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SOME RESULTS ON THE COMPOSITION OF DISTRIBUTIONS

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Abstract. Let F be a distribution, f be a locally summable function and $F_n = (F * \delta_n)(x)$, where $\delta_n(x)$ is a certain sequence converging to the Dirac delta-function $\delta(x)$. The distribution $F(f)$ is defined as the neutrix limit of the sequence $\{F_n(f)\}$, provided its limit h exists in the sense that

$$\operatorname*{N-lim}_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle$$

for all test functions $\varphi(x)$ in $\mathcal{D}(a,b)$. The composition of the distributions $x_-^{-s}\ln x_-$ and x_+^r is evaluated for $r = 0, 1, \dots$ and the composition of the distributions $x_-^{-s}\ln^m x_-$ and $H(x)$ is evaluated for $s, m = 1, 2, \dots$.

1. INTRODUCTION

The theory of distribution was first formulated by Sobolev and then developed systematically by L. Shwartz, [15]. It is a subject that is used not only in many mathematical disciplines such as functional analysis and applied mathematics but also in physics and engineering science, see [3, 4, 8].

Together with Shwartz's theory, some singular functions such as the Dirac's delta function $\delta(x)$ and operations with them can be defined mathematically. But besides this, the theory has become insufficient to satisfy all the needs of the physicists. For example to evaluate δ^2 or $\sqrt{\delta}$, because in general the meanings of the expressions of the form $f.g$ or $f(g(x))$ can not be given in Shwartz's theory where f and g are distributions [8]. Thus mathematicians started to develop new mathematical approaches. For example B. Fisher gave a definition to express the composition of the distributions via the theory of neutrices, see [5, 11].

Besides the Dirac's delta function, the concept of finite parts of divergent integrals are at the origin of the theory. The Hadamard's finite part is a

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way of resulting finite values to the integral of functions with the technique of neglecting appropriately defined infinite quantities, see [10]. Since taking the neutrix limit of a function is to extract a finite part from a divergent quantity, this method can be regarded as an application of the neutrix calculus. A neutrix is an additive group of negligible functions which satisfies the condition that does not contain any constant except zero, see [2].

Besides being used in the context of distributions, the theory of neutrices has had an impact on physical and engineering disciplines. For example, it has rendered perturbative quantum field theory mathematically meaningful by obtaining finite re-normalizations, see [13].

2. PRELIMINARIES

Before giving the definition of compositions, we need to construct a regular sequence of infinitely differentiable functions converging to a distribution. For this, firstly we shall use a sequence of regular functions which converges to Dirac's delta function $\delta(x)$ in the distributional sense. There are many ways to do this, see [9]. In this paper we use Temple's $\delta-$sequence $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, where $\rho(x)$ is a fixed infinitely differentiable function on $\mathbb{R}$ with the following four properties:

(i) $\rho(x) \geq 0$,
(ii) $\rho(x) =0$ for $|x| \geq 1$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1} \rho(x)\, dx = 1$.

Now let us denote by $\mathcal{D}$ the test function space of all infinitely differentiable functions $\varphi(x)$ $(x \in \mathbb{R})$ with compact support and by $\mathcal{D}'$ the space of distributions defined on $\mathcal{D}$. Also let $\mathcal{D}(a, b)$ be the space of infinitely differentiable functions with compact support contained in the interval (a, b) and let $\mathcal{D}'(a, b)$ be the space of distributions defined on $\mathcal{D}(a, b)$. Now if f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x - t)\rangle$$

for $n = 1, 2, \ldots$. Then we get a regular sequence $\{f_n(x)\}$ of infinitely differentiable functions converging to the distribution $f(x)$ in $\mathcal{D}'$.

The definition of the composition $F(f)$, where F and f are distributions, is defined as the double neutrix limit of the sequence $\{F_n(f_m(x))\}$, see [11, 12]. This definition is an extension of the definition of the composition of distributions given by Antosik, see [1]. The following definition, originally called the composition of distributions, was given in [5].

Throughout the paper, $\mathcal{N}$ stands for the neutrix, having domain N' the positive integers and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^{\lambda} \ln^{r-1} n,\ \ln^{r} n: \qquad \lambda > 0,\ r = 1, 2, \ldots$$

and all functions which converge to zero in the usual sense as n tends to infinity.

Definition 2.1. *Let F be a distribution in $\mathcal{D}'$ and let f be a locally summable function. We say that the distribution $F(f(x))$ exists and is equal to h on the open interval (a, b) if*

$$\mathrm{N}\!-\!\lim_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle$$

for all test functions $\varphi(x)$ in $\mathcal{D}(a, b)$, where $F_n(x) = (F * \delta_n)(x)$ for $n = 1, 2, \ldots$.

In particular, we say that the composition $F(f(x))$ exists and is equal to h on the open interval (a, b) if

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle$$

for all test functions $\varphi(x)$ in $\mathcal{D}(a, b)$.

The Heaviside function $H(x)$, a locally summable function which is equal to zero for $x < 0$ and to 1 for $x > 0$, has an influence in distribution theory. Given any $\varphi(x)$ in $\mathcal{D}$, we have

$$\langle H^{'}(x), \varphi(x)\rangle = \langle H(x), \varphi^{'}(x)\rangle = -\int_{0}^{\infty} \varphi^{'}(x)dx = \varphi(0),$$

and we get that its distributional derivative is the Dirac's delta function $\delta(x)$.

The s-th power of the Heaviside function for negative integers has been defined by the following theorem in [14].

Theorem 2.2. *The distribution $[H(x)]^{-s}$ exists and*

$$[H(x)]^{-s} = H(x) \tag{1}$$

for $s = 1, 2, \ldots$.

Theorem 2.3. *The distributions $[H(x)]_{-}^{-s}$ and $[H(x)]_{+}^{-s}$ exist and*

$$[H(x)]_{-}^{-s} = 0 \tag{2}$$

$$[H(x)]_{+}^{-s} = H(x) \tag{3}$$

for $s = 1, 2, \ldots$, see [14].

The distributions x_-^{-s} and $x_-^{-1}\ln x_-$ were defined respectively by

$$x_-^{-s} = \frac{(\ln x_-)^{(s)}}{(s-1)!} \quad \text{and} \quad x_-^{-1}\ln x_- = -\frac{1}{2}(\ln^2 x_-)' \tag{4}$$

and the distributions x_+^{-s} and $x_+^{-1}\ln x_+$ were defined respectively by

$$x_+^{-s} = \frac{(-1)^{s-1}(\ln x_+)^{(s)}}{(s-1)!} \quad \text{and} \quad x_+^{-1}\ln x_+ = \frac{1}{2}(\ln^2 x_+)' \tag{5}$$

for $s = 1, 2, \ldots$. Note that the definitions of x_-^{-s} and x_+^{-s} are not the same as in [9] and the distributions $x_-^{-r-1}\ln x_-$ and $x_+^{-r-1}\ln x_+$ were defined respectively by

$$x_-^{-r-1}\ln x_- = \frac{x_-^{-r-1} + (x_-^{-r}\ln x_-)'}{r} \tag{6}$$

and

$$x_+^{-r-1}\ln x_+ = \frac{x_+^{-r-1} - (x_+^{-r}\ln x_+)'}{r} \tag{7}$$

for $r = 1, 2, \ldots$. It follows by induction that

$$x_-^{-r-1}\ln x_- = \phi(r)\, x_-^{-r-1} - \frac{(\ln^2 x_-)^{(r+1)}}{2r!}\ , \tag{8}$$

$$x_+^{-r-1}\ln x_+ = \phi(r)\, x_+^{-r-1} + \frac{(-1)^r(\ln^2 x_+)^{(r+1)}}{2r!} \tag{9}$$

where

$$\phi(r) = \begin{cases} \sum_{i=1}^r i^{-1}, & r = 1, 2, \ldots, \\ 0, & r = 0 \end{cases}$$

see [7].

The following definition was given in [6].

Theorem 2.4. *The distribution $(x_+^r)_-^{-s}$ exists and*

$$(x_+^r)_-^{-s} = \frac{(-1)^{rs+s}c(\rho)}{r(rs-1)!}\delta^{(rs-1)}(x) \tag{10}$$

for $r, s = 1, 2, \ldots$, where

$$c(\rho) = \int_0^1 \ln t \rho(t)\, dt.$$

Before giving our main results we need the following lemmas which can be easily proved.

Lemma 2.5.

$$\int_{-1}^{1} v^i \rho^{(r)}(v)\, dv = \begin{cases} 0, & 0 \le i < r, \\ (-1)^r r!, & i = r \end{cases} \tag{11}$$

for $r = 0, 1, 2, \ldots$.

Lemma 2.6.

$$\int_0^1 v^r \ln v \rho^{(r)}(v) dv = \frac{1}{2}(-1)^r r! \phi(r) + (-1)^r r! c(\rho) \tag{12}$$

for $r = 0, 1, 2, \ldots$.

Lemma 2.7.

$$\int_0^1 v^r \ln^2 v \rho^{(r)}(v) dv = (-1)^r r! \left[\phi_1(r) + 2\phi(r)c(\rho) + c_1(\rho)\right] \tag{13}$$

for $r = 0, 1, 2, \ldots$, *where*

$$c_1(\rho) = \int_0^1 \ln^2 t \rho(t)\, dt$$

and

$$\phi_1(r) = \begin{cases} 0, & r = 0, 1, \\ \sum_{i=1}^{r-1} \frac{\phi(i)}{i+1}, & r = 2, 3, \ldots. \end{cases}$$

Lemma 2.8.

$$\int_0^1 v^r \ln(1 - v)\, dv = -\frac{\phi(r+1)}{r+1} \tag{14}$$

for $r = 0, 1, 2, \ldots$.

Lemma 2.9.

$$\int_0^1 v^r \ln^2(1 - v)\, dv = \frac{2\phi_2(r+1)}{r+1} \tag{15}$$

where

$$\phi_2(r) = \sum_{i=1}^{r+1} \frac{\phi(i)}{i}$$

for $r = 0, 1, 2, \ldots$.

3. MAIN RESULTS

In this section first we give the generalization of Theorem 2.3.

Theorem 3.1. *The compositions of the distributions* $x_-^{-s}\ln x_-$ *and* $H(x)$ *and of the distributions* $x_+^{-s}\ln x_+$ *and* $H(x)$ *exist and*

$$[H(x)]_-^{-s}\ln[H(x)]_- = 0, \tag{16}$$

$$[H(x)]_+^{-s}\ln[H(x)]_+ = \phi(s-1)H(x) \tag{17}$$

for $s = 1, 2, \ldots$.

Proof. Since we have

$$[H(x)]_-^{-s}\ln[H(x)]_- = \phi(s-1)[H(x)]_-^{-s} - \frac{(\ln^2[H(x)]_-)^{(s)}}{2(s-1)!}$$

and we know that $[H(x)]_-^{-s} = 0$ from the equation (2), it is necessary to evaluate $(\ln^2[H(x)]_-)^{(s)}$. Since

$$(\ln^2[H(x)]_-)_n^{(s)} = \ln^2[H(x)]_- * \delta_n^{(s)}(x),$$

we can write

$$I = \ln^2[H(x)]_- * \delta_n^{(s)}(x) = \begin{cases} \int_{-\frac{1}{n}}^{\frac{1}{n}} \ln^2(1-t)_-\, \delta_n^{(s)}(t)\, dt, & x > 0, \\ \int_0^{\frac{1}{n}} \ln^2 t\, \delta_n^{(s)}(t)\, dt, & x < 0 \end{cases}$$

$$= \begin{cases} 0, & x > 0, \\ \int_0^{\frac{1}{n}} \ln^2 t\, \delta_n^{(s)}(t)\, dt, & x < 0. \end{cases}$$

Then we have

$$\begin{aligned} \underset{n\to\infty}{\text{N}-\lim}\, I &= \underset{n\to\infty}{\text{N}-\lim} \int_0^{\frac{1}{n}} \ln^2 t\, \delta_n^{(s)}(t)\, dt \\ &= \underset{n\to\infty}{\text{N}-\lim}\, n^s \int_0^1 (\ln v - \ln n)^2 \rho^{(s)}(v)\, dv = 0 \end{aligned}$$

by using the substitution $nt = v$. Hence for any $\varphi(x) \in \mathcal{D}(a,b)$ we get

$$\begin{aligned} \underset{n\to\infty}{\text{N}-\lim}\langle I, \varphi\rangle &= \underset{n\to\infty}{\text{N}-\lim}\langle [[H(x)]_-^{-s}\ln[H(x)]_-]_n, \varphi(x)\rangle \\ &= \int_a^b \varphi(x) \int_0^1 (\ln v - \ln n)^2 \rho^{(s)}(v)\, dv\, dx = 0 \end{aligned}$$

and equation (16) follows.

Now from equation (9) we need to evaluate $(\ln^2[H(x)]_+)^{(s)}$. We can write

$$(\ln^2[H(x)]_+)^{(s)} * \delta_n(x) = \begin{cases} \int_{-\frac{1}{n}}^{\frac{1}{n}} \ln^2(1-t)\,\delta_n^{(s)}(t)\,dt, & x > 0, \\ \int_{-\frac{1}{n}}^{0} \ln^2 t\,\delta_n^{(s)}(t)\,dt, & x < 0. \end{cases}$$

Then by using the substitution $nt = v$ we have

$$\begin{aligned} \int_{-\frac{1}{n}}^{\frac{1}{n}} \ln^2(1-t)\delta_n^{(s)}(t)\,dt &= n^s \int_{-1}^{1} \Big(\sum_{i=1}^{\infty} \frac{v^i}{i\,n^i}\Big)^2 \rho^{(s)}(v)\,dv \\ &= 2\sum_{i=1}^{\infty} \frac{\phi(i)}{i+1}\, n^{s-i-1} \int_{-1}^{1} v^{i+1}\rho^{(s)}(v)\,dv. \end{aligned}$$

It follows that

$$\lim_{n\to\infty} \int_{-\frac{1}{n}}^{\frac{1}{n}} \ln^2(1-t)\,\delta_n^{(s)}(t)\,dt = 0 \tag{18}$$

for $i+1 > s$, and from equation (11) we have

$$\mathrm{N-}\lim_{n\to\infty} \int_{-\frac{1}{n}}^{\frac{1}{n}} \ln^2(1-t)\delta_n^{(s)}(t)\,dt = 0 \tag{19}$$

for $i+1 < s$.

Finally for the case $i+1 = s$, it is immediately seen from the equation (11) that

$$\int_{-\frac{1}{n}}^{\frac{1}{n}} \ln^2(1-t)\,\delta_n^{(s)}(t)\,dt = 2(-1)^s(s-1)!\phi(s-1). \tag{20}$$

Similarly by calculating the neutrix limit of the integral I we have

$$\mathrm{N-}\lim_{n\to\infty} \int_{-\frac{1}{n}}^{0} \ln^2 t\,\delta_n^{(s)}(t)\,dt = 0. \tag{21}$$

If $\varphi(x)$ is an arbitrary function in $\mathcal{D}(a,b)$ with $a < 0 < b$ then we have

$$\begin{aligned} \mathrm{N-}\lim_{n\to\infty}\langle(\ln^2[H(x)]_+)^{(s)} * \delta_n(x), \varphi(x)\rangle &= \mathrm{N-}\lim_{n\to\infty} \int_a^0 \varphi(x) \int_{-\frac{1}{n}}^{0} \ln^2 t\,\delta_n^{(s)}(t)\,dt \\ &\quad + \mathrm{N-}\lim_{n\to\infty} \int_0^b \varphi(x) \int_{-\frac{1}{n}}^{\frac{1}{n}} \ln^2(1-t)\,\delta_n^{(s)}(t)\,dt. \end{aligned}$$

Since we have from equation (9) that

$$[H(x)]_+^{-s} \ln[H(x)]_+ = \phi(s-1)[H(x)]_+^{-s} + \frac{(-1)^{s-1}(\ln^2[H(x)]_+)^{(s)}}{2(s-1)!},$$

then by using equations (3) and (18) through (21) equation (17) follows, completing the proof of the theorem. □

Let $F_s(x)$ denote the distribution $x_-^{-s}\ln x_-$. In the previous theorem, we evaluated the composition $F_s(x_+^r)$ when $r=0$. Now we will evaluate this composition for $r=1,2\ldots$..

Theorem 3.2. *The distribution $F_s(x_+^r)$ exists and*

$$F_s(x_+^r) = \frac{(-1)^{s+rs}}{2\,r\,(rs-1)!}[2\,c(\rho)\,\phi(s-1)+\phi_1(s) \\ +\phi_2(s)+c_1(\rho)-\phi^2(s)]\,\delta^{(rs-1)}(x) \quad (22)$$

for $r,s=1,2,\ldots$.

Proof. Let $\varphi(x)$ be an arbitrary function in $\mathcal{D}(a,b)$. Then after substituting $F_s(x_+^r)=(x_+^r)_-^{-s}\ln(x_+^r)_-$ we must evaluate $\text{N}-\lim_{n\to\infty}\langle[F_s(x_+^r)]_n,\varphi(x)\rangle$.

Since we have

$$(x_+^r)_-^{-s}\ln(x_+^r)_- = \phi(s-1)\,(x_+^r)_-^{-s} - \frac{(\ln^2(x_+^r)_-)^{(s)}}{2(s-1)!} \quad (23)$$

from equation (8), we will evaluate $(\ln^2(x_+^r)_-)^{(s)}$. Then

$$(\ln^2(x_+^r)_-)*\delta_n^{(s)}(x) = \int_{-\frac{1}{n}}^{\frac{1}{n}} \ln^2(x_+^r-t)_-\,\delta_n^{(s)}(t)\,dt \\ = \begin{cases} \int_{x^r}^{\frac{1}{n}} \ln^2(t-x^r)\,\delta_n^{(s)}(t)\,dt, & 0\le x\le n^{-\frac{1}{r}}, \\ \int_0^{\frac{1}{n}} \ln^2 t\,\delta_n^{(s)}(t)\,dt, & x<0 \\ 0, & x>n^{-\frac{1}{r}} \end{cases}$$

for $r,s=1,2,\ \ldots$. By Taylor's Theorem we have

$$\varphi(x) = \sum_{i=0}^{rs-1}\frac{x^i}{i!}\varphi^{(i)}(0) + \frac{x^{rs}}{r!}\varphi^{(rs)}(\xi x),$$

where $0<\xi<1$ and for $a<0<b$, $\text{supp}\varphi(x)\subset(a,b)$.

Now we have to evaluate

$$\underset{n\to\infty}{\text{N}-\lim}\int_a^0\int_0^{\frac{1}{n}} \ln^2 t\,\delta_n^{(s)}(t)\varphi(x)\,dt\,dx \\ + \underset{n\to\infty}{\text{N}-\lim}\int_0^{n^{-\frac{1}{r}}}\int_{x^r}^{\frac{1}{n}} \ln^2(t-x^r)\,\delta_n^{(s)}(t)\varphi(x)\,dt\,dx \\ = \underset{n\to\infty}{\text{N}-\lim} I_1 + \underset{n\to\infty}{\text{N}-\lim} I_2.$$

For I_1, we can write

$$I_1=\int_a^0\int_0^{\frac{1}{n}}\ln^2 t\,\delta_n^{(s)}(t)\varphi(x)\,dt\,dx=\left(\int_0^{\frac{1}{n}}\ln^2 t\,\delta_n^{(s)}(t)\,dt\right)\left(\int_a^0\varphi(x)\,dx\right).\tag{24}$$

Then as in the proof of Theorem 3.2, we have

$$\mathrm{N}\!-\!\lim_{n\to\infty} I_1=0.\tag{25}$$

Now, for I_2 we have

$$\begin{aligned}
I_2 &= \sum_{i=0}^{rs-1}\frac{\varphi^{(i)}(0)}{i!}\int_0^{n^{-\frac{1}{r}}}x^i\int_{x^r}^{\frac{1}{n}}\ln^2(t-x^r)\,\delta_n^{(s)}(t)\,dt\,dx\\
&\qquad +\frac{1}{(rs)!}\int_0^{n^{-\frac{1}{r}}}\int_{x^r}^{\frac{1}{n}}\ln^2(t-x^r)\,\delta_n^{(s)}(t)\varphi^{(rs)}(\xi x)x^{rs}\,dt\,dx\\
&= \sum_{i=0}^{rs-1}\frac{\varphi^{(i)}(0)}{r\,i!}n^{s-(i+1)/r}\int_0^1\rho^{(s)}(v)v^{(i+1)/r}\\
&\times\int_0^1[\ln(v-uv)-\ln n]^2u^{(i+1)/r-1}\,du\,dv+\frac{n^{-1/r}}{(rs)!}\int_0^1\rho^{(s)}(v)v^{(i+1)/r}\\
&\qquad\times\int_0^1[\ln(v-uv)-\ln n]^2u^{(i+1)/r-1}\varphi^{(rs)}(\xi\,(uv/n)^{1/r})\,du\,dv
\end{aligned}$$

by using the substitutions $nt=v$ and $x^r=ut$. Then if $i=0,1,\ldots,rs-2$, we have

$$\mathrm{N}\!-\!\lim_{n\to\infty}\int_0^{n^{-\frac{1}{r}}}x^i\int_{x^r}^{\frac{1}{n}}\ln^2(t-x^r)\,\delta_n^{(s)}(t)\,dt\,dx=0\tag{26}$$

and

$$\int_0^{n^{-\frac{1}{r}}}x^{rs}\int_{x^r}^{\frac{1}{n}}|\ln^2(t-x^r)\,\delta_n^{(s)}(t)|\,dt\,dx=O(n^{-\frac{1}{r}})$$

so

$$\lim_{n\to\infty}\int_0^{n^{-\frac{1}{r}}}x^{rs}\int_{x^r}^{\frac{1}{n}}\ln^2(t-x^r)\,\delta_n^{(s)}(t)\,dt\,dx=0.\tag{27}$$

When $i = rs - 1$, we have

$$\int_0^{n^{-\frac{1}{r}}} x^{rs-1} \int_{x^r}^{\frac{1}{n}} \ln^2(t - x^r)\, \delta_n^{(s)}(t)\, dt\, dx$$
$$= \frac{1}{r} \int_0^1 \rho^{(s)}(v) v^s \int_0^1 [\ln(v - uv) - \ln n]^2 u^{s-1}\, du\, dv$$
$$= \frac{1}{r} \int_0^1 \rho^{(s)}(v) v^s \int_0^1 \ln^2(v - uv) u^{s-1}\, du\, dv$$
$$- \frac{2}{r} \ln n \int_0^1 \rho^{(s)}(v) v^s \int_0^1 \ln(v - uv) u^{s-1}\, du\, dv$$
$$+ \frac{\ln^2 n}{r} \int_0^1 \rho^{(s)}(v) v^s\, dv \int_0^1 u^{s-1}\, du = J_1 + J_2 + J_3.$$

Then

$$rJ_1 = \int_0^1 \rho^{(s)}(v) v^s \ln^2 v\, dv \int_0^1 u^{s-1}\, du$$
$$+ 2 \int_0^1 \rho^{(s)}(v) v^s \ln v\, dv \int_0^1 \ln(1 - u) u^{s-1}\, du$$
$$+ \int_0^1 \rho^{(s)}(v) v^s\, dv \int_0^1 \ln^2(1 - u) u^{s-1}\, du.$$

Consequently, by using Lemma 2.5 to Lemma 2.9 we get

$$rJ_1 = (-1)^s (s-1)!\, [\phi_1(s) + 2\phi(s) c(\rho) + c_1(\rho)]$$
$$+ (-1)^{s-1} (s-1)!\, \phi(s)[\phi(s) + 2c(\rho)] + (-1)^s (s-1)!\, \phi_2(s)$$
$$= (-1)^s (s-1)! [\varphi_1(s) + c_1(\rho) - \phi^2(s) + \phi_2(s)].$$

Further, it is immediately seen that

$$\mathrm{N}\!-\!\lim_{n\to\infty} J_2 = \mathrm{N}\!-\!\lim_{n\to\infty} J_3 = 0.$$

Then

$$\mathrm{N}\!-\!\lim_{n\to\infty} \langle \int_{-\frac{1}{n}}^{\frac{1}{n}} \ln^2(x_+^r - t)\, \delta_n^{(s)}(t)\, dt, \varphi(x) \rangle$$
$$= \mathrm{N}\!-\!\lim_{n\to\infty} \sum_{i=0}^{rs-1} \frac{\varphi^{(i)}(0)}{i!} \int_0^{n^{-\frac{1}{r}}} \int_{x^r}^{\frac{1}{n}} x^i \ln^2(t - x^r)\, \delta_n^{(s)}(t)\, dt\, dx$$
$$+ \mathrm{N}\!-\!\lim_{n\to\infty} \frac{\varphi^{(rs)(\xi x)}}{(rs)!} \int_0^{n^{-\frac{1}{r}}} \int_{x^r}^{\frac{1}{n}} x^{rs} \ln^2(t - x^r)\, \delta_n^{(s)}(t)\, dt\, dx$$

$$= \frac{(-1)^s (s-1)!}{r\,(rs-1)!}\,\varphi^{(rs-1)}(0)[\phi_1(s)+\phi_2(s)+c_1(\rho)-\phi^2(s)].$$

Then we have

$$\mathrm{N}\!-\!\lim_{n\to\infty} \frac{1}{(s-1)!}\langle \int_{-\frac{1}{n}}^{\frac{1}{n}} \ln^2(x_+^r - t)_-\,\delta_n^{(s)}(t)\,dt, \varphi(x)\rangle$$
$$= \frac{(-1)^s\,[\phi_1(s)+\phi_2(s)+c_1(\rho)-\phi^2(s)]}{r\,(rs-1)!}\,\varphi^{(rs-1)}(0).$$

Hence using equations (10) and (23) we get

$$F_s\,(x_+^r) = \frac{(-1)^{s+rs}[\phi_1(s)+\phi_2(s)+c_1(\rho)-\phi^2(s)]}{2\,r\,(rs-1)!}\,\delta^{(rs-1)(x)}$$
$$+\,\phi(s-1)\frac{(-1)^{rs+s}c(\rho)}{r(rs-1)!}\delta^{(rs-1)}(x)$$

and equation (22) follows. □

Replacing x by $-x$ in equation (22), we get the following corollary.

Corollary 3.3. *The distribution $F_s(x_-^r)$ exists and*

$$F_s(x_-^r) = \frac{(-1)^{s-1}}{2\,r\,(rs-1)!}[2\,c(\rho)\,\phi(s-1)+\phi_1(s)$$
$$+\,\phi_2(s)+c_1(\rho)-\phi^2(s)]\,\delta^{(rs-1)}(x) \quad (28)$$

for $r, s = 1, 2, \ldots$.

Theorem 3.4. *The composition of the distributions $x_-^{-s}\ln^m x_-$ and $H(x)$ exists and*

$$(H(x))_-^{-s}\ln^m(H(x))_- = 0 \quad (29)$$

for $s, m = 1, 2, \ldots$.

Proof. First let $F_{s,m}(x) = x_-^{-s}\ln^m x_-$ and put $G_{s,m}(x) = (\ln^{m+1} x_-)^{(s)}$. Note that $G_{s,m}(x)$ can be written in the form

$$G_{s,m}(x) = \sum_{i=0}^{m} c_{s,m,i} x_-^{-s}\ln^i x_-,$$

where $c_{s,m,i} = 0$ if $i \le m-s$.

We will now evaluate

$$\mathrm{N}\!-\!\lim_{n\to\infty}\langle G_{s,m,n}(H(x)), \varphi(x)\rangle$$

for arbitrary $\varphi \in \mathcal{D}(a,b)$. We have

$$G_{s,m,n}(x) = (\ln^{m+1} x_-)^{(s)} * \delta_n(x) = \int_{-\frac{1}{n}}^{\frac{1}{n}} \ln^{m+1}(x-t)_- \delta_n^{(s)}(t)\,dt.$$

Then

$$\begin{aligned} G_{s,m,n}(H(x)) &= \begin{cases} \int_{-\frac{1}{n}}^{\frac{1}{n}} \ln^{m+1}(1-t)_-\,\delta_n^{(s)}(t)\,dt, & x > 0, \\ \int_0^{\frac{1}{n}} \ln^{m+1} t\,\delta_n^{(s)}(t)\,dt, & x < 0 \end{cases} \\ &= \begin{cases} 0, & x > 0, \\ \int_0^{\frac{1}{n}} \ln^{m+1} t\,\delta_n^{(s)}(t)\,dt, & x < 0. \end{cases} \end{aligned}$$

Since

$$\begin{aligned} \int_0^{\frac{1}{n}} \ln^{m+1} t\,\delta_n^{(s)}(t)\,dt &= n^s \int_0^1 \ln^{m+1}\left(\frac{v}{n}\right)\rho(v)\,dv \\ &= n^s \int_0^1 [\ln v - \ln n]^{m+1}\rho(v)\,dv, \end{aligned}$$

we have

$$\begin{aligned} \operatorname*{N-lim}_{n\to\infty}\langle G_{s,m,n}(H(x)), \varphi(x)\rangle &= \operatorname*{N-lim}_{n\to\infty} \int_a^0 \varphi(x) \int_0^{\frac{1}{n}} \ln^{m+1} t\delta_n^{(s)}(t)\,dt \\ &= 0. \end{aligned}$$

This proves that $G_{s,m}(H(x))$ exists and

$$G_{s,m}(H(x)) = 0. \tag{30}$$

Now suppose that $F_{s,i}(H(x)) = 0$ for $i = 0, 1, 2, \ldots, m-1$ for some m and $s = 1, 2, \ldots$. This is true when $m = 1$ from equation (2).

Since we can write

$$G_{s,m}(H(x)) = \sum_{i=0}^{m-1} c_{s,m,i}F_{s,i}(H(x)) + c_{s,m,m}F_{s,m}(H(x))$$

and since $c_{s,m,m}$ is not zero, we get $F_{s,m}(H(x)) = 0$ from our assumption and by using equation (30). We have therefore proved equation (29). $\square$

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