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ON GENERALIZED LIPSCHITZIAN SEMITOPOLOGICAL SEMIGROUP OF SELF-MAPPINGS WITH APPLICATIONS

M. IMDAD, AHMED H. SOLIMAN AND MOHAMED A. BARAKAT

Abstract. In this paper, we use a generalized Lipschitzian type condition for a semigroup of self-mappings as employed in Imdad and Soliman (Fixed Point Theory Appl. Vol. (2010), Article ID 692401, 1-14) to prove a fixed point theorem for a generalized Lipschitzian left reversible semitopological semigroup of self-mappings defined on a p-uniformly convex Banach space, besides indicating some possible applications to our main result. Our results generalize and extend some results due to J. S. Jung and B. S. Thakur (Inter. Jour. Math. Math. Sci., 28 (1)(2001), 41-50).

1. Introduction

Many new fixed-point theorems involving mathematical structures can be utilized to describe many general economic equilibrium problems besides some real world problems. Also, there are many treatises dealing with fixed-point and equilibrium problems under the banner of "homology theory" so therein the arguments based on the convex as well as differentiable approaches are unified.

Let G be a semigroup equipped with a Hausdorff topology such that for each $a \in G$, the mappings $s \to s.a$ and $s \to a.s$ from G to G are continuous real valued functions on G, which is also sometimes referred to as a semitopological semigroup. Further, G is called left reversible if any two closed right ideals of S have a non-void intersection. In this case, $(G, \preceq)$ is a directed system wherein the binary relation "$\preceq$" on G is defined as $a \preceq b$ if and only if $\{a\} \cup \overline{aG} \supseteq \{b\} \cup \overline{bG}.$ The natural examples of left reversible semigroups include commutative as well as left amenable semigroups.

Let C be a non-empty subset of a Banach space E. *A mapping* $T : C \to C$ is said to be a uniformly Lipschitzian mapping if for each integer $n \geq 1$, there

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exists a constant $k_n > 0$ such that

$$||T^n x - T^n y|| \leq k_n ||x - y|| \text{ for all } x, y \in C.$$

A Lipschitzian mapping is said to be *k-uniformly Lipschitzian* if $k_n \equiv k$ for all $n \geq 1$. *These* kinds of mappings were studied by Goebel and Kirk [4] *as well as* Goebel et al. Lifshitz [10] showed that a uniformly k-Lipschitzian mapping T defined on the Hilbert space H with $k < \sqrt{2}$ has a fixed point. Downing and Ray [3] (also Ishihara and Takahashi [7]) showed that Lifshitz's theorem remains valid for a uniformly Lipschitzian semigroup of self-mappings defined on Hilbert spaces. Casini and Maluta[2] (also Ishihara and Takahashi [7]) proved that a uniformly k-Lipschitzian semigroup of self-mappings defined on a Banach space E has a common fixed point if $k < \sqrt{N(E)}$ wherein $N(E)$ stands for the uniform normal structure coefficient.

In 2001, J. S. Jung and B. S. Thakur [6] *considered* a relatively larger class of mappings as compared to the class of Lipschitzian mappings and also showed that a uniformly generalized Lipschitzian semigroup of self mappings defined on a p-uniformly convex Banach space (also on uniformly convex Banach space) has a common fixed point. Recently, Imdad and Soliman [9] employed yet another class of uniformly generalized Lipschitzian semigroup of self-mappings which is larger than the one studied by J. S. Jung and B. S. Thakur [6]. In this paper, we use the uniformly generalized Lipschitzian semitoplogical semigroup of self mappings as employed in Imdad and Soliman [9] to extend certain results contained in J. S. Jung and B. S. Thakur [6].

2. Preliminaries

In what follows, we recall some relevant definitions and results with respect to a uniformly generalized Lipschitzian semigroup of self-mappings defined on Banach spaces.

Let C be a closed convex subset of a Banach space E. Then the collection $\tau = \{T_s : s \in G\}$ of self-mappings defined on C is said to be a semigroup on C if the following conditions are satisfied:

(i) $T_{st}x = T_s T_t x$ for all $s, t \in G$ and $x \in C$;
(ii) The mapping $(s, x) \to T_s(x)$ from $G \times C$ into C is continuous when $G \times C$ has the product topology;
(iii) For each $t \in G$, $T_t : C \to C$ is continuous on C.

In [6], J. S. Jung and B. S. Thakur introduced and studied the following class of uniformly generalized Lipschitzian semigroups of self-mappings:

Definition 2.1. [6]. *A family of self mappings $\tau = \{T_s : s \in G\}$ defined on X is called a generalized Lipschitzian (for the sake of symmetry, we call it*

G1-Lipschitzian in the sequel) semigroup if it is a semigroup (i.e. (i), (ii) *and* (iii) *hold), and also satisfies the condition*

$$||T_tx - T_ty|| \leq a_t||x - y|| + b_t(||x - T_tx|| + ||y - T_ty||) + c_t(||x - T_ty|| + ||y - T_tx||)$$

for each $x, y \in X$, *where* a_t, b_t *and* c_t *are nonnegative constants such that* $b_t + c_t < 1$, $\sup\{a_t : t \in G\} = a < \infty$, $\sup\{b_t : t \in G\} = b < \infty$ *and* $\sup\{c_t : t \in G\} = c < \infty$ *with* $b + c < 1$.

The simplest example of a G1-Lipschitzian semigroup is the semigroup of iterates of a mapping $T : X \to X$ whenever $\sup\{n : n \in N\} = a < \infty$, $\sup\{n : n \in N\} = b < \infty$, and $\sup\{n : n \in N\} = c < \infty$ with $b + c < 1$.

In [17], Ahmed H. Soliman considered yet another class of mappings which is termed as a G2-Lipschitzian semigroup of self-mappings.

Definition 2.2. *A family of self mappings* $\tau = \{T_s : s \in G\}$ *defined on* X *is called a G2-Lipschitzian semigroup if it is a semi group (i.e.*(i), (ii) *and* (iii) *hold) and*

$$\sup\{k_t : t \in G\} = k < \infty,$$

where

$$||T_tx - T_ty|| \leq k_t \max\left\{||x - y||, \frac{1}{2}||x - T_tx||, \frac{1}{2}||y - T_ty||\right\}$$

for each $x, y \in X$ *and* $\max\left\{||x - y||, \frac{1}{2}||x - T_tx||, \frac{1}{2}||y - T_ty||\right\} \neq 0$.

Most recently, Imdad and Soliman [9] further enlarged the preceeding classes of semigroups of self-mappings by introducing the following definition.

Definition 2.3. *A family of self mappings* $\tau = \{T_s : s \in G\}$ *defined on* X *is called a G3-Lipschitzian semigroup if it is a semi group (i.e.*(i), (ii) *and* (iii) *hold) and*

$$\sup\{k_t : t \in G\} = k < \infty,$$

where

$$||T_tx - T_ty|| \leq \underline{k}_t M(x, y)$$

for each $x, y \in X$ *and* $M(x, y) = \max\left\{||x-y||, \frac{1}{2}||x-T_tx||, \frac{1}{2}||y-T_ty||, \frac{1}{2}||x-T_ty||, \frac{1}{2}||y - T_tx||\right\} \neq 0$.

Remark 2.4. The class of G3-Lipschitzian semigroups of self-mappings is relatively larger than the other classes which include *k-Lipschitzian* semigroups, G1-Lipschitzian semigroups and G2-Lipschitzian semigroups (of self-mappings).

Let $\{B_\alpha : \alpha \in \wedge\}$ be a decreasing net of bounded subsets of a Banach space E. For a non-empty subset C of E, define

$$r(\{B_\alpha\}, x) = \inf_\alpha \sup\{||x - y|| : y \in B_\alpha\};$$

$$r(\{B_\alpha\}, C) = \inf\{r(\{B_\alpha\}, x) : x \in C\};$$

$$A(\{B_\alpha\}, C) = \{x \in C : r(\{B_\alpha\}, x) = r(\{B_\alpha\}, C)\}.$$

It is well known that $r(\{B_\alpha\}, .)$ is a continuous convex function on E which satisfies the following:

$$|r(\{B_\alpha\}, x) - r(\{B_\alpha\}, y)| \leq ||x - y|| \leq r(\{B_\alpha\}, x) + r(\{B_\alpha\}, y)$$

for each $x, y \in E$. Further, if E is reflexive and C is closed convex set, then $A(\{B_\alpha\}, C)$ is nonempty whereas the same (i.e.$A(\{B_\alpha\}, C)$) is singleton provided E is uniformly convex (cf.[11]).

Let λ be a real number in $[0, 1]$ with $p > 1$, and $W_p(\lambda)$ be a function described by $\lambda.(1-\lambda)^p+\lambda^p.(1-\lambda)$. The functional $||.||^p$ is said to be uniformly convex (cf. Zălinescu [19]) on a Banach space E if there exists a positive constant c_p such that for all $\lambda \in [0, 1]$ and $x, y \in E$ the following inequality holds:

$$||\lambda x + (1 - \lambda)y||^p \leq \lambda||x||^p + (1 - \lambda)||y||^p - W_p(\lambda).c_p.||x - y||^p \tag{1}$$

Xu [18] proved that the functional $||.||^p$ is uniformly convex on the (whole) Banach space E if and only if E is p-uniformly convex.

The normal structure coefficient $N(E)$ (cf. [1]) is defined by

$$N(E) = \inf\left\{\frac{\text{diam}(C)}{r_C(C)}\right\}$$

where C is a bounded convex subset of E consisting of more than one point and $diam(C) = \sup\{||x - y|| : x, y \in C\}$ is the diameter of C and $r_C(C) = \inf_{x\in C}\{\sup_{y\in C} ||x - y||\}$ is the Chebyshev radius of C relative to itself. The space E is said to have uniform normal structure if $N(E) > 1$. It is known that a uniformly convex Banach space has uniform normal structure and with respect to a Hilbert space H, $N(H) = \sqrt{2}$. Recently, Pichugov [14] (also Prus [15]) calculated that

$$N(L^p) = \min\{2^{1/p}, 2^{(p-1)/p}\}, \;\; 1 < p < \infty.$$

3. Results

Our main result runs as follows.

Theorem 3.1. *Let K be a nonempty subset of a p-uniformly convex Banach space E (with $p > 1$). Also, let G be a left reversible semitopological* *semigroup whereas* $\Im = \{T_t : t \in G\}$ *be a G3-Lipschitzian semigroup on* K *such that*

$$\Big[\frac{3^p\alpha^p(\alpha^p - 2^{1-p})}{(2c_p - \alpha^p)N^p}\Big]^{\frac{1}{P}} < 1$$

where

$$\alpha = \limsup_s \frac{2k(s)}{2-k(s)}.$$

If $\{T_t y : t \in G\}$ *is bounded for some* $y \in K$ *and also if there is a closed subset* C *of* K *such that* $\cap_s \overline{co}\{T_t x : t \succeq s\} \subseteq C$ *for all* $x \in K$*, then there exists* $z \in C$ *such that* $T_s z = z$ *for all* $s \in G$*.*

Proof. Let $B_s(x) = \overline{co}\{T_t x : t \succeq s\}$, $B(x) = \cap_s B_s(x)$ (wherein $s \in G$) and $x \in K$. Define $\{x_n : n \geq 0\}$ (by induction) as follows:

$$x_0 = y, \ \ x_n = A(\{B_s(x_{n-1})\}, B(x_{n-1})), \quad \text{for } n \geq 1.$$

Since $B_s(x) \subseteq C \subseteq K$ for all $x \in K$, $\{x_n\}$ is well defined. Let

$$\begin{aligned} r_m &= r(\{B_s(x_m)\}, B(x_{m+1})), \\ D_m &= r(\{B_s(x_m)\}, B(x_m)), \quad m \geq 1. \end{aligned} \tag{2}$$

Now for each $s, t \in G$ and $x, y \in K$, we have

$$\begin{aligned} ||T_sT_tx - T_sy|| &\leq k(s).\max\Big\{||T_tx-y||, \frac{1}{2}||T_tx - T_sT_tx||, \frac{1}{2}||y-T_sy||, \\ &\qquad \frac{1}{2}||T_tx - T_sy||, \frac{1}{2}||y - T_sT_tx||\Big\} \\ &\leq k(s).\max\Big\{||T_tx-y||, \frac{1}{2}(||T_tx-y|| + ||y-T_sy|| + ||T_sT_tx - T_sy||), \\ \frac{1}{2}||y-T_sy||, &\frac{1}{2}(||T_tx-y|| + ||y-T_sy||), \frac{1}{2}(||y-T_sy|| + ||T_sT_tx - T_sy||)\Big\}, \end{aligned}$$

which in turn yields

$$||T_sT_tx - T_sy|| \leq k(s)\Big\{||T_tx-y|| + \frac{1}{2}||y-T_sy|| + \frac{1}{2}||T_sT_tx - T_sy|\Big\},$$

so that

$$||T_sT_tx - T_sy|| \leq \frac{2k(s)}{2-k(s)}\Big\{||T_tx-y|| + \frac{1}{2}||y-T_sy||\Big\}. \tag{3}$$

Then owing to $x_m \in B(x_{m-1}) = \cap_s B_s(x_{m-1})$ and a result of Ishihara and Takahashi [7], we have

$$\begin{aligned} r_m = r(\{B_s(x_m)\}, B_s(x_m)) &\le \frac{1}{N}.\inf_s \operatorname{diam}(B_s(x_m)) \\ &\le \frac{1}{N}\inf_s \sup\{||T_a x_m - T_b x_m|| : a, b \succeq s\} \\ &\le \frac{1}{N}\limsup_t(\limsup_s ||T_s x_m - T_t x_m||) \\ &\le \frac{1}{N}\limsup_t(\limsup_s ||T_t T_s x_m - T_t x_m||). \end{aligned} \tag{4}$$

Making use of (3) in (4), we have

$$\begin{aligned} r_m &\le \frac{1}{N}\limsup_t \Big(\limsup_s \frac{2k(s)}{2-k(s)}\Big\{||T_s x_m - x_m|| + \frac{1}{2}||x_m - T_t x_m||\Big\}\Big) \\ &\le \frac{3\alpha}{2N} D_m \end{aligned} \tag{5}$$

where $\alpha = \limsup_s \frac{2k(s)}{2-k(s)}$ and N is the normal structure coefficient of E. On the other hand, in view of (1) and (3), we have

$$\begin{aligned} &||\lambda x_{m+1} + (1-\lambda)T_t x_{m+1} - T_s x_m||^p + W_p(\lambda).c_p.||x_{m+1} - T_t x_{m+1}||^p \\ &\le \lambda||x_{m+1} - T_s x_m||^p + (1-\lambda)||T_t x_{m+1} - T_s x_m||^p \\ &\le \lambda||x_{m+1} - T_s x_m||^p + (1-\lambda)||T_t x_{m+1} - T_t T_s x_m||^p. \end{aligned} \tag{6}$$

Due to (3) and (6), we can write

$$\begin{aligned} ||\lambda x_{m+1} + (1-\lambda)T_t x_{m+1} - T_s x_m||^p &+ W_p(\lambda).c_p.||x_{m+1} - T_t x_{m+1}||^p \\ \le \lambda||x_{m+1} - T_s x_m||^p &+ (1-\lambda)\Big[\frac{2k(t)}{2-k(t)}||x_{m+1} - T_s x_m|| \\ &+ \frac{k(t)}{2-k(t)}||x_{m+1} - T_t x_{m+1}||\Big]^p. \end{aligned}$$

By taking the $\limsup_s$ and $\limsup_t$, we obtain

$$\begin{aligned} r_m^p + c_p.W_p(\lambda)D_{m+1}^p &\le \lambda r_m^p + (1-\lambda)\Big[\alpha r_m + \frac{\alpha}{2}D_{m+1}\Big]^p \\ &\le \lambda r_m^p + (1-\lambda)2^{p-1}\Big[\alpha^p r_m^p + (\frac{\alpha}{2})^p D_{m+1}^p\Big], \end{aligned} \tag{7}$$

which due to (4) gives rise to,

$$\begin{aligned}D_{m+1}^p &\leq \Big[\frac{(1-\lambda)2^p\alpha^p - 2(1-\lambda)}{2c_p.W_p(\lambda)-(1-\lambda)\alpha^p}\Big] r_m^p \\ &\leq \Big[\frac{(1-\lambda)(2^p\alpha^p - 2)}{2c_p.W_p(\lambda)-(1-\lambda)\alpha^p}\Big]\Big(\frac{3\alpha}{2N}\Big)^p D_m^p.\end{aligned}$$

Letting $\lambda \to 1$, we conclude that

$$D_{m+1} \leq \Big[\frac{3^p\alpha^p(\alpha^p - 2^{1-p})}{(2c_p-\alpha^p)N^p}\Big]^{\frac{1}{p}} D_m = AD_m, \ \ m \geq 1,$$

where

$$A = \Big[\frac{3^p\alpha^p(\alpha^p - 2^{1-p})}{(2c_p-\alpha^p)N^p}\Big]^{\frac{1}{p}} < 1.$$

Since

$$\begin{aligned}||x_{m+1}-x_m|| &\leq r(\{B_s(x_m)\}, x_{m+1}) + r(\{B_s(x_m)\}, x_m) \\ &\leq r_m + D_m \leq 2D_m \\ &\leq 2.A^{m-1}D_1 \to 0 \ \text{ as } \ m \to \infty,\end{aligned}$$

$\{x_m\}$ is a Cauchy sequence. Let $z = \lim_{m\to\infty} x_m$. Then we have

$$\begin{aligned}&||z - T_sz|| \leq ||z-x_m|| + ||x_m - T_sx_m|| + ||T_sx_m - T_sz|| \\ &\leq ||z-x_m|| + ||x_m - T_sx_m|| + \frac{2k(s)}{2-k(s)}\Big\{||x_m - z|| + \frac{1}{2}||x_m - T_sx_m||\Big\} \\ &= \Big(1+\frac{2k(s)}{2-k(s)}\Big)||z-x_m|| + \Big(1+\frac{k(s)}{2-k(s)}\Big)||x_m - T_sx_m|| \\ &\leq \frac{2+k(s)}{2-k(s)}||z-x_m|| + \frac{2}{2-k(s)}D_m.\end{aligned}$$

By taking the limit as $m \to \infty$ we have,

$$||z - T_sz|| \leq \lim_{m\to\infty}\Big(\frac{2+k(s)}{2-k(s)}||z-x_m|| + \frac{2}{2-k(s)}D_m\Big) = 0$$

for all $s \in G$. Hence we have $T_sz = z$. This completes the proof. □

Since the simplest G3-Lipschitzian semigroup is a semigroup of iterates of a mapping $T : X \to X$ with $\sup\{k_n : n \in N\} = k < \infty$, we have the following corollary:

Corollary 3.2. *Let K be a nonempty subset of a p-uniformly convex Banach space E (with $p > 1$) and let T be a self-mapping of K such that T is a G3-Lipschitzian mapping on K with*

$$\Big[\frac{3^p\alpha^p(\alpha^p - 2^{1-p})}{(2c_p-\alpha^p)N^p}\Big]^{\frac{1}{P}} < 1$$

where

$$\alpha = \limsup_{n} \frac{2k_n}{2-k_n}.$$

Then there exists $z \in K$ such that $T_s z = z$.

In what follows, we indicate certain possible applications of Theorem 3.1. In this respect, the following properties are needed.

(I) In a Hilbert space H, the following equality holds:

$$||\lambda x + (1-\lambda)y||^2 = \lambda||x||^2 + (1-\lambda)||y||^2 - \lambda(1-\lambda)||x-y||^2, \quad (8)$$

for all $x, y \in H$ and $\lambda \in [0,1]$.

(II) If $1 < p \leq 2$, then we have for all $x, y \in L^p$ (see [13, 16]) and $\lambda \in [0,1]$,

$$||\lambda x - (1-\lambda)y||^2 \leq \lambda||x||^2 + (1-\lambda)||y||^2 - \lambda(1-\lambda)(p-1)||x-y||^2. \quad (9)$$

(III) Assume that $2 < p < \infty$ and t_p is the unique zero of the function $g(x) = -x^{p-1} + \underline{(p-1)x + p - 2}$ in the interval $(1,\infty)$. Let $c_p = (p-1)(1+t_p)^{2-p} = \frac{1+t_p^{p-1}}{(1+t_p)^{p-1}}$. Then we have the following inequality:

$$||\lambda x + (1-\lambda)y||^p \leq \lambda||x||^p + (1-\lambda)||y||^p - W_p(\lambda).c_p.||x-y||^p, \quad (10)$$

for all $x, y \in L^p$ and $\lambda \in [0,1]$ (see [12]).

In view of Theorem 3.1 and equation (8), we have the following:

Theorem 3.3. *Let K be a nonempty subset of a Hilbert space E (with $p > 1$). Also, let G be a left reversible semitopological semigroup whereas $\Im = \{T_t : t \in G\}$ be a G3-Lipschitzian semigroup of self-mappings defined on K with*

$$\Big[\frac{3^2\alpha^2(\alpha^2 - \frac{1}{2})}{2(2-\alpha^2)}\Big]^{\frac{1}{2}} < 1$$

where

$$\alpha = \limsup_{s} \frac{2k(s)}{2-k(s)}.$$

Suppose that $\{T_t y : t \in G\}$ is bounded for some $y \in K$ and also there is a closed subset C of K such that $\cap_s \overline{co}\{T_t x : t \succeq s\} \subseteq C$ for all $x \in K$. Then there exists $z \in C$ such that $T_s z = z$ for all $s \in G$.

In view of Theorem 3.1 and equations (9) and (10), we also have the following:

Theorem 3.4. *Let K be a nonempty subset of L^p, $1 < p < \infty$ whereas G be a left reversible semitopological semigroup on K. Also, let $\Im = \{T_t : t \in G\}$ be a G3-Lipschitzian semigroup of self-mappings defined on K with*

$$\Big[\frac{3^2\alpha^2(\alpha^2 - \frac{1}{2})}{2^{(p-1)/p}(2(p-1)-\alpha^2)}\Big]^{\frac{1}{2}} < 1 \quad \textit{for } 1 < p \leq 2,$$

$$\Big[\frac{3^p\alpha^p(\alpha^p - 2^{1-p})}{2(2c_p - \alpha^p)}\Big]^{\frac{1}{p}} < 1 \qquad \text{for } 2 < p < \infty,$$

where

$$\alpha = \limsup_s \frac{2k(s)}{2 - k(s)}.$$

Suppose that $\{T_t y : t \in G\}$ *is bounded for some* $y \in K$ *and also there is a closed subset* C *of* K *such that* $\cap_s \overline{co}\{T_t x : t \succeq s\} \subseteq C$ *for all* $x \in K$. *Then there exists* $z \in C$ *such that* $T_s z = z$ *for all* $s \in G$.

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M. Imdad
Department of Mathematics
Aligarh Muslim University
Aligarh, India, 202002
E–mail: mhimdad@yahoo.co.in

Mohamed A. Barakat
Department of Mathematics
Faculty of Science
Al-Azhar University
Assiut 71524, Egypt
E–mail: Barakat14285@yahoo.com

Ahmed H. Soliman
Department of Mathematics
Faculty of Science
King Khalid University
P.O.Box. 9004, Abha, Saudi Arabia
E–mail: ahsolimanm@gmail.com