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COMPLEXIFIED LIE ALGEBROIDS FROM A GENERALIZED STIELTJES ACTION APPROACH TO THE CALCULUS OF VARIATIONS

AHMAD RAMI EL-NABULSI, MUBASHER JAMIL AND GUO-CHENG WU

Abstract. In this work, we communicate the issue of Lie algebroids. More precisely, we discuss the subject based on the generalized Stieltjes fractal-like approach of the calculus of variations. We derived the corresponding Euler-Lagrange, geodesics and Wongs equations and we illustrate then, by discussing, the resulted dynamics of a colored particle in Yang-Mills. Many motivating consequences are explored in particular, the emergence of complexified Lie algebroids with its corresponding complexified Lagrangian and Hamiltonian dynamics from the fractal approach.

1. Introduction

Currently, it is recognized that for nonconservative dynamical systems, traditional Lagrangian and Hamiltonian mechanics based on the calculus of variations do not hold and do not succeed to describe correctly the decaying behavior. In recent years numerous works have been dedicated to this problem. One potential approach is based on the fractional calculus of variations (FCOV) [1,2]. This approach deals in fact with Riemann-Liouville fractional derivatives, in some cases with Caputo or Riesz derivatives [3,4,5,6,7,8] and more recently with the Riemann-Liouville fractional integral, entitled "Fractional ActionLike Variational Approach (FALVA)" [9,10]. Depending on the type of fractional functional being considered, different fractional Euler-Lagrange type equations are obtained. The FCOV was proved recently to be a practical device for portrayal of weak dissipative and nonconservative dynamical systems. In fact, fractional calculus is an interdisciplinary area of applied mathematics, with many applications in several fields of science and engineering, ranging from hydrology to viscoelastivity, heat conduction, polymer physics, chaos and fractals, control theory, plasma physics, wave

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propagation in complex and porous media, astrophysics, cosmology, quantum field theory, potential theory, etc. [11-17].

Despite the success of the new formalism, the formulation of the fractional problems of the calculus of variations still needs more amplification as the problem is deeply related to the fractional quantization procedure and to the presence of non-local fractional differential operators. More recently, we proposed a new kind of functional based on a generalized fractal version of the FALVA, namely the generalized singletime Stieltjes fractal-fractional action [18,19]. The new class of generalized fractional Hamiltonian systems obtained has two parameters and is wider than the standard class of Hamiltonian dynamical systems. Our main aim in this work is to address the new fractal version for Lie algebroids which was proved to be a powerful tool in the investigation of many fundamental and elementary properties in applied mathematics, dynamical systems and differential geometry.

The paper is organized as follows: In Section 2, we review the basic concepts of the generalized fractal actionlike variational approach. In Section 3, we introduce the Lagrangian concepts on Lie algebroids. The resulting Euler-Lagrange, geodesics and Wong's equations are discussed in the same section. In Section 4 we initiate the Hamiltonian formalism on Lie algebroids. Conclusions and perspectives are discussed in Section 5.

2. Generalized fractal action-like variational approach

In this section, we review the basic Lagrangian and Hamiltonian concepts of the generalized fractal version of the fractional actionlike integral approach, namely the generalized single-time Stieltjes fractal-fractional action [18,19]. Usually, the standard single-time Stieltjes action is defined for two given functions $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\times\mathbb{R}^+\to\mathbb{R}$ by $S_\tau f=\int_0^\tau f(t)dg_\tau(t)$ with $g(t,\tau)=g_\tau(t)$, $\tau>0$. The following three cases are interesting:

$g:\mathbb{R}\times\mathbb{R}^+\to\mathbb{R}, g_\tau(t)$ $=[\tau^\alpha-(\tau-t)^\alpha]/\Gamma(1+\alpha),\ \alpha\in(0,1]$	$I_\tau[f]=\frac{1}{\Gamma(\alpha)}\int_0^\tau f(t)(\tau-t)^{\alpha-1}dt$ fractional Riemann-Liouville integral
$g:\mathbb{R}\times\mathbb{R}^+\to\mathbb{R}, g_\tau(t)=-e^{-t\tau}/\tau$	$I_\tau[f]=\int_0^\tau f(t)e^{-t\tau}dt$ is used in economics when we speak about discounted $f(t)$ at rate τ
$g:\mathbb{R}\times\mathbb{R}^+\to\mathbb{R}, g_\tau(t)=t^\tau/\tau$	$I_\tau[f]=\int_0^\tau f(t)t^{\tau-1}dt$ is a fractional integral used in fractal action.

In [35], we introduced a more generalized 4th case in the following form:

$$g_\tau(t)=\frac{\tau^{d/2\alpha}e^{-b^2/\tau}}{2\alpha},\quad b,d\in\mathbb{R},\alpha\in(0,1],$$

from which resulted the following generalized fractal action:

$$I[f:\mathbb{R}\to\mathbb{R}]=\frac{1}{\Gamma(d/2\alpha)\Gamma(d/2)}\int_0^\tau f(t)^{d-1}\exp(-\tau(t^{2\alpha}+b^2))dt. \quad (1)$$

Here τ can be a real or a complex parameter. The following problem was proposed:

Problem 2.1. *For a smooth Lagrangian function $L(\cdot,\cdot,\cdot)\in C^1([0,\tau]\times\mathbb{R}^2;\mathbb{R})$, find the stationary points of the functional*

$$I[L:L(\cdot,\cdot,\cdot)\in C^1([0,\tau]\times\mathbb{R}^2;\mathbb{R})]$$
$$=\frac{1}{\Gamma(d/2\alpha)\Gamma(d/2)}\int_0^\tau L(\dot{q}(t),q(t),t)t^{d-1}\exp(-\tau(t^{2\alpha}+b^2))dt\to\min, \quad (2)$$

where $\dot{q}=dq/dt$ and Γ is the Euler gamma function.

Theorem 2.1. (The generalized Euler-Lagrange equation) *If $q(\cdot)$ is a local minimizer of Problem 2.1, then $q(\cdot)$ satisfies the following Euler-Lagrange equation:*

$$\frac{\partial L}{\partial q}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)=\left[\frac{d-1}{t}-2\alpha\tau t^{2\alpha-1}\right]\frac{\partial L}{\partial \dot{q}}. \quad (3)$$

Proof. In fact, we may write $q(t)=q^0(t)+\sigma(t)$ where $q^0(t)$ is the minimum solution and $\sigma(\tau)$ describes the deviation of $q_k(t)$ for the minimum path $q^0(t)$. By substitution into equation (2) we have:

$$I[L]=\frac{1}{\Gamma(d/2\alpha)\Gamma(d/2)}\int_0^\tau L(\dot{q}^0(t)+\dot{\sigma}(t),q(t)+\sigma(t),t)t^{d-1}\exp(-\tau(t^{2\alpha}+b^2))dt$$

and performing Taylor expansions of the first order in $\dot{\sigma}(t)$ and $\sigma(t)$, yields

$$I[L]=\frac{1}{\Gamma(d/2\alpha)\Gamma(d/2)}$$
$$\times\int_0^\tau\left\{L(\dot{q}^0(t),q(t),t)+\frac{\partial L}{\partial \dot{q}}\dot{\sigma}(t)+\frac{\partial L}{\partial q}\sigma(t)\right\}t^{d-1}\exp(-\tau(t^{2\alpha}+b^2))dt.$$

Integrating the term in $\sigma(t)$ by parts gives without difficulty:

$$I[L]=\frac{1}{\Gamma(d/2\alpha)\Gamma(d/2)}\int_0^\tau\left\{L(\dot{q}^0(t),q(t),t)t^{d-1}\exp(-\tau(t^{2\alpha}+b^2))\right\}dt$$
$$-\frac{1}{\Gamma(d/2\alpha)\Gamma(d/2)}\int_0^\tau\sigma(t)\Bigg[t^{d-1}\exp(-\tau(t^{2\alpha}+b^2))\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{q}}\right)$$
$$+(d-1)t^{d-2}\exp(-\tau(t^{2\alpha}+b^2))\frac{\partial L}{\partial \dot{q}}-2\alpha\tau t^{2\alpha-1}\exp(-\tau(t^{2\alpha}+b^2))\frac{\partial L}{\partial \dot{q}}$$

$$- \frac{\partial L}{\partial q} t^{d-1} \exp(-\tau(t^{2\alpha} + b^2))\Big] dt,$$

and we get the required result. □

Remark 2.1. a) For $\alpha = 1/2$ and $b = 0$ the integral fractal functional

$$I[L] = \frac{1}{\Gamma(d/2\alpha)\Gamma(d/2)} \int_0^\tau L(\dot{q}^0(t), q(t), t) t^{d-1} \exp(-\tau t) dt \tag{4}$$

and accordingly, the associated Euler-Lagrange equation is

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}}\Big) = \Big[\frac{d-1}{t} - \tau\Big] \frac{\partial L}{\partial \dot{q}} \,. \tag{5}$$

b) For $\alpha = 1$ and $b = 0$, the integral fractal functional is

$$I[L] = \frac{1}{\Gamma(d/2\alpha)\Gamma(d/2)} \int_0^\tau L(\dot{q}^0(t), q(t), t) t^{d-1} \exp(-\tau t^2) dt, \tag{6}$$

and therefore, the associated Euler-Lagrange equation is

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{q}}\Big) = \Big[\frac{d-1}{t} - 2t\tau\Big] \frac{\partial L}{\partial \dot{q}} \,. \tag{7}$$

These special cases may have interesting consequences when applied to dynamical systems, in particular the Bateman-Feshbach-Tikochinsky harmonic oscillator and the inverted Caldirola-Kanai oscillator which has been explored largely in literature to comprehend dissipation in quantum dynamical systems [20].

The corresponding Hamiltonian formalism was introduced mainly by letting

$$F(q, p, \dot{q}, \dot{p}, t) = \sum_{k=1}^{N} p_k \dot{q}_k - H(q, p, t), \tag{8}$$

where q, p, $\dot{q}$, $\dot{p}$ are four sets of independent variables. By usually requiring that

$$\delta\Big\{ \frac{1}{\Gamma(d/2\alpha)\Gamma(d/2)} \int_0^\tau L(q, p, \dot{q}, \dot{p}, t) t^{d-1} \exp(-\tau(t^{2\alpha} + b^2)) dt \Big\} = 0, \tag{9}$$

and varying the variables q_k and p_k independently, we get the generalized canonical equations $\delta F/\delta q_k = 0$ and $\delta F/\delta p_k = 0$ or more explicitly:

$$\dot{p}_k = -\frac{\partial H}{\partial q_k} - \Big[\frac{d-1}{t} - 2\alpha\tau t^{2\alpha-1}\Big] p_k, \tag{10}$$

$$\dot{q}_k = \frac{\partial H}{\partial p_k}. \tag{11}$$

The second term on the RHS of equation (11) may be identified to external forces violating the Lie algebra conditions and verify instead the Lie-admissible algebra. The following table summarizes some particular dynamics obtained for different values of the parameter α and in particular for $k = 1$:

$\alpha = 1/2$	$\dot{p} = -\frac{\partial H}{\partial q} - \left(\frac{d-1}{t} - \tau\right)p$
$\alpha = 1$	$\dot{p} = -\frac{\partial H}{\partial q} - \left(\frac{d-1}{t} - 2t\tau\right)p$
$\alpha = -1/2$	$\dot{p} = -\frac{\partial H}{\partial q} - \left(\frac{d+\tau-1}{t}\right)p$
$\alpha = -1$	$\dot{p} = -\frac{\partial H}{\partial q} - \left(\frac{d-1}{t} - \frac{2\tau}{t^3}\right)p$

Remarkably for $\alpha = 1/2$, $\dot{p}$ is dominated by the second "constant" term inside the bracket at very large times, while for $\alpha = 1$, $\dot{p}$ is dominated by the second constant "increasing" term inside the bracket at very large times. Accordingly, the new formalism introduced here may be used to model dynamical systems not only exhibiting dissipative behavior but also revealing opposite behavior, e.g. increasing mass and energy.

3. Generalized lagrangian on Lie algebroids

To define a variational problem on Lie algebroids, we have to specify an infinite-dimensional manifold M of paths whose tangent space (TM, π, M) represents all possible variations and an action functional on M. Then we have to choose a submanifold of admissible paths and a set of admissible variations of admissible path. In fact, given a vector bundle $\pi : E \to M$ and a smooth Lagrangian function $L : E \to R$. A Lie algebroid given by a vector bundle morphism $\rho : E \to TM$ over the identity in M, is called the anchor map, jointly with the Lie algebra structure on the space $C^\infty(M)-$ module of sections of E determined by the Lie bracket. In its turn the Lie bracket induces a Lie algebra homomorphism $\bar{\rho}$ of $C^\infty(M)$ from $Sec(E) \to \chi(M)$ by the anchor $\rho : E \to TM$ given by:

$$\sigma \in Sec(E) \to \bar{\rho}(\sigma)(q) \in \chi(M), \quad \bar{\rho}(\sigma)(q) = \rho(s(q)), \ \forall q \in M$$

satisfying the Leibniz compatibility identity [21,22,23,24]:

$$[\sigma, f\eta]_E = (\rho(\sigma)f)\eta + f[\sigma,\eta]_E, \ \forall f \in C^\infty(M), \ \sigma,\eta \in Sec(E). \tag{12}$$

We also have the relation $[\rho(\sigma), \rho(\eta)]_E = \rho([\sigma,\eta]_E)$ and $[\sigma,[\eta,\xi]]_E + [\eta,[\xi,\sigma]] + [\xi,[\sigma,\eta]] = 0, \ \xi,\sigma,\eta \in Sec(E)$.

Definition 3.1. *A vector bundle (E, ξ, M) endowed with a Lie algebroid structure $([\cdot,\cdot]_E, \rho)$ is called a Lie algebroid on M and is denoted by the triple $(E, [\cdot,\cdot]_E, \rho)$.*

Given local coordinates (q^i) in the base manifold M and a local basis of the section $\{e_\gamma\}$ of the bundle E, the local coordinates of a point $a \in E$ are (q^i, y^n) where $a = y^\eta e_\eta(\pi(a))$ in the given base. The anchor and the bracket are locally determined by the structure functions

$$\rho_i^\gamma,\ C_{ij}^k \in C^\infty(M) \text{ of } (E, [\cdot,\cdot]_E, \rho),$$

which contain the local information of the Lie algebroid structure with

$$\bar{\rho}(e_i) = \rho_i^\eta \partial_\eta,\ \partial_\eta \doteq \frac{\partial}{\partial q^\eta},\ \{e_i, e_j\} = C_{ij}^k e_k,$$

which satisfies the following relations resulting from the Leibniz identity and the Jacobi identity (usually called the structure equations):

$$\rho_i^\eta \frac{\partial \rho_j^\sigma}{\partial q^\eta} - \rho_j^\eta \frac{\partial \rho_i^\sigma}{\partial q^\eta} = \rho_k^\sigma C_{ij}^k, \quad \eta = \overline{1, N}, \quad i = \overline{1, N} \tag{13}$$

$$\sum_{cyclic(i,j,k)} \left(\rho_i^\eta \frac{\partial C_{jk}^l}{\partial q^\eta} + C_{jk}^h C_{ih}^l \right) = 0. \tag{14}$$

This new formalism was proved in recent years to be a powerful tool for exploring many intrinsic properties of a given dynamical systems. FALVA was recently defined on Lie algebroids [25] and it was conjectured so that the set of admissible curves on a Lie algebroid with fixed endpoints can be endowed with a structure of a Banach manifold, the fractional action integral in FALVA is continuously differentiable and so that the equations for the critical points are precisely those of the fractional Euler-Lagrange equations obtained from FALVA for the given Lagrange system. We consider now the space of E-paths on the Lie algebroid denoted by $\mathrm{P}(J, E), J = [a, b]$, which is a differentiable Banach manifold [25].

Remark 3.1. In fact, since a Lie algebroid is a notion which merely unifies tangent bundles and Lie algebras, one can expect their relation to Lagrangian and Hamiltonian classical mechanics [26].

Theorem 3.1. *Let the manifold M be the space of all C^1 paths $[0, \tau] \to E$ in E and $L : E \to \mathbb{R}$ be a smooth regular and non-degenerate Lagrangian function on the Lie manifold $E(L \in C^\infty(E))$ with admissible curve q in E and with two end fixed points*

$$A, B \in M \subset \mathrm{P}(J, E)_A^B = \{\mathrm{P}(J, E) | \pi(q(a)) = A \text{ and } \pi(q(b)) = B\}.$$

Let $\tau_L = (\partial L/\partial y^i)dy^i$ in local coordinates. The critical points of the fractional action integral $I : \mathrm{P}(J, E) \to R$ defined by equation (2) on the Banach *manifold* $\mathrm{P}(J,E)_A^B$ *of* $\mathrm{P}(J,E)$ *are precisely those elements of that space that satisfy the following Euler-Lagrange equation:*

$$\delta^r L(\dot{q}(t), q(t), t) = 0,$$

where

$$< \delta^r L(\dot{q}(t), q(t), t) >=< dL(\dot{q}(t), q(t), t), \sum_q(\sigma) >$$
$$- \frac{d}{dt} < \tau_{L(\dot{q}(t),q(t),t)} \circ q, \sigma > - \Big[\frac{d-1}{t} - 2\alpha\tau t^{2\alpha-1}\Big] < \tau_{L(D_\gamma^{\alpha,\beta} q(t),t)} \circ q, \sigma > . \tag{15}$$

Proof. Let $q \in \mathrm{P}(J,E)_A^B$, the tangent space on $\mathrm{P}(J,E)_A^B$ is the set of vector fields along q are of the form: $\Xi_q(\sigma) : \sigma \in Sec(E)/\sigma(a) = \sigma(b) = 0$.

The action (2) is smooth, so then:

$$0 =< dS, \Xi_q(f\sigma) > = \frac{1}{\Gamma(d/2\alpha)\Gamma(d/2)} \int_0^\tau < dL(\dot{q}, q, t), \Xi_q(f\sigma) > t^{d-1}$$
$$\exp(-\tau(t^{2\alpha} + b^2))dt = \frac{1}{\Gamma(d/2\alpha)\Gamma(d/2)} \Bigg(\int_0^\tau \Big[f(t) < dL(\dot{q}, q, t), \Xi_q(\sigma) >$$
$$- \frac{d}{dt} < \tau_L \circ q, \sigma > t^{d-1} \exp(-\tau(t^{2\alpha} + b^2)) + \frac{d}{dt}(t^{d-1} \exp(-\tau(t^{2\alpha} + b^2)))$$
$$< \tau_L \circ q, \sigma > \Big] dt + f(t) < \tau_L \circ q, \sigma > t^{d-1} \exp(-\tau(t^{2\alpha} + b^2))|_0^\tau \Bigg). \tag{16}$$

Using the fact that

$$\Xi_q(f\sigma) = f \sum_q(\sigma) + \frac{df}{dt}\sigma_q^\nu,$$

we obtain after simple algebraic manipulation

$$< dS, \Xi_q(f\sigma) >= \int_0^\tau f(t) < \delta^r L(\dot{q}, q, t), \sigma(t) > dt.$$

Here $v = \rho(q)$ is the actual velocity and $d\sigma_q^\nu/d\tau$ is the Riemann-Liouville fractional derivative of the canonical vertical lift of σ. This equation is satisfied for every $f \in C^\infty(\mathbb{R})$ and every section $\sigma \in Sec(E)$. Thus the critical points are satisfied by $\delta^r L(\dot{q}, q, t) = 0$. □

Corollary 3.1. *The Euler-Lagrange equation in local coordinates is:*

$$\rho_i^\eta \frac{\partial L}{\partial x^\eta}(\dot{x}^\eta, x^\eta, t) - \frac{d}{dt}\frac{\partial L}{\partial y^i}(\dot{x}^\eta, x^\eta, t) - \frac{\partial L}{\partial y^k} C_{ij}^k y^j$$
$$- \Big[\frac{d-1}{t} - 2\alpha\tau t^{2\alpha-1}\Big] \frac{\partial L}{\partial y^i}(\dot{x}^\eta, x^\eta, t) = 0, \ \dot{x}^\eta \equiv \frac{dx^\eta}{dt} = \rho_i^\eta y^i. \tag{17}$$

Remark 3.2. If for instance $\tau = a + \iota c \in \mathbb{C}, \iota = \sqrt{-1}, a = \Re\tau, c = \Im\tau$ and $b = 0$ for mathematical simplicity, then the Euler-Lagrange equation in complexified coordinates is complexified as follows:

$$\rho_i^\eta \frac{\partial L}{\partial x^\eta}(\dot{x}^\eta, x^\eta, t) - \frac{d}{dt}\frac{\partial L}{\partial y^i}(\dot{x}^\eta, x^\eta, t) - \frac{\partial L}{\partial y^k} C_{ij}^k y^j - \Big[\frac{d-1}{t} - 2\alpha(a + \iota c)t^{2\alpha-1}\Big]\frac{\partial L}{\partial y^i}(\dot{x}^\eta, x^\eta, t) = 0, \quad (18)$$

whereas the action is complexified as:

$$\begin{aligned} I[f] &= \frac{1}{\Gamma(d/2\alpha)\Gamma(d/2)}\int_0^\tau L(\dot{x}^\eta, x^\eta, t)\exp\left(-at^{2\alpha}\right)\Big[\cos(ct^{2\alpha}) - \iota\sin(ct^{2\alpha})\Big]dt \\ &= \frac{1}{\Gamma(d/2\alpha)\Gamma(d/2)}\Big[\int_0^\tau L(\dot{x}^\eta, x^\eta, t)\exp\left(-at^{2\alpha}\right)\cos(ct^{2\alpha})dt \\ &\quad - \int_0^\tau L(\dot{x}^\eta, x^\eta, t)\exp\left(-at^{2\alpha}\right)\sin(ct^{2\alpha})dt\Big]. \end{aligned} \quad (19)$$

The complex part of the action integral may conduct to an exotic-type dynamics which may differ completely from the classical mechanics resulting from the states being complexified. For this, we let $x^\eta = x_1^\eta + \iota x_2^\eta$ and $y^\eta = y_1^\eta + \iota y_2^\eta$. Similarly the Lagrangian $L : E \to \mathbb{C}$ takes the special form:

$$L(x_1^\eta + \iota x_2^\eta) = L_1(x_1^\eta + \iota x_2^\eta) + \iota L(x_1^\eta + \iota x_2^\eta), \quad (20)$$

with

$$\frac{\partial}{\partial x^\eta} = \frac{1}{2}\Big(\frac{\partial}{\partial x_1^\eta} - \iota\frac{\partial}{\partial x_2^\eta}\Big), \quad (21)$$

$$\frac{\partial}{\partial y^i} = \frac{1}{2}\Big(\frac{\partial}{\partial y_1^i} - \iota\frac{\partial}{\partial y_2^i}\Big). \quad (22)$$

For that case

$$\begin{aligned} \rho_i^\eta\frac{\partial\rho_j^\sigma}{\partial q_1^\eta} - \rho_j^\eta\frac{\partial\rho_i^\sigma}{\partial q_1^\eta} - \iota\Big(\rho_i^\eta\frac{\partial\rho_j^\sigma}{\partial q_2^\eta} - \rho_j^\eta\frac{\partial\rho_i^\sigma}{\partial q_2^\eta}\Big) &\equiv [\rho_k^\sigma C_{ij}^k]_1 - \iota[\rho_k^\sigma C_{ij}^k]_2, \\ &\equiv \rho_k^\sigma\Big(\Re[C_{ij}^k]_1 - \iota\Im[C_{ij}^k]_2\Big) \equiv \rho_k^\sigma\bar{C}_{ij}^k \end{aligned} \quad (23)$$

where

$$\bar{C}_{ij}^k = \Re[C_{ij}^k]_1 - \iota\Im[C_{ij}^k]_2. \quad (24)$$

Lemma 3.1. *The Euler-Lagrange equation in complexified coordinates is split into two parts:*

$$\rho_i^\eta\left(\frac{\partial L_1}{\partial x_1^\eta} + \frac{\partial L_2}{\partial x_2^\eta}\right) - \frac{d}{dt}\left(\frac{\partial L_1}{\partial y_1^i} + \frac{\partial L_2}{\partial y_2^i}\right) - \Big(y_1^j\Re[C_{ij}^k]_1 + y_2^j\Im[C_{ij}^k]_2\Big)$$

$$\times\left(\frac{\partial L_1}{\partial y_1^k}+\frac{\partial L_2}{\partial y_2^k}\right)+\left(y_2^j\Re[C_{ij}^k]_1-y_1^j\Im[C_{ij}^k]_2\right)\left(\frac{\partial L_2}{\partial y_1^k}-\frac{\partial L_1}{\partial y_2^k}\right) \tag{25}$$
$$-\left[\frac{d-1}{t}-2\alpha a t^{2\alpha-1}\right]\left(\frac{\partial L_1}{\partial y_1^i}+\frac{\partial L_2}{\partial y_2^i}\right)+2\alpha a t^{2\alpha-1}\left(\frac{\partial L_2}{\partial y_1^k}-\frac{\partial L_1}{\partial y_2^k}\right)=0$$

$$\rho_i^\eta\left(\frac{\partial L_2}{\partial x_1^\eta}-\frac{\partial L_1}{\partial x_2^\eta}\right)-\frac{d}{dt}\left(\frac{\partial L_2}{\partial y_1^i}+\frac{\partial L_1}{\partial y_2^i}\right)-\left(y_2^j\Re[C_{ij}^k]_1-y_1^j\Im[C_{ij}^k]_2\right)$$
$$\times\left(\frac{\partial L_1}{\partial y_1^k}+\frac{\partial L_2}{\partial y_2^k}\right)-\left(y_1^j\Re[C_{ij}^k]_1+y_2^j\Im[C_{ij}^k]_2\right)\left(\frac{\partial L_2}{\partial y_1^k}-\frac{\partial L_1}{\partial y_2^k}\right) \tag{26}$$
$$-\left[\frac{d-1}{t}-2\alpha a t^{2\alpha-1}\right]\left(\frac{\partial L_2}{\partial y_1^i}-\frac{\partial L_1}{\partial y_2^i}\right)+2\alpha a t^{2\alpha-1}\left(\frac{\partial L_1}{\partial y_1^k}+\frac{\partial L_2}{\partial y_2^k}\right)=0.$$

As an exemplification, we will treat the geodesics for Lie algebroids. In complexified coordinates, the complexified Lagrangian $L=(1/2)g_{ij}(x)y^iy^j=L_1+\iota L_2$ is expected to induce an isomorphism of the vector (dual E^*) bundles $\tilde{g}:E\to E^*$. Here the metric $g=g_{ij}(x)e^i\otimes e^j$ is complexified, i.e. $g=g_1+\iota g_2$. The Euler–Lagrange equations (25) and (26) read:

$$\frac{d}{dt}\left[(g_{ij})_1y_1^j(\dot{x}^\eta,x^\eta,t)-(g_{ij})_2y_2^j(\dot{x}^\eta,x^\eta,t)\right]=\left[\frac{d-1}{t}-2\alpha a t^{2\alpha-1}\right]$$
$$\times\left[(g_{ik})_1y_1^j(\dot{x}^\eta,x^\eta,t)-(g_{ij})_2y_2^j(\dot{x}^\eta,x^\eta,t)\right]$$
$$+2\alpha a t^{2\alpha-1}\left[(g_{ij})_1y_2^j(\dot{x}^\eta,x^\eta,t)+(g_{ij})_2y_1^j(\dot{x}^\eta,x^\eta,t)\right]$$
$$+\left(\Re[C_{ij}^k]_1(g_{sj})_1+\Im[C_{ij}^k]_2(g_{sj})_2+\frac{1}{4}\rho_k^\eta\left(\frac{\partial(g_{ij})_1}{\partial x_1^\eta}+\frac{\partial(g_{ij})_2}{\partial x_2^\eta}\right)\right)$$
$$\times\left(y_1^j(\dot{x}^\eta,x^\eta,t)y_1^j(\dot{x}^\eta,x^\eta,t)-y_2^j(\dot{x}^\eta,x^\eta,t)y_2^j(\dot{x}^\eta,x^\eta,t)\right)$$
$$-\left(\Re[C_{ij}^k]_1(g_{sj})_2-\Im[C_{ij}^k]_2(g_{sj})_1+\frac{1}{4}\rho_k^\eta\left(\frac{\partial(g_{ij})_2}{\partial x_1^\eta}-\frac{\partial(g_{ij})_1}{\partial x_2^\eta}\right)\right)$$
$$\times\left(y_1^j(\dot{x}^\eta,x^\eta,t)y_2^j(\dot{x}^\eta,x^\eta,t)+y_2^j(\dot{x}^\eta,x^\eta,t)y_1^j(\dot{x}^\eta,x^\eta,t)\right) \tag{27}$$

$$\frac{d}{dt}\left[(g_{ij})_1y_2^j(\dot{x}^\eta,x^\eta,t)-(g_{ij})_2y_1^j(\dot{x}^\eta,x^\eta,t)\right]=\left[\frac{d-1}{t}-2\alpha a t^{2\alpha-1}\right]$$
$$\times\left[(g_{ik})_1y_2^j(\dot{x}^\eta,x^\eta,t)-(g_{ij})_2y_1^j(\dot{x}^\eta,x^\eta,t)\right]$$
$$+2\alpha a t^{2\alpha-1}\left[(g_{ij})_1y_1^j(\dot{x}^\eta,x^\eta,t)-(g_{ij})_2y_2^j(\dot{x}^\eta,x^\eta,t)\right]$$
$$+\left(\Re[C_{ij}^k]_1(g_{sj})_1+\Im[C_{ij}^k]_2(g_{sj})_2+\frac{1}{4}\rho_k^\eta\left(\frac{\partial(g_{ij})_1}{\partial x_1^\eta}+\frac{\partial(g_{ij})_2}{\partial x_2^\eta}\right)\right)$$

$$\times\left(y_1^j(\dot{x}^\eta,x^\eta,t)y_2^j(\dot{x}^\eta,x^\eta,t)+y_2^j(\dot{x}^\eta,x^\eta,t)y_1^j(\dot{x}^\eta,x^\eta,t)\right)$$
$$+\left(\Re[C_{ij}^k]_1(g_{sj})_2-\Im[C_{ij}^k]_2(g_{sj})_1+\frac{1}{4}\rho_k^\eta\left(\frac{\partial(g_{ij})_2}{\partial x_1^\eta}-\frac{\partial(g_{ij})_1}{\partial x_2^\eta}\right)\right)$$
$$\times\left(y_1^j(\dot{x}^\eta,x^\eta,t)y_1^j(\dot{x}^\eta,x^\eta,t)-y_2^j(\dot{x}^\eta,x^\eta,t)y_2^j(\dot{x}^\eta,x^\eta,t)\right), \quad (28)$$

and these are the complexified geodesics equations. Another illustration concerns the Wong's equations which arise in the dynamics of a colored particle in the Yang–Mills field and on the falling cat theorem [27, 28]. The complexified Lagrangian and Hamiltonian of the theory on the Lie algebroid E are given by:

$$\begin{aligned}L(x^\eta,\dot{x}^{eta},\bar{v}^i)&=\frac{1}{2}h_{ij}\bar{v}^i\bar{v}^j+\frac{1}{2}g_{\eta\sigma}u^\eta u^\sigma,\\&=\frac{1}{2}(h_{ij}^1(\bar{v}_1^i\bar{v}_1^j-\bar{v}_2^i\bar{v}_2^j)-h_{ij}^2(\bar{v}_1^i\bar{v}_2^j+\bar{v}_2^i\bar{v}_1^j))\\&+i[(h_{ij}^1(\bar{v}_1^i\bar{v}_2^j+\bar{v}_2^i\bar{v}_1^j)+h_{ij}^2(\bar{v}_1^i\bar{v}_1^j-\bar{v}_2^i\bar{v}_2^j))]\\&+\frac{1}{2}(g_{\sigma\eta}^1(u_1^\eta u_1^\sigma-u_2^\eta u_2^\sigma)-g_{\sigma\eta}^2(u_1^\eta u_2^\sigma+u_2^\eta u_1^\sigma))\\&+i[g_{\sigma\eta}^1(u_1^\eta u_2^\sigma+u_2^\eta u_1^\sigma)+g_{\sigma\eta}^2(u_1^\eta u_1^\sigma-u_2^\eta u_2^\sigma)]\\&=L_1(h)+L_1(g)+i[L_2(h)+L_2(g)],\end{aligned} \quad (29)$$

where $L_1(h)=\frac{1}{2}(h_{ij}^1(\bar{v}_1^i\bar{v}_1^j-\bar{v}_2^i\bar{v}_2^j)-h_{ij}^2(\bar{v}_1^i\bar{v}_2^j+\bar{v}_2^i\bar{v}_1^j))$, $L_1(g)=\frac{1}{2}(g_{\sigma\eta}^1(u_1^\eta u_1^\sigma-u_2^\eta u_2^\sigma)-g_{\sigma\eta}^2(u_1^\eta u_2^\sigma+u_2^\eta u_1^\sigma))$, $L_2(h)=\frac{1}{2}(h_{ij}^1(\bar{v}_1^i\bar{v}_2^j+\bar{v}_2^i\bar{v}_1^j)+h_{ij}^2(\bar{v}_1^i\bar{v}_1^j-\bar{v}_2^i\bar{v}_2^j))$ and $L_2(g)=\frac{1}{2}(g_{\sigma\eta}^1(u_1^\eta u_2^\sigma+u_2^\eta u_1^\sigma)+g_{\sigma\eta}^2(u_1^\eta u_1^\sigma-u_2^\eta u_2^\sigma))$ for $x^\eta=x_1^\eta+ix_2^\eta;\bar{v}^i=\bar{v}_1^i+i\bar{v}_2^i;u^\eta=u_1^\eta+iu_2^\eta$ and

$$\begin{aligned}H(x^\eta,p_\eta,\bar{p}_i)&=\frac{1}{2}h^{ij}\bar{p}_i\bar{p}_j+\frac{1}{2}g^{\eta\sigma}\bar{p}_\eta\bar{p}_\sigma,\\&=H_1(h)+H_1(g)+i[H_2(h)+H_2(g)],\end{aligned} \quad (30)$$

where $H_1(h)=\frac{1}{2}(h_1^{ij}(\bar{p}_i^1\bar{p}_j^1-\bar{p}_i^2\bar{p}_j^2)-h_2^{ij}(\bar{p}_i^1\bar{p}_j^2+\bar{p}_i^2\bar{p}_j^1))$, $H_1(g)=\frac{1}{2}(g_1^{\eta\sigma}(\bar{p}_\eta^1\bar{p}_\sigma^1-\bar{p}_\eta^2\bar{p}_\sigma^2)-g_2^{\eta\sigma}(\bar{p}_\eta^1\bar{p}_\sigma^2+\bar{p}_\eta^2\bar{p}_\sigma^1))$, $H_2(h)=\frac{1}{2}(h_2^{ij}(\bar{p}_i^1\bar{p}_j^2+\bar{p}_i^2\bar{p}_j^1)+h_2^{ij}(\bar{p}_i^1\bar{p}_j^1-\bar{p}_i^2\bar{p}_j^2))$ and $H_2(g)=\frac{1}{2}(g_1^{\eta\sigma}(\bar{p}_\eta^1\bar{p}_\sigma^2+\bar{p}_\eta^2\bar{p}_\sigma^1)-g_2^{\eta\sigma}(\bar{p}_\eta^1\bar{p}_\sigma^1-\bar{p}_\eta^2\bar{p}_\sigma^2))$ for $x^\eta=x_1^\eta+ix_2^\eta$ and $\bar{p}^i=\bar{p}_1^i+i\bar{p}_2^i$ where $u=\dot{x},(x^\eta,\dot{x}^\eta,\bar{v}^i)\equiv(x_1^\eta+ix_2^\eta,\dot{x}_1^\eta+i\dot{x}_2^\eta,\bar{v}_1^i+i\bar{v}_2^i)$ is the corresponding dual fibered complexified coordinate on TQ/G and $(x^\eta,p_\eta,\bar{p}_i)\equiv(x_1^\eta+ix_2^\eta,p_\eta^1+ip_\eta^2,\bar{p}_i^1+i\bar{p}_i^2)$ is the dual complexified coordinate on $T^*Q/G,G$ being a compact Lie group, $E=TM\times A$ with $dimTM=m$ and $dimA=n,A$ being an arbitrary Lie C– algebra of complex dimension n. h is a metric on A and g is assumed to be Riemannian metric on M. Both are assumed to be complexified ,i.e. $h=h_1+ih_2$ and $g=g_1+ig_2$.

Replacing equation (29) into equations (27) and (28) yields the complexified Wong's equation (real and complex parts respectively):

$$\begin{aligned}
&\frac{d}{dt}[h_{ik}^1\overline{v}_2^j - h_{ik}^2\overline{v}_2^j] + \frac{d}{dt}[g_{\eta\rho}^1 u_1^\sigma - g_{\eta\rho}^2 u_2^\sigma] \\
&= \left[\frac{d-1}{t} - 2\alpha a t^{2\alpha-1}\right][h_{ik}^1\overline{v}_1^j + h_{ik}^2\overline{v}_1^j + g_{\eta\rho}^1 u_2^\sigma + g_{\eta\rho}^2 u_1^\sigma] \\
&\quad +2\alpha c t^{2\alpha-1}[h_{ik}^1\overline{v}_2^j + h_{ik}^2\overline{v}_1^j + g_{\eta\rho}^1 u_2^\sigma + g_{\eta\rho}^2 u_1^\sigma] \\
&\sum(\Re[C_{ij}^k(h_{sj})^{1,2} \pm Im\left[C_{ij}^k h_{sj}^{2,1} + \frac{1}{4}\rho_k^\eta\left(\frac{\partial h_{ij}^{1,2}}{\partial x_1^\eta} \pm \frac{\partial h_{ij}^{2,1}}{\partial x_2^\eta}\right)\right](\overline{v}_1^j\overline{v}_2^j \mp \overline{v}_2^j\overline{v}_1^j) \\
&\sum \pm\Re[C_{ij}^k(g_{sj})^{1,2} \pm Im\Bigg[C_{ij}^k(g_{sj})^{2,1} \\
&\quad +\frac{1}{4}\rho_k^\eta\left(\frac{\partial(g_{ij})^{1,2}}{\partial x_1^\eta} \pm \frac{\partial(g_{ij})^{2,1}}{\partial x_2^\eta}\right)\Bigg](\overline{u}_1^j\overline{u}_2^j \mp \overline{u}_2^j\overline{u}_1^j). \qquad (31)
\end{aligned}$$

4. Generalized Hamiltonian for Lie algebroids

Let $L(\dot{q}(t), q(t), t)$ be a hyperregular Lagrangian on the Lie algebroid E with admissible curve q in E and with two fixed endpoints

$$A, B \in P(J,E)_A^B = \{P(J,E)|\pi(q(a)) = A \text{ and } \pi(q(b)) = B\} \qquad (32)$$

and $H : E^* \to \mathbb{R}$ be the Hamiltonian function defined on the dual bundle by:

$$H(\dot{q}(t), p(t), q(t), t) = p_i y^i - L(\dot{q}(t), q(t), t), \qquad (33)$$

where

$$p_i = \frac{\partial L\dot{q}(t), q(t), t)}{\partial y^i} \qquad (34)$$

is the corresponding Hamiltonian. It is notable that a Lie algebroids structure on a vector bundle can be viewed as a linear Poisson structure on the dual bundle. If $(q, \dot{q})$ is the corresponding minimizer, then $\exists p$ such that the generalized Hamiltonian system for Lie algebroids:

$$\frac{x^\alpha(t)}{dt} = \rho_i^\eta \frac{\partial H}{\partial p_i}(\dot{q}(t), p(t), q(t), t), \qquad (35)$$

$$\begin{aligned}
\frac{p_i(t)}{dt} &= -\rho_i^\eta \frac{\partial H}{\partial x^\eta}(\dot{q}(t), p(t), q(t), t) \\
-\left[\frac{d-1}{t} - 2\alpha\tau t^{2\alpha-1}\right]p_i &- C_{ij}^k p_k(t)\frac{\partial H}{\partial p_j}(\dot{q}(t), p(t), q(t), t). \qquad (36)
\end{aligned}$$

If E is a standard Lie algebroid $\tau_{TQ} : TQ \to Q$one may recover the standard fractional Hamilton equations [18,19]. Besides, the fractional Lie-Poisson equations (when E is a Lie algebra as a Lie algebroid over a one point manifold on the vector bundle), the fractional Hamilton-Poincare equations (when E is the Atiyah algebroid associated with a principal G-bundle with bracket induced by the Lie bracket of vector fields), the fractional Lie Poisson equations on the dual of a semidirect product of Lie algebras (when E is an action Lie algebroid over a real vector space for a given Lie algebra) are all recovered [29,30]. If for instance $\tau = a + ic \in \mathbb{C}$, $i = \sqrt{-1}$, $a = \Re\tau$, $c = \Im\tau$, then the generalized Hamiltonian system is complexified:

$$\frac{dx^{\alpha}(t)}{dt} = \rho_i^{\eta}\left[\frac{\partial H_1}{\partial p_j^1} + \frac{\partial H_2}{\partial p_j^2} + i\left(\frac{\partial H_2}{\partial p_j^1} - \frac{\partial H_1}{\partial p_j^2}\right)\right] \tag{37}$$

$$\begin{aligned}
\frac{dp_i(t)}{dt} &= -\frac{1}{2}\rho_i^{\eta}\left(\frac{\partial H_1}{\partial x_1^{\eta}} + \frac{\partial H_2}{\partial x_2^{\eta}}\right) - \left[\frac{d-1}{t} - 2\alpha\tau t^{2\alpha-1}\right]p_i \\
&-\left[\Re[C_{ij}^k]_1 p_k(t)\left(\frac{\partial H_1}{\partial p_j^1} + \frac{\partial H_2}{\partial p_j^2}\right) - \Im[C_{ij}^k]_2 p_k(t)\left(\frac{\partial H_2}{\partial p_j^1} - \frac{\partial H_2}{\partial p_j^2}\right)\right] \\
&+i\left[2\alpha c t^{2\alpha-1}p_i - \Re[C_{ij}^k]_1 p_k(t)\left(\frac{\partial H_2}{\partial p_j^1} - \frac{\partial H_1}{\partial p_j^2}\right)\right. \\
&\left.+\Im[C_{ij}^k]_2 p_k(t)\left(\frac{\partial H_1}{\partial p_j^1} + \frac{\partial H_2}{\partial p_j^2}\right) - \frac{1}{2}\rho_i^{\eta}\left(\frac{\partial H_2}{\partial x_1^{\eta}} - \frac{\partial H_1}{\partial x_2^{\eta}}\right)\right].
\end{aligned} \tag{38}$$

Lemma 4.1. *The fractional Hamilton dynamics on the dual bundle E^* is represented by the vector field:*

$$D^r(x,\xi) = \rho_i^{\eta}\frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^{\eta}} - \left[\frac{d-1}{t} - 2\alpha\tau t^{2\alpha-1}\right]\frac{\partial}{\partial \xi_i} - \left(\rho_i^{\eta}\frac{\partial H}{\partial x^{\eta}} + C_{ij}^k p_k(t)\frac{\partial H}{\partial p_j}\right). \tag{39}$$

The integral curves of the vector field satisfy in general the fractional Hamilton equations (37) *and* (38) *for H. This vector field is complexified if for instance $\tau = a + ic \in \mathbb{C}$, $i = \sqrt{-1}$, $a = \Re\tau$, $c = \Im\tau$ and accordingly:*

$$\begin{aligned}
D^r(x,\xi) &= \frac{1}{2}\rho_i^{\eta}\left[\frac{\partial H_1}{\partial p_j^2}\frac{\partial}{\partial x_1^{\eta}} + \left(\frac{\partial H_2}{\partial p_j^1} - \frac{\partial H_1}{\partial p_j^2}\right)\frac{\partial}{\partial x_2^{\eta}}\right] \\
&-\frac{1}{2}\left[\frac{d-1}{t} - 2\alpha\tau t^{2\alpha-1}\right]\frac{\partial}{\partial \xi_i} + [2\alpha c t^{2\alpha-1}]\frac{\partial}{\partial x_i^2} \\
&-\frac{1}{2}\rho_i^{\eta}\left(\frac{\partial H_1}{\partial x_1^{\eta}} + \frac{\partial H_2}{\partial x_2^{\eta}}\right)\frac{\partial}{\partial \xi_i^1} - \frac{1}{2}\rho_i^{\eta}\left(\frac{\partial H_2}{\partial x_1^{\eta}} - \frac{\partial H_1}{\partial x_2^{\eta}}\right)\frac{\partial}{\partial \xi_i^2} \\
&-\frac{1}{2}\Re[C_{ij}^k]_1\left[\left(\frac{\partial H_1}{\partial p_j^1} + \frac{\partial H_2}{\partial p_j^2}\right)p_k^1 - \left(\frac{\partial H_2}{\partial p_j^1} - \frac{\partial H_1}{\partial p_j^2}\right)p_k^2\right]\frac{\partial}{\partial \xi_i^1} \\
&-\frac{1}{2}\tau[C_{ij}^k]_2(p_k^1\left(\frac{\partial H_2}{\partial p_j^1} - \frac{\partial H_1}{\partial p_j^2}\right) + p_k^2\left(\frac{\partial H_1}{\partial p_j^1} + \frac{\partial H_2}{\partial p_j^2}\right)\frac{\partial}{\partial \xi_i^1}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\Re[C_{ij}^{k}]_{1}\left(p_{k}^{1}\left(\frac{\partial H_{2}}{\partial p_{j}^{1}}-\frac{\partial H_{1}}{\partial p_{j}^{2}}\right)+p_{k}^{2}\left(\frac{\partial H_{1}}{\partial p_{j}^{1}}+\frac{\partial H_{2}}{\partial p_{j}^{2}}\right)\right)\frac{\partial}{\partial \xi_{i}^{2}}\\
&+\frac{1}{2}Im[C_{ij}^{k}]_{2}\left[\left(\frac{\partial H_{1}}{\partial p_{j}^{1}}+\frac{\partial H_{2}}{\partial p_{j}^{2}}\right)p_{k}^{1}-\left(\frac{\partial H_{2}}{\partial p_{j}^{1}}-\frac{\partial H_{2}}{\partial p_{j}^{2}}\right)p_{k}^{2}\right]\frac{\partial}{\partial \xi_{i}^{2}}\\
&+i\left[\frac{1}{2}\rho_{i}^{\eta}\left(\frac{\partial H_{2}}{\partial p_{j}^{1}}-\frac{\partial H_{2}}{\partial p_{j}^{2}}\right)\frac{\partial}{\partial x_{1}^{\eta}}-\frac{1}{2}\rho_{i}^{\eta}\left(\frac{\partial H_{1}}{\partial p_{j}^{1}}+\frac{\partial H_{2}}{\partial p_{j}^{2}}\right)\frac{\partial}{\partial x_{1}^{\eta}}\right.\\
&+\frac{1}{2}\left[\frac{d-1}{t}-2\alpha c t^{2\alpha-1}\right]\frac{\partial}{\partial \xi_{i}^{2}}-2\alpha c t^{2\alpha-1}\frac{\partial}{\partial \xi_{i}^{1}}\\
&-\frac{1}{2}\rho_{i}^{\eta}\left(\frac{\partial H_{1}}{\partial x_{1}^{\eta}}+\frac{\partial H_{2}}{\partial x_{2}^{\eta}}\right)\frac{\partial}{\xi_{i}^{2}}-\frac{1}{2}\rho_{i}^{\eta}\left(\frac{\partial H_{2}}{\partial x_{1}^{\eta}}-\frac{H_{1}}{\partial x_{2}^{\eta}}\right)\frac{\partial}{\xi_{i}^{2}}\\
&-\frac{1}{2}\Re[C_{ij}^{k}]_{1}\left[\left(\frac{H_{1}}{\partial p_{j}^{1}}+\frac{\partial H_{2}}{\partial p_{j}^{2}}\right)p_{k}^{1}-\left(\frac{\partial H_{2}}{\partial p_{j}^{1}}-\frac{\partial H_{1}}{\partial p_{j}^{2}}\right)p_{k}^{2}\right]\frac{\partial}{\partial \xi_{i}^{2}}\\
&+\frac{1}{2}Im[C_{ij}^{k}]_{2}(p_{k}^{1}\left(\frac{\partial H_{2}}{\partial p_{j}^{1}}-\frac{\partial H_{1}}{\partial p_{j}^{2}}\right)+p_{k}^{2}\left(\frac{\partial H_{1}}{\partial p_{j}^{1}})\right)\frac{\partial}{\partial \xi_{i}^{2}}\\
&-\frac{1}{2}\Re[C_{ij}^{k}]\left(p_{k}^{1}\left(\frac{\partial H_{2}}{\partial p_{j}^{1}}-\frac{\partial H_{1}}{\partial p_{j}^{2}}\right)+p_{k}^{2}\left(\frac{\partial H_{1}}{\partial p_{j}^{1}}+\frac{\partial H_{2}}{\partial p_{j}^{2}}\right)\right)\frac{\partial}{\partial x_{i}^{1}}\\
&\left.-\frac{1}{2}Im[C_{ij}^{k}]_{2}\left[\left(\frac{\partial H_{1}}{\partial p_{j}^{1}}+\frac{\partial H_{2}}{\partial p_{j}^{2}}\right)p_{k}^{1}-\left(\frac{\partial H_{2}}{\partial p_{j}^{1}}-\frac{\partial H_{1}}{\partial p_{j}^{2}}\right)p_{k}^{2}\right]\frac{\partial}{\partial x_{i}^{1}}\right]. \qquad (40)
\end{aligned}
$$

One may extend all the previous formalisms for the case of complex Lie algebroids structures on complex vector bundles [31]. Besides making use of the extended FALVA formalism [32,33,34,35,36] this is also motivation for further work in this direction.

5. CONCLUSIONS

To the best of our knowledge, this work represents the first effort to build the theoretical framework of complex Lie algebroids based on fractal action-like calculus of variations. That will be the beginning, we believe, of new exciting investigations. Our contribution is, however, only theoretical and, in that sense, more modest. The complexified geodesics equations and complexified Wongs equations characterized for some particular values of the parameter α and in particular for $\alpha > 1/2$by growing mass/energy could have interesting consequences on quantum field theory and astrophysics. Accordingly, the new class of generalized Lagrangian and Hamiltonian systems obtained on Lie algebroids is wider than the standard class of Lagrangian and Hamiltonian dynamical systems. We expect they will open up in the future a new stimulating research area in different branch of mathematical physics and provide us by a potent tool to comprehend many fundamental problems in the area of complexified dynamical systems. Future research efforts may be directed towards formulating predictions that can be tracked tested numerically. De-complexification of the dynamical equations, in particular the fractional Euler-Lagrange equations, geodesic equations and Wongs equations could be realized if the generalized fractal action introduced in the theory is complexified in the theory. From a phenomenological standpoint, we can go further and construct a fractional Hamiltonian approach to holomorphic Poisson manifolds and holomorphic Lie algebroids from the viewpoint of real Poisson geometry [37,38,39,40,41]. Follow-up studies may be devoted to explore, how fractional derivatives encoded by the fractional action-like variational approach may be mapped into the Lie algebroids. The results in this work declare further generalization and recommend additional study.

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Ahmad Rami El-Nabulsi and Guo-Cheng Wu
Key Laboratory of Numerical Simulation
of Sichuan Province and
College of Mathematics and Information Science
Neijiang Normal University
Neijiang, Sichuan 641112, China
E–mail: nabulsiahmadrami@yahoo.fr
E–mail: frac.calc@china.com

Mubasher Jamil
Center for Advanced Mathematics and Physics
National University of Sciences and Technology
H-12, Islamabad, Pakistan and
Eurasian National University, Astana, Kazakhstan
E–mail: mjamil@camp.nust.edu.pk