda_1}}(t),$$

$$g_+(t) := \frac{\beta-1}{\beta^{\frac{\beta}{\beta-1}}} \int_{t_4}^{t} a(t,s)Q^{\beta}(s)N^{\frac{\beta}{\beta-1}}(s)m^{\frac{1}{1-\beta}}(s)\Delta s - b_1^{\alpha}\sum_{i=1}^{n} q_i(t) \min_{s\in[\tau_i(t),\xi_i(t)]} Q^{\alpha}(s),$$

$$g_-(t) := \frac{\beta-1}{\beta^{\frac{\beta}{\beta-1}}}\int_{t_4}^{t} a(t,s)Q^{\beta}(s)N^{\frac{\beta}{\beta-1}}(s)m^{\frac{1}{1-\beta}}(s)\Delta s - b_1^{\alpha}\sum_{i=1}^{n} q_i(t)\max_{s\in[\tau_i(t),\xi_i(t)]} Q^{\alpha}(s),$$

$$g^*(t) := \max\{g_-(t), g_+(t), 0\}$$

and

$$Q(t) := [1 - \frac{g_1(t)}{b_1} - h_1(t) - p_2(t)b_2^{\lambda_2-1}L^{\lambda_2-1}(\eta_2(t),t_3)\psi(t,t_4)] > 0.$$

Proof. Let $x(t)$ be a non-oscillatory solution of Eq. (1.1). Then, we may assume that there exists $t_4 \geq t_0$ sufficiently large such that $x(t) > 0$, $x(\eta_1(t)) > 0$ and $x(\tau_i(t)) > 0$ for all $t \geq t_4$, where t_4 is as specified in Lemma 2.4. From the definition of $z(t)$ (see(1.2)) and (2.3), we see that

$$\begin{aligned}
x(t) &= z(t) - p_1(t)x^{\lambda_1}(\eta_1(t)) - p_2(t)x^{\lambda_2}(\eta_2(t)) \geq z(t) - p_1(t)z^{\lambda_1}(\eta_1(t)) - p_2(t)z^{\lambda_2}(\eta_2(t))\\
&= z(t) - [p_1(t)z^{\lambda_1}(\eta_1(t)) - h_1(t)z(\eta_1(t))] - h_1(t)z(\eta_1(t)) - p_2(t)z^{\lambda_2-1}(\eta_2(t))z(\eta_2(t))\\
&\geq z(t) - [p_1(t)z^{\lambda_1}(\eta_1(t)) - h_1(t)z(\eta_1(t))] - h_1(t)z(\eta_1(t)) - p_2(t)b_2^{\lambda_2-1}L^{\lambda_2-1}(\eta_2(t),t_3)z(\eta_2(t)),
\end{aligned}$$

where $h_1(t)$ is a positive rd-continuous function. Applying Lemma 2.1, with $\lambda = \lambda_1, X = (p_1)^{\frac{1}{\lambda_1}} z(\eta_1(t)$ and $Y = [\frac{h_1 p_1^{\frac{-1}{\lambda_1}}}{\lambda_1}]^{\frac{1}{\lambda_1-1}}$, we have

$$p_1(t)z^{\lambda_1}(\eta_1(t)) - h_1(t)z(\eta_1(t)) \leq \frac{1-\lambda_1}{\lambda_1^{\frac{\lambda_1}{\lambda_1-1}}} h_1^{\frac{\lambda_1}{\lambda_1-1}}(t)p_1^{\frac{1}{1-\lambda_1}}(t) := g_1(t), \tag{3.2}$$

so that

$$\begin{aligned}
x(t) &\geq z(t) - g_1(t) - h_1(t)z(\eta_1(t)) - p_2(t)b_2^{\lambda_2-1}L^{\lambda_2-1}(\eta_2(t),t_3)z(\eta_2(t))\\
&\geq (1 - \frac{g_1(t)}{z(t)} - h_1(t)\frac{z(\eta_1(t))}{z(t)} - p_2(t)b_2^{\lambda_2-1}L^{\lambda_2-1}(\eta_2(t),t_3)\frac{z(\eta_2(t))}{z(t)})z(t).
\end{aligned}$$

Since $z(t)$ is increasing, (2.3) and (2.9) imply that

$$x(t) \geq [1 - \frac{g_1(t)}{b_1} - h_1(t) - p_2(t)b_2^{\lambda_2-1}L^{\lambda_2-1}(\eta_2(t),t_3)\psi(t,t_4)]z(t) := Q(t)z(t), \tag{3.3}$$

where $Q(t) := [1 - \frac{g_1(t)}{b_1} - h_1(t) - p_2(t)b_2^{\lambda_2-1}L^{\lambda_2-1}(\eta_2(t),t_3)\psi(t,t_4)] > 0$. Then

$$\max_{s\in[\tau_i(t),\xi_i(t)]} x^{\alpha}(s) \geq \max_{s\in[\tau_i(t),\xi_i(t)]} b_1^{\alpha}Q^{\alpha}(s) = b_1^{\alpha}\max_{s\in[\tau_i(t),\xi_i(t)]} Q^{\alpha}(s). \tag{3.4}$$

For a sufficiently large t_4, using H_4 in (2.5), we get

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} \leq -\int_{t_4}^{t} a(t,s)m(s)x^{\beta}(s)\Delta s - \sum_{i=1}^{n} q_i(t)\max_{s\in[\tau_i(t),\xi_i(t)]} x^{\alpha}(s).$$

Substituting from (3.3) and (3.4) in the previous inequality, we get

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} \leq -\int_{t_4}^{t} a(t,s)m(s)Q^{\beta}(s)z^{\beta}(s)\Delta s - b_1^{\alpha}\sum_{i=1}^{n} q_i(t) \max_{s\in[\tau_i(t),\xi_i(t)]} Q^{\alpha}(s). \quad (3.5)$$

Since $N(t)$ is a positive rd-continuous function, (3.5) can be written as

$$\begin{aligned}(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} \leq & \int_{t_4}^{t} a(t,s)Q^{\beta}(s)[N(s)z(s) - m(s)z^{\beta}(s)]\Delta s \\ & - \int_{t_4}^{t} a(t,s)Q^{\beta}(s)N(s)z(s)\Delta s - b_1^{\alpha}\sum_{i=1}^{n} q_i(t) \max_{s\in[\tau_i(t),\xi_i(t)]} Q^{\alpha}(s). \quad (3.6)\end{aligned}$$

Applying Lemma 2.1, with $\lambda = \beta, X = m^{\frac{1}{\beta}}(s)z(s)$ and $Y = [\frac{N(s)}{\beta m^{\frac{1}{\beta}}(s)}]^{\frac{1}{\beta-1}}$, we have

$$N(s)z(s) - m(s)z^{\beta}(s) \leq \frac{\beta-1}{\beta^{\frac{\beta}{\beta-1}}} N^{\frac{\beta}{\beta-1}}(s)m^{\frac{1}{1-\beta}}(s).$$

Substituting from the previous inequality into (3.6), gives

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} \leq g_-(t) - \int_{t_4}^{t} a(t,s)Q^{\beta}(s)N(s)z(s)\Delta s, \quad (3.7)$$

where

$$g_-(t) = \frac{\beta-1}{\beta^{\frac{\beta}{\beta-1}}} \int_{t_4}^{t} a(t,s)Q^{\beta}(s)N^{\frac{\beta}{\beta-1}}(s)m^{\frac{1}{1-\beta}}(s)\Delta s - b_1^{\alpha}\sum_{i=1}^{n} q_i(t) \max_{s\in[\tau_i(t),\xi_i(t)]} Q^{\alpha}(s).$$

Integrating (3.7) from t_4 to t, leads to

$$(z^{\Delta}(t))^{\gamma} \leq \frac{r(t_4)(z^{\Delta}(t_4))^{\gamma}}{r(t)} - \frac{1}{r(t)}\int_{t_4}^{t}\int_{t_4}^{u} a(u,s)N(s)Q^{\beta}(s)z(s)\Delta s\Delta u + \frac{1}{r(t)}\int_{t_4}^{t} g_-(s)\Delta s.$$

Using Lemma 2.2 and taking $g^*(t) = \max\{g_-(t), g_+(t), 0\}$, we have

$$z^{\Delta}(t) \leq \Big[\frac{c_4}{r(t)} + \frac{1}{r(t)}\int_{t_4}^{t} g^*(s)\Delta s + \frac{1}{r(t)}\int_{t_4}^{t} \sigma(s)a(\sigma(s),s)N(s)Q^{\beta}(s)z(s)\Delta s\Big]^{\frac{1}{\gamma}}, \quad (3.8)$$

where $c_4 = r(t_4)[z^{\Delta}(t_4)]^{\gamma}$. Using $(a+b)^{\lambda} \leq a^{\lambda} + b^{\lambda}$ for $a \geq 0$, $b \geq 0$ and $\lambda \leq 1$, (3.8) becomes

$$z^{\Delta}(t) \leq \Big(\frac{c_4}{r(t)}\Big)^{\frac{1}{\gamma}} + \Big(\frac{1}{r(t)}\int_{t_4}^{t} g^*(s)\Delta s\Big)^{\frac{1}{\gamma}} + \Big(\frac{1}{r(t)}\int_{t_4}^{t} \sigma(s)a(\sigma(s),s)N(s)Q^{\beta}(s)z(s)\Delta s\Big)^{\frac{1}{\gamma}}. \quad (3.9)$$

Integrating the above inequality from t_4 to t and taking A_1 as upper bound for

$$z(t_4)+\int_{t_4}^{t}(\frac{1}{r(u)}\int_{t_4}^{u}g^*(s)\Delta s)^{\frac{1}{\gamma}}\Delta u,$$

we have

$$z(t)\leq A_1+\int_{t_4}^{t}(\frac{c_4}{r(s)})^{\frac{1}{\gamma}}\Delta s+\int_{t_4}^{t}(\frac{1}{r(u)}\int_{t_4}^{u}\sigma(s)a(\sigma(s),s)Q^{\beta}(s)N(s)z(s)\Delta s)^{\frac{1}{\gamma}}\Delta u.$$

Again using Lemma 2.1, with $X=\frac{1}{r(u)}\int_{t_4}^{u}\sigma(s)a(\sigma(s),s)Q^{\beta}(s)N(s)z(s)\Delta s$, $\lambda=\frac{1}{\gamma}\leq 1$, and $Y=1$, the previous inequality can be written as

$$\begin{aligned}z(t)&\leq A_1+\int_{t_4}^{t}(\frac{c_4}{r(s)})^{\frac{1}{\gamma}}\Delta s+(1-\frac{1}{\gamma})\int_{t_4}^{t}\Delta u+\int_{t_4}^{t}\frac{1}{\gamma r(u)}\int_{t_4}^{u}\sigma(s)a(\sigma(s),s)N(s)Q^{\beta}(s)z(s)\Delta s\Delta u\\&\leq A_1+\int_{t_4}^{t}(\frac{c_4}{r(s)})^{\frac{1}{\gamma}}\Delta s+(1-\frac{1}{\gamma})t+\int_{t_4}^{t}\frac{1}{\gamma r(u)}\int_{t_4}^{u}\sigma(s)a(\sigma(s),s)Q^{\beta}(s)N(s)z(s)\Delta s\Delta u.\end{aligned}$$

Let $u(t)$ equals the right hand side of the previous inequality. Then, we have

$$u^{\Delta}(t)=(\frac{c_4}{r(t)})^{\frac{1}{\gamma}}+(1-\frac{1}{\gamma})+\frac{1}{\gamma r(t)}\int_{t_4}^{t}\sigma(s)a(\sigma(s),s)N(s)Q^{\beta}(s)z(s)\Delta s,u(t_4)=A_1.$$

Therefore $u(t)$ is increasing and since $z(t)\leq u(t)$, we have

$$u^{\Delta}(t)\leq f(t)+B(t)u(t),$$

where $f(t):=(\frac{c_4}{r(t)})^{\frac{1}{\gamma}}+(1-\frac{1}{\gamma})$ and $B(t)=\frac{1}{\gamma r(t)}\int_{t_4}^{t}\sigma(s)a(\sigma(s),s)Q^{\beta}(s)N(s)\Delta s$. Using Lemma 2.3, leads to

$$x(t)\leq z(t)\leq u(t)\leq A_1e_{B(t)}(t,t_4)+\int_{t_4}^{t}e_{B(t)}(t,\sigma(v))f(v)\Delta v.$$

Then, $x(t)=O[A_1e_{B(t)}(t,t_4)+\int_{t_4}^{t}e_{B(t)}(t,\sigma(v))f(v)\Delta v]$.

If $x(t)$ is an eventually negative solution of Eq. (1.1), then we can see that using the transformation $y(t)=-x(t), y(t)>0$, Eq. (1.1) becomes

$$(r(t)(v^{\Delta}(t))^{\gamma})^{\Delta}-\int_{0}^{t}a(t,s)f(s,-y(s))\Delta s+\sum_{i=1}^{n}q_i(t)\min_{s\in[\tau_i(t),\xi_i(t)]}y^{\alpha}(s)=0,$$

where

$$v(t)=y(t)+p_1(t)y^{\lambda_1}(\eta_1(t))+p_2(t)y^{\lambda_2}(\eta_2(t)).$$

Thus,

$$
\begin{aligned}
(r(t)(v^{\Delta}(t))^{\gamma})^{\Delta} &= \int_{0}^{t} a(t,s)f(s,-y(s))\Delta s - \sum_{i=1}^{n} q_i(t) \min_{s\in[\tau_i(t),\xi_i(t)]} y^{\alpha}(s) \\
&= \int_{0}^{t_2} a(t,s)f(s,x(s))\Delta s + \int_{t_2}^{t} a(t,s)f(s,x(s))\Delta s - \sum_{i=1}^{n} q_i(t) \min_{s\in[\tau_i(t),\xi_i(t)]} y^{\alpha}(s). \qquad (3.10)
\end{aligned}
$$

Choosing $t_4 > t_2$ sufficiently large, and in view of H_4, we can find $k_1 \le 0$ such that

$$
k_1 := \int_{0}^{t_2} a(t,s)f(s,x(s))\Delta s - \int_{t_2}^{t_4} a(t,s)m(s)y^{\beta}(s)\Delta s.
$$

So (3.10) can be written as

$$
\begin{aligned}
(r(t)(v^{\Delta}(t))^{\gamma})^{\Delta} \le &\int_{0}^{t_2} a(t,s)f(s,x(s))\Delta s - \int_{t_2}^{t_4} a(t,s)m(s)y^{\beta}(s)\Delta s \\
&- \int_{t_4}^{t} a(t,s)f(s,x(s))\Delta s - \sum_{i=1}^{n} q_i(t) \min_{s\in[\tau_i(t),\xi_i(t)]} y^{\alpha}(s) \\
=&k_1 + \int_{t_4}^{t} a(t,s)f(s,x(s))\Delta s - \sum_{i=1}^{n} q_i(t) \min_{s\in[\tau_i(t),\xi_i(t)]} y^{\alpha}(s) \\
<&- \int_{t_4}^{t} a(t,s)m(s)y^{\beta}(s)\Delta s - \sum_{i=1}^{n} q_i(t) \min_{s\in[\tau_i(t),\xi_i(t)]} y^{\alpha}(s) < 0.
\end{aligned}
$$

Similarly, it follows that $-x(t) = O[A_1 e_{B(t)}(t,t_4) + \int_{t_4}^{t} e_{B(t)}(t,\sigma(v))f(v)\Delta v]$. This completes the proof. $\square$

Corollary 3.1. *Let all assumptions of Theorem 3.1 hold and*

$$
\limsup_{t\to+\infty} \frac{1}{e_{B(t)}(t,t_4)} \int_{t_4}^{t} e_{B(t)}(t,\sigma(v))[(\frac{c_4}{r(v)})^{\frac{1}{\gamma}} + (1-\frac{1}{\gamma})]\Delta v < +\infty. \qquad (3.11)
$$

Then every nonoscillatory solution of (1.1) satisfies

$$
|x(t)| = O(e_{B(t)}(t,t_4)).
$$

Definition 3.1. *Let* $\mathbb{D} \equiv \{(t,s) \in \mathbb{T}^2 : t \ge s \ge t_0\}$. *The function* $H \in C_{rd}(\mathbb{D},\mathbb{R}^+)$ *belongs to the class* Ω *if*

(i) $H(t,s) > 0$, *on* $\mathbb{D}$ *and* $\lim_{t\to+\infty} \frac{H^{\Delta_s}(t,t_0)}{H(t,t_0)} = O(1)$.

Theorem 3.2. *Assume(1.3) and conditions* $H_1 - H_5$ *hold with* $\beta \ge 1, \gamma > 1, 0 < \lambda_1 < 1$ *and* $\lambda_2 > 1$. *Also, suppose that there exist a kernel function* $H(t,s) \in \Omega$

and positive rd- continuous functions $N(t)$ and $h_1(t)$ such that for all t_4 sufficiently large ($t_4 \geq t_3 > t_0$), the following hold:

$$(H^{\Delta_s}(t,s)r(s))^{\Delta_s} \geq 0,\ H^{\Delta_s}(t,s) < 0, \tag{3.12}$$

$$\liminf_{t\to\infty} \frac{1}{H(t,t_4)} \int_{t_4}^{t} (H^{\Delta_s}(t,s)r(s))^{\Delta_s} - (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}} H^{\Delta_s}(t,s)r(s)\Delta s < \infty, \tag{3.13}$$

$$\liminf_{t\to\infty} \frac{1}{H(t,t_4)} \int_{t_4}^{t} H(t,\sigma(s))g_-(s)\Delta s = -\infty \tag{3.14}$$

and

$$\liminf_{t\to\infty} \frac{1}{H(t,t_4)} \int_{t_4}^{t} H(t,\sigma(s)) \int_{t_4}^{s} a(t,u)Q^{\beta}(u)N(u)\Delta u\Delta s < \infty, \tag{3.15}$$

where

$$g_1(t) := \frac{1-\lambda_1}{\lambda_1^{\frac{\lambda_1}{\lambda_1-1}}} h_1^{\frac{\lambda_1}{\lambda_1-1}}(t) p_1^{\frac{1}{1-\lambda_1}}(t),$$

$$Q(t) := [1 - \frac{g_1(t)}{b_1} - h_1(t) - p_2(t)b_2^{\lambda_2-1}L^{\lambda_2-1}(\eta_2(t),t_3)\psi(t,t_4)] > 0$$

and

$$g_-(t) := \frac{\beta-1}{\beta^{\frac{\beta}{\beta-1}}} \int_{t_4}^{t} a(t,s)Q^{\beta}(s)N^{\frac{\beta}{\beta-1}}(s)m^{\frac{1}{1-\beta}}(s)\Delta s - b_1^{\alpha}\sum_{i=1}^{n} q_i(t) \max_{s\in[\tau_i(t),\xi_i(t)]} Q^{\alpha}(s),$$

then, every solution of (1.1) is oscillatory.

Proof. Let $x(t)$ be a non-oscillatory solution of Eq. (1.1). Then, we may assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$ for all $t \geq t_1$ and $t_2 \geq t_1 + \max\{\eta_1, \tau_i, i = 1,2,...,n\}$, such that $x(\eta_1(t)) > 0$ and $x(\tau_i(t)) > 0$ for all $t \geq t_2$. We choose $t_4 > t_2$ and proceed as in the proof of Theorem 3.1, untill we get (3.7). Then we can write

$$g_-(t) \geq (r(t)(z^{\Delta}(t))^{\gamma})^{\Delta} + \int_{t_4}^{t} a(t,s)Q^{\beta}(s)N(s)z(s)\Delta s. \tag{3.16}$$

Multiplying (3.16) by $H(t,\sigma(s))$, and integrating from t_4 to t, we obtain

$$\begin{aligned}\int_{t_4}^{t} H(t,\sigma(s))g_-(s)\Delta s \geq& \int_{t_4}^{t} H(t,\sigma(s))(r(s)(z^{\Delta}(s))^{\gamma})^{\Delta}\Delta s \\ &+ \int_{t_4}^{t} H(t,\sigma(s)) \int_{t_4}^{s} a(t,u)Q^{\beta}(u)N(u)z(u)\Delta u\Delta s.\end{aligned} \tag{3.17}$$

Using integration by parts and lemma 2.1, we get

$$\int_{t_4}^{t} H(t,\sigma(s))(r(s)(z^{\Delta}(s))^{\gamma})^{\Delta}\Delta s = H(t,t)r(t)(z^{\Delta}(t))^{\gamma} - H(t,t_4)r(t_4)(z^{\Delta}(t_4))^{\gamma} - \int_{t_4}^{t} H^{\Delta_s}(t,s)r(s)(z^{\Delta}(s))^{\gamma}\Delta s$$

$$> -H(t,t_4)r(t_4)(z^{\Delta}(t_4))^{\gamma} - \int_{t_4}^{t} H^{\Delta_s}(t,s)r(s)[(z^{\Delta}(s))^{\gamma} - z^{\Delta}(s)]\Delta s - \int_{t_4}^{t} H^{\Delta_s}(t,s)r(s)z^{\Delta}(s)\Delta s$$

$$> -H(t,t_4)r(t_4)(z^{\Delta}(t_4))^{\gamma} - (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}} \int_{t_4}^{t} H^{\Delta_s}(t,s)r(s)\Delta s - \int_{t_4}^{t} H^{\Delta_s}(t,s)r(s)z^{\Delta}(s)\Delta s$$

$$> -H(t,t_4)r(t_4)(z^{\Delta}(t_4))^{\gamma} - (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}} \int_{t_4}^{t} H^{\Delta_s}(t,s)r(s)\Delta s - H^{\Delta_s}(t,t)r(t)z(t) + H^{\Delta_s}(t,t_4)r(t_4)z(t_4) + \int_{t_4}^{t} (H^{\Delta_s}(t,s)r(s))^{\Delta_s} z(\sigma(s))\Delta s$$

$$> -H(t,t_4)r(t_4)(z^{\Delta}(t_4))^{\gamma} - (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}} \int_{t_4}^{t} H^{\Delta_s}(t,s)r(s)\Delta s + H^{\Delta_s}(t,t_4)r(t_4)z(t_4 + \int_{t_4}^{t} (H^{\Delta_s}(t,s)r(s))^{\Delta_s} z(\sigma(s))\Delta s.$$

Since $z(t)$ is increasing, we have that

$$\int_{t_4}^{t} H(t,\sigma(s))(r(s)(z^{\Delta}(s))^{\gamma})^{\Delta}\Delta s \geq A(t,t_4) + z(\sigma(t_4)) \int_{t_4}^{t} (H^{\Delta_s}(t,s)r(s))^{\Delta_s}\Delta s \quad (3.18)$$

where

$$A(t,t_4) = H^{\Delta_s}(t,t_4)r(t_4)z(t_4) - H(t,t_4)r(t_4)(z^{\Delta}(t_4))^{\gamma} - (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}} \int_{t_4}^{t} H^{\Delta_s}(t,s)r(s)\Delta s.$$

Substituting from (3.18) into (3.17), we obtain

$$\int_{t_4}^{t} H(t,\sigma(s))g_{-}(s)\Delta s \geq A(t,t_4) + z(\sigma(t_4)) \int_{t_4}^{t} (H^{\Delta_s}(t,s)r(s))^{\Delta_s}\Delta s + \int_{t_4}^{t} H(t,\sigma(s)) \int_{t_4}^{s} a(t,u)Q^{\beta}(u)N(u)z(u)\Delta u\Delta s.$$

Since $z^{\Delta}(t) > 0$, we have

$$\int_{t_4}^{t} H(t,\sigma(s))g_-(s)\Delta s \geq A(t,t_4) + z(\sigma(t_4)) \int_{t_4}^{t} (H^{\Delta_s}(t,s)r(s))^{\Delta_s}\Delta s + z(t_4)\int_{t_4}^{t} H(t,\sigma(s)) \int_{t_4}^{s} a(t,u)Q^{\beta}(u)N(u)\Delta u\Delta s.$$

Multiplying both sides of the previous inequality by $\frac{1}{H(t,t_4)}$, then taking the lower limits we obtain a contradiction. This completes the proof. □

Theorem 3.3. *Assume all conditions of Theorem 3.2 hold with* $\lambda_1 > 1$ *and* $0 < \lambda_2 < 1$. *Let*

$$Q(t) := [1 - \frac{g_2(t)}{b_1} - p_1(t)b_2^{\lambda_1-1}L^{\lambda_1-1}(\eta_1(t),t_3) - h_2(t)\psi(t,t_3)] > 0,$$

where

$$g_2(t) := \frac{1-\lambda_2}{\lambda_2^{\frac{\lambda_2}{\lambda_2-1}}} h_2^{\frac{\lambda_2}{\lambda_2-1}}(t) p_2^{\frac{1}{1-\lambda_2}}(t)$$

and $h_2(t)$ *is a positive rd-continuous function. Then, every solution of (1.1) is oscillatory.*

Proof. The proof is similar to that of Theorem 3.2, so it is omitted. □

Theorem 3.4. *Assume all conditions of Theorem 3.2 hold with* $0 < \lambda_1 < 1$ *and* $0 < \lambda_2 < 1$. *Let*

$$Q(t) := [1 - \frac{g_1(t)+g_2(t)}{b_1} - h_1(t) - h_2(t)\psi(t,t_3)] > 0,$$

where

$$g_1(t) = \frac{1-\lambda_1}{\lambda_1^{\frac{\lambda_1}{\lambda_1-1}}} h_1^{\frac{\lambda_1}{\lambda_1-1}}(t) p_1^{\frac{1}{1-\lambda_1}}(t),$$

$$g_2(t) = \frac{1-\lambda_2}{\lambda_2^{\frac{\lambda_2}{\lambda_2-1}}} h_2^{\frac{\lambda_2}{\lambda_2-1}}(t) p_2^{\frac{1}{1-\lambda_2}}(t)$$

and $h_1(t), h_2(t)$ *are positive rd-continuous functions. Then, every solution of (1.1) is oscillatory.*

Proof. The proof is similar to that of Theorem 3.2, so it is omitted. □

Theorem 3.5. *Assume all conditions of Theorem 3.2 hold with* $\lambda_1 > 1$ *and* $\lambda_2 > 1$. *Let*

$$Q(t) := [1 - p_1(t)b_2^{\lambda_1-1}L^{\lambda_1-1}(\eta_1(t),t_3) - p_2(t)b_2^{\lambda_2-1}L^{\lambda_2-1}(\eta_2(t),t_3)\psi(t,t_3)] > 0.$$

Then, every solution of (1.1) is oscillatory.

Proof. The proof is similar to that of Theorem 3.2, so it is omitted. □

Theorem 3.6. *Let condition (1.4) holds. Assume that all the assumptions of Theorem 3.2 hold except condition (1.3). If for all sufficiently large t_4,*

$$\int_{t_4}^{\infty}\frac{1}{r^{\frac{1}{\gamma}}(v)}\Big[\int_{t_4}^{v}\sum_{i=1}^{n}q_i(s)\max_{u\in[\tau_i(s),\xi_i(s)]}Q_*^{\alpha}(u)A^{\alpha}(u)-\sigma(s)a(\sigma(s),s)m(s)Q_*^{\beta}(s)A^{\beta}(s)\Delta s\Big]^{\frac{1}{\gamma}}\Delta v=\infty, \tag{3.19}$$

where

$$Q_*(t):=\Big[1-\frac{g_1(t)}{k_3A(t)}-h_1(t)\frac{A(\eta_1(t))}{A(t)}-b_3^{\lambda_2-1}p_2(t)\Big]>0,$$

and

$$A(t):=\int_t^{\infty}\frac{1}{r^{\frac{1}{\gamma}}(v)}\Delta v,$$

then every solution of (1.1) is oscillatory.

Proof. Let $x(t)$ be a non-oscillatory solution of Eq. (1.1). Then, we may assume that there exists $t_1\geq t_0$ such that $x(t)>0$ for all $t\geq t_1$ and $t_2\geq t_1+\max\{\eta_1,\tau_i,i=1,2,...,n\}$, such that $x(\eta_1(t))>0$ and $x(\tau_i(t))>0$ for all $t\geq t_2$. In view of Eq. (1.2) and (2.5), we have $z(t)>0$ for $t\in[t_2,\infty)_{\mathbb{T}}$ and moreover

$$(r(t)(z^{\Delta}(t))^{\gamma})^{\Delta}<-\int_{t_3}^{t}a(t,s)f(s,x(s))\Delta s-\sum_{i=1}^{n}q_i(t)\max_{s\in[\tau_i(t),\xi_i(t)]}x^{\alpha}(s)<0. \tag{3.20}$$

Then, $r(t)(z^{\Delta}(t))^{\gamma}$ is strictly decreasing on $[t_3,+\infty)_{\mathbb{T}}$, and there exists $t_5\in[t_3,\infty)_{\mathbb{T}}$ such that:

$$z^{\Delta}(t)>0\ \text{ or }\ z^{\Delta}(t)<0\ \text{ for }\ t\in[t_5,\infty)_{\mathbb{T}}.$$

We consider each of the following two cases separately.

Case 1. $z^{\Delta}(t)>0$ for $t\in[t_5,\infty)_{\mathbb{T}}$.

This case is similar to the proof of Theorem 3.2 and thus, is omitted.

Case 2. $z^{\Delta}(t)<0$ for $t\in[t_5,\infty)_{\mathbb{T}}$.

In this case, we have

$$\lim_{t\to\infty}z(t)=l,\ \ l\geq 0.$$

Since $r(t)(z^{\Delta}(t))^{\gamma}$ is strictly decreasing, then

$$z^{\Delta}(s)\leq\frac{r^{\frac{1}{\gamma}}(t)z^{\Delta}(t)}{r^{\frac{1}{\gamma}}(s)},\text{for } s\geq t.$$

Integrating from t to s and letting $s\to\infty$, we have

$$\begin{aligned} z(t)&\geq -r^{\frac{1}{\gamma}}(t)z^{\Delta}(t)\int_t^{\infty}\frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}=-r^{\frac{1}{\gamma}}(t)z^{\Delta}(t)A(t)\\ &\geq -r^{\frac{1}{\gamma}}(t_5)z^{\Delta}(t_5)A(t)=k_3A(t), \end{aligned} \tag{3.21}$$

where $k_3 = -r^{\frac{1}{\gamma}}(t_5)z^{\Delta}(t_5) > 0$ and $A(t) = \int\limits_t^{\infty} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}$. On the other hand, from (3.21), it follows that

$$\left(\frac{z(t)}{A(t)}\right)^{\Delta} = \frac{r^{\frac{1}{\gamma}}(t)A(t)z^{\Delta}(t) + z(t)}{r^{\frac{1}{\gamma}}(t)A(t)A(\sigma(t))} \geq 0, t \geq t_5. \tag{3.22}$$

Using the fact that $z(t)$ is decreasing, we obtain

$$z(t) < z(t_5) := b_3,\ t \geq t_5.$$

Using the above inequality and substituting (3.21) and (3.22) into (1.2), we get

$$\begin{aligned}
x(t) &= z(t) - p_1(t)x^{\lambda_1}(\eta_1(t)) - p_2(t)x^{\lambda_2}(\eta_2(t)) \\
&\geq z(t) - p_1(t)z^{\lambda_1}(\eta_1(t)) - p_2(t)z^{\lambda_2}(\eta_2(t)) \\
&= z(t) - [p_1(t)z^{\lambda_1}(\eta_1(t)) - h_1(t)z(\eta_1(t))] - h_1(t)z(\eta_1(t)) - p_2(t)z^{\lambda_2 - 1}(\eta_2(t))z(\eta_2(t)) \\
&\geq z(t) - [p_1(t)z^{\lambda_1}(\eta_1(t)) - h_1(t)z(\eta_1(t))] - h_1(t)z(\eta_1(t)) - b_3^{\lambda_2 - 1}p_2(t)z(\eta_2(t)),
\end{aligned}$$

where $h_1(t)$ is a positive rd-continuous function. Applying Lemma 2.1, with $\lambda = \lambda_1$, $X = (p_1)^{\frac{1}{\lambda_1}}z(\eta_1(t)$ and $Y = [\frac{h_1 p_1^{\frac{-1}{\lambda_1}}}{\lambda_1}]^{\frac{1}{\lambda_1 - 1}}$, we have

$$p_1(t)z^{\lambda_1}(\eta_1(t)) - h_1(t)z(\eta_1(t)) \leq \frac{1-\lambda_1}{\lambda_1^{\frac{\lambda_1}{\lambda_1 - 1}}} h_1^{\frac{\lambda_1}{\lambda_1 - 1}}(t)p_1^{\frac{1}{1-\lambda_1}}(t) := g_1(t). \tag{3.23}$$

Thus

$$\begin{aligned}
x(t) &> z(t) - g_1(t) - h_1(t)z(\eta_1(t)) - b_3^{\lambda_2 - 1}p_2(t)z(\eta_2(t)) \\
&> [1 - \frac{g_1(t)}{k_3 A(t)} - h_1(t)\frac{A(\eta_1(t))}{A(t)} - b_3^{\lambda_2 - 1}p_2(t)]z(t) := Q_*(t)z(t) \geq k_3 Q_*(t)A(t),
\end{aligned} \tag{3.24}$$

where $Q_*(t) = [1 - \frac{g_1(t)}{k_3 A(t)} - h_1(t)\frac{A(\eta_1(t))}{A(t)} - b_3^{\lambda_2 - 1}p_2(t)]$. Now substituting from (3.24) into (3.20), we obtain

$$\begin{aligned}
(r(t)(-z^{\Delta}(t))^{\gamma})^{\Delta} &> \int_{t_5}^{t} a(t,s)m(s)k_3^{\beta}Q_*^{\beta}(s)A^{\beta}(s)\Delta s + k_3^{\alpha}\sum_{i=1}^{n} q_i(t) \max_{s\in[\tau_i(t),\xi_i(t)]} Q_*^{\alpha}(s)A^{\alpha}(s) \\
&> C[\int_{t_5}^{t} a(t,s)m(s)Q_*^{\beta}(s)A^{\beta}(s)\Delta s + \sum_{i=1}^{n} q_i(t) \max_{s\in[\tau_i(t),\xi_i(t)]} Q_*^{\alpha}(s)A^{\alpha}(s)],
\end{aligned}$$

where $C = \min\{k_3^{\beta}, k_3^{\alpha}\}$. Integrating the previous inequality from t_5 to t and using Lemma 2.2, we obtain

$$r(t)(-z^{\Delta}(t))^{\gamma} > C\int_{t_5}^{t}\sum_{i=1}^{n}q_i(s)\max_{u\in[\tau_i(s),\xi_i(s)]}Q_*^{\alpha}(u)A^{\alpha}(u)+C\int_{t_5}^{t}\int_{t_5}^{u}a(u,s)m(s)Q_*^{\beta}(s)A^{\beta}(s)\Delta s\Delta u$$

$$> C\int_{t_5}^{t}[\sum_{i=1}^{n}q_i(s)\max_{u\in[\tau_i(s),\xi_i(s)]}Q_*^{\alpha}(u)A^{\alpha}(u)-\sigma(s)a(\sigma(s),s)m(s)Q_*^{\beta}(s)A^{\beta}(s)]\Delta s,$$

then

$$z^{\Delta}(t)\leq\frac{-C^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(t)}[\int_{t_5}^{t}\sum_{i=1}^{n}q_i(s)\max_{u\in[\tau_i(s),\xi_i(s)]}Q_*^{\alpha}(u)A^{\alpha}(u)-\sigma(s)a(\sigma(s),s)m(s)Q_*^{\beta}(s)A^{\beta}(s)\Delta s]^{\frac{1}{\gamma}}.$$

Again integrating the previous inequality from t_5 to t, we get

$$z(t)\leq z(t_5)-c^{\frac{1}{\gamma}}\int_{t_5}^{t}\frac{1}{r^{\frac{1}{\gamma}}(v)}\int_{t_5}^{v}\sum_{i=1}^{n}q_i(s)\max_{u\in[\tau_i(s),\xi_i(s)]}Q_*^{\alpha}(u)A^{\alpha}(u)$$
$$-\sigma(s)a(\sigma(s),s)m(s)Q_*^{\beta}(s)A^{\beta}(s)\Delta s]^{\frac{1}{\gamma}}\Delta v,$$

which means that $\lim_{t\to\infty}z(t)=-\infty$. This is a contradiction. □

Theorem 3.7. *Assume that all assumptions of Theorem 3.3 hold except condition (1.3). Let conditions (1.4) and (3.19) hold with*

$$Q_*(t):=[1-\frac{g_2(t)}{k_3A(t)}-b_3^{\lambda_1-1}p_1(t)\frac{A(\eta_1(t))}{A(t)}-h_2(t)]>0.$$

Then every solution of (1.1) is oscillatory.

Proof. The proof is similar to that of Theorem 3.6, so it is omitted. □

Theorem 3.8. *Assume that all assumptions of Theorem 3.4 hold except condition (1.3). Let conditions (1.4) and (3.19) hold with*

$$Q_*(t):=[1-\frac{g_1(t)+g_2(t)}{k_3A(t)}-h_1(t)\frac{A(\eta_1(t)}{A(t)}-h_2(t)]>0.$$

Then every solution of (1.1) is oscillatory.

Proof. The proof is similar to that of Theorem 3.6, so it is omitted. □

Theorem 3.9. *Assume that all assumptions of Theorem 3.5 hold except condition (1.3). Let conditions (1.4) and (3.19) hold with*

$$Q_*(t):=[1-b_3^{\lambda_1-1}p_1(t)\frac{A(\eta_1(t)}{A(t)}-b_3^{\lambda_2-1}p_2(t)]>0.$$

Then every solution of (1.1) is oscillatory.

Proof. The proof is similar to that of Theorem 3.6, so it is omitted. □

4. EXAMPLE

To illustrate our results, we provide an example of a second order neutral integro-dynamic equation with maxima. Note that this equation cannot be studied using the existing results in literature.

Example 4.1. *Let $t \in [t_0, +\infty)_{\mathbb{T}}$ with $t_3 = 2, t_4 = 4$, and take $\mathbb{T} = \mathbb{R}$. Consider the following neutral integro dynamic equation with maxima*

$$\big[t[x(t) + \frac{b_1^{\frac{1}{2}}}{t} x^{\frac{1}{2}}(\eta_1(t)) + \frac{t-2}{t b_2^{\lambda_2 - 1} L^{\lambda_2 - 1}(2t, t_3)\psi(t, t_4)} x^{\lambda_2}(2t)]^{\Delta}\big]^{\Delta}$$
$$+ \int_0^t \frac{1}{t^2 s^3} f(s, x(s))\Delta s + t^3 \max_{s \in [t, t+1]} x^{\alpha}(s) = 0. \tag{4.1}$$

Here, $r(t) = t,\ p_1(t) = \frac{b_1^{\frac{1}{2}}}{t},\ p_2(t) = \frac{t-2}{t b_2^{\lambda_2 - 1} L^{\lambda_2 - 1}(2t, t_3)\psi(t, t_4)},\ a(t,s) = \frac{1}{t^2 s^3},\ n = 1,$ $\tau_1(t) = t,\ \xi_1(t) = t+1,\ \eta_1(t) \le t, \alpha = 1,\ \beta = 2,\ \gamma = 1,\ \lambda_1 = \frac{1}{2},\ \lambda_2 > 1,\ q_1(t) = t^3$ and $m(t) = t$. Hence

$$lim_{t \to +\infty} L(t, t_0) := \int_{t_0}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)} = \int_{t_0}^{t} \frac{\Delta s}{s} = \infty,$$

which means that (1.3) holds. Taking $H(t,s) = \frac{t}{s}$, it is easy to show that conditions (3.12) and (3.13) hold, and

$$g_1(t) = \frac{1-\lambda_1}{\lambda_1^{\frac{\lambda_1}{\lambda_1 - 1}}} h_1^{\frac{\lambda_1}{\lambda_1 - 1}}(s) p_1^{\frac{1}{1-\lambda_1}}(s) = \frac{p_1^2}{4h_1}.$$

Letting $h_1(t) = \frac{1}{2t}$, we get

$$g_1(t) = \frac{b_1}{4t^2 * \frac{1}{2t}} = \frac{b_1}{2t}$$

and thus

$$Q(t) := [1 - \frac{g_1(t)}{b_1} - h_1(t) - p_2(t) b_2^{\lambda_2 - 1} L^{\lambda_2 - 1}(\eta_2(t), t_3)\psi(t, t_4)] = 1 - \frac{1}{t} - \frac{t-2}{t} = \frac{1}{t} > 0.$$

By taking $N(t) = 1$, we obtain

$$\begin{aligned} g_-(t) &= \frac{\beta - 1}{\beta^{\frac{\beta}{\beta - 1}}} \int_{t_4}^{t} a(t,s) Q^{\beta}(s) N^{\frac{\beta}{\beta - 1}}(s) m^{\frac{1}{1-\beta}}(s)\Delta s - b_1^{\alpha} \sum_{i=1}^{n} q_i(t) \max_{u \in [\tau_i(t), \xi_i(t)]} Q^{\alpha}(u) \\ &= \frac{1}{2^2} \int_4^t \frac{1}{t^2 s^5} s^{-1} ds - b_1 t^3 \max_{u \in [t, t+1]} \frac{1}{u} = \frac{1}{20}[\frac{1}{4^5 t^2} - \frac{1}{t^7}] - b_1 t^2. \end{aligned} \tag{4.2}$$

Moreover,

$$g_+(t) = \frac{\beta-1}{\beta^{\frac{\beta}{\beta-1}}}\int_{t_4}^{t} a(t,s)Q^{\beta}(t)N^{\frac{\beta}{\beta-1}}(s)m^{\frac{1}{1-\beta}}(s)\Delta s - b_1^{\alpha}\sum_{i=1}^{n} q_i(t)\min_{u\in[\tau_i(t),\xi_i(t)]} Q^{\alpha}(u)$$
$$= \frac{1}{20}\left[\frac{1}{4^5t^2}-\frac{1}{t^7}\right]-b_1\frac{t^3}{t+1}. \tag{4.3}$$

Now,

$$\liminf_{t\to\infty}\frac{1}{H(t,t_4)}\int_{t_4}^{t}H(t,\sigma(s))g_-(s)\Delta s=\liminf_{t\to\infty}\frac{4}{t}\int_{4}^{t}\frac{t}{s}\left(\frac{1}{20}\left[\frac{1}{4^5s^2}-\frac{1}{s^7}\right]-b_1s^2\right)\Delta s\to-\infty \text{ as } t\to\infty,$$

$$\liminf_{t\to\infty}\frac{1}{H(t,t_4)}\int_{t_4}^{t}H(t,\sigma(s))\int_{t_4}^{s}a(t,u)Q^{\beta}(u)N(u)\Delta u\Delta s = \liminf_{t\to\infty}\frac{4}{t}\int_{4}^{t}\frac{t}{s}\int_{4}^{s}\frac{1}{t^2u^3}\frac{1}{u^2}\Delta u\Delta s$$
$$= \liminf_{t\to\infty}\frac{4}{t^2}\int_{4}^{t}\frac{1}{s}\left[\frac{-1}{4u^4}\right]_4^s\Delta s = \liminf_{t\to\infty}\frac{-1}{t^2}\int_{4}^{t}s\left[\frac{1}{s^5}-\frac{1}{4^4s}\right]\Delta s = \liminf_{t\to\infty}\frac{1}{t^2}\left[\frac{1}{4s^4}+\frac{\ln s}{4^4}\right]_4^t<\infty.$$

Thus, conditions (3.14) and (3.15) hold. Now using Theorem 3.2, we conclude that every solution of Eq. (4.1) is oscillatory.

Remark 4.1. The results of [4], [8] and [9] can't be applied to (4.1) as $p_2(t)\neq 0\neq a(t,s)$, $\lambda_1\neq 1\neq\lambda_2$. The results of [7] too cannot be applied to (4.1) as $p_1(t)\neq 0\neq p_2(t)\neq q_i(t)$. But according to Theorem 3.2, we obtain that every solution of (4.1) is oscillatory.

Remark 4.2. The results of [3] can't be applied to (4.1) as $\lambda_1\neq 1\neq\lambda_2$. But according to Theorem 3.2, we obtain that every solution of (4.1) is oscillatory. So our work generalize and extend some recent published results in 2019 see [3].

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