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ON THE FAINTLY e-CONTINUOUS FUNCTIONS

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Abstract. A new class of functions, called faintly e-continuous functions, has been defined and studied. Relationships among faintly e-continuous functions and e-connected spaces, e-normal spaces and e-compact spaces are investigated. Furthermore, the relationships between faintly e-continuous functions and graphs are investigated.

1. Introduction

Recent progress in the study of characterizations and generalizations of continuity, compactness, connectedness, separation axioms etc. has been made by means of several generalized closed sets. The first step of generalizing closed sets was done by Levine in 1970 [12]. The notion of generalized closed sets has been studied extensively in recent years by many topologist because generalized closed sets are the only natural generalization of closed sets. More importantly, they also suggest several new properties of topological spaces. As a generalization of closed sets, e-closed sets and the related sets were introduced and studied by E. Ekici ([2], [3], [4], [5], [6], [7]). In this paper the faintly e-continuous functions is introduced and studied. Moreover, basic properties and preservation theorems of faintly e-continuous functions are investigated and the relationships between faintly e-continuous functions and graphs are also investigated.

2. Preliminaries

Throughout the paper (X,τ) and (Y,σ) (or simply X and Y) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X,τ), $cl(A)$, $int(A)$ and $X\backslash A$ denote the closure of A, the interior of A and the complement of A in X, respectively. A point $x \in X$ is called a θ-cluster [13] (resp. δ-cluster [18]) point of A if $cl(V) \cap A \neq \emptyset$ (resp. $int(cl(V)) \cap A \neq \emptyset$) for every open set

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V of X containing x. The set of all θ-cluster (resp. δ-cluster) points of A is called the θ-closure (resp. δ-closure) of A and is denoted by $cl_\theta(A)$ (resp. $\delta cl(A)$). If $A = cl_\theta(A)$ (resp. $A = \delta cl(A)$), then A is said to be θ-closed (resp. δ-closed). The complement of a θ-closed (resp. δ-closed) set is said to be θ-open (resp. δ-open). The union of all θ-open sets contained in a subset A is called the θ-interior of A and is denoted by $int_\theta(A)$. It follows from [13] that the collection of θ-open sets in a topological space (X, τ) forms a topology τ_θ on X. The family of all θ-open (resp. θ-closed) subsets of X is denoted by $\theta O(X)$ (resp. $\theta C(X)$).

We recall the following definitions, which are useful in the sequel.

Definition 1. *A subset A of a space (X, τ) is called semi-open (resp. α-open, preopen, β-open, γ-open or b-open, δ-preopen) if $A \subset cl(int(A))$ (resp. $A \subset int(cl(int(A)))$, $A \subset int(cl((A))$, $A \subset cl(int(cl(A)))$, $A \subset cl(int(A)) \cup int(cl(A))$, $A \subset int(\delta cl(A))$).*

A subset A of a topological space X is said to be e-open [2] if $A \subset int(\delta cl(A)) \cup cl(\delta int(A))$. The complement of an e-open set is said to be e-closed [2]. The intersection (union) of all e-closed (e-open) sets containing (contained in) A in X is called the e-closure [2] (e-interior [2]) of A and is denoted by $e\text{-}cl(A)$ (resp. $e\text{-}int(A)$). By $eO(X)$ or $eO(X, \tau)$ (resp. $eC(X)$), we denote the collection of all e-open (resp. e-closed) sets of X. We set $eO(X, x) = \{U : x \in U \in eO(X)\}$. Similarly we denote the collection of all γ-open, semiopen, preopen and β-open sets by $\gamma O(X, \tau)$, $SO(X, \tau)$, $PO(X, \tau)$ and $\beta O(X, \tau)$, respectively.

Lemma 2.1. ([2], [4]). *The following properties hold for the e-closure of a set in a space X:*

1. *Arbitrary union (intersection) of e-open (e-closed) sets in X is e-open (resp. e-closed).*
2. *A is e-closed in X if and only if $A = e\text{-}cl(A)$.*
3. *$e\text{-}cl(A) \subset e\text{-}cl(B)$ whenever $A \subset B (\subset X)$.*
4. *$e\text{-}cl(A)$ is e-closed in X.*
5. *$e\text{-}cl(e\text{-}cl(A)) = e\text{-}cl(A)$.*
6. *$e\text{-}cl(A) = \{x \in X \mid U \cap A \neq \emptyset$ for every e-open set U containing $x\}$.*

Definition 2. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be faintly continuous* [13] *(resp. faintly semicontinuous* [15]*, faintly precontinuous* [15]*, faintly β-continuous* [15]*, faintly α-continuous* [14]*, faintly γ-continuous* [14]*) if $f^{-1}(V)$ is open (resp. semiopen, preopen, β-open, α-open, γ-open) in X for every θ-open set V of Y.*

We have the following diagram in which the converses of implications need not be true, as is shown in [2].

$$\begin{tikzcd}
 & \text{open} \arrow[r] & \alpha\text{-open} \arrow[r] & \text{preopen} \arrow[r] \arrow[dd] & b\text{-open} \arrow[r] & \beta\text{-open} \\
\text{regular open} \arrow[ur] \arrow[dr] & & & & & \\
 & \delta\text{-open} \arrow[uu] \arrow[dr] & & \delta\text{-preopen} \arrow[d] & & \\
 & & \delta\text{-semiopen} \arrow[r] & e\text{-open} & &
\end{tikzcd}$$

3. The faintly e-continuous functions

Definition 3. *A function $f : (X, \tau) \to (Y, \sigma)$ is called faintly e-continuous at a point $x \in X$ if for each θ-open set V of Y containing $f(x)$, there exists $U \in eO(X, x)$ such that $f(U) \subset V$. If f has this property at each point of X, then it is said to be faintly e-continuous.*

Definition 4. *A function $f : (X, \tau) \to (Y, \sigma)$ is called e-continuous* [2] *if for every open subset A of Y, $f^{-1}(A) \in eO(X, \tau)$.*

Theorem 3.1. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

(i) *f is faintly e-continuous;*
(ii) *$f^{-1}(V)$ is e-open in X for every θ-open set V of Y;*
(iii) *$f^{-1}(F)$ is e-closed in X for every θ-closed subset F of Y.*

Proof. (i)⇒(ii): Let V be a θ-open set of Y and $x \in f^{-1}(V)$. Since $f(x) \in V$ and f is faintly e-continuous, there exists $U \in eO(X, x)$ such that $f(U) \subset V$. It follows that $x \in U \subset f^{-1}(V)$. Hence $f^{-1}(V)$ is e-open in X.
(ii)⇒(i): Let $x \in X$ and V be a θ-open set of Y containing $f(x)$. By (ii), $f^{-1}(V)$ is a e-open set containing x. Take $U = f^{-1}(V)$. Then $f(U) \subset V$. This shows that f is faintly e-continuous.
(ii)⇒(iii): Let V be any θ-closed set of Y. Since $Y \setminus V$ is an θ-open set, by (ii), it follows that $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is e-open. This shows that $f^{-1}(V)$ is e-closed in X.
(iii)⇒(ii): Let V be a θ-open set of Y. Then $Y \setminus V$ is θ-closed in Y. By (iii), $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$ is e-closed and thus $f^{-1}(V)$ is e-open. □

Definition 5. *A function $f : (X, \tau) \to (Y, \sigma)$ is called δ-semicontinuous* [8] *if $f^{-1}(A)$ is δ-semiopen for every open set A of Y.*

Remark 3.2. The below diagram holds for a function f. The reverse of the implications are not true in general as shown in [2] and the following example.

δ-semicontinuous $\Rightarrow$ e-continuous $\Rightarrow$ faintly e-continuous

Example 3.3. Let $X = \{a, b, c, d, e\}$ with topologies $\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{c, d\}, \{a, c, d\}, X\}$ and $\sigma = \{\emptyset, \{b, c, d\}, X\}$. Then the identity function $f : (X, \tau) \to (Y, \sigma)$ is faintly e-continuous. But f is not e-continuous.

If (Y, σ) is a regular space, we have $\sigma = \sigma_\theta$ and the next theorem follows immediately.

Theorem 3.4. *Let (Y, σ) be a regular space. Then for a function $f : (X, \tau) \to (Y, \sigma)$ the following properties are equivalent:*

(i) *f is e-continuous,*
(ii) *f is faintly e-continuous.*

Definition 6. *A function $f : (X, \tau) \to (Y, \sigma)$ is called contra e-continuous* [4] *if $f^{-1}(B)$ is e-closed in X for every open set B of Y.*

Theorem 3.5. ([4]) *If a function $f : (X, \tau) \to (Y, \sigma)$ is contra e-continuous and Y is regular, then it is e-continuous.*

Theorem 3.6. *If a function $f : (X, \tau) \to (Y, \sigma)$ is contra e-continuous and Y is regular, then it is faintly e-continuous.*

Proof. It follows from Theorem 3.5 and Remark 3.2. □

Definition 7. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be weakly e-continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in eO(X, x)$ such that $f(U) \subset cl(V)$.*

Definition 8. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be almost e-continuous* [6] *if $f^{-1}(B)$ is e-open in X for every regular open set B of Y.*

Remark 3.7. Every almost e-continuous function is weakly e-continuous.

Definition 9. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be almost δ-precontinuous* [9] *if for each $x \in X$ and each regular open set V of Y containing $f(x)$, there exists a δ-preopen set U in X containing x such that $f(U) \subset V$.*

Definition 10. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be almost δ-semicontinuous* [10] *if for each $x \in X$ and each regular open set V of Y containing $f(x)$, there exists a δ-semiopen set U in X containing x such that $f(U) \subset V$.*

Theorem 3.8. *If a function $f : (X, \tau) \to (Y, \sigma)$ is weakly e-continuous, then it is faintly e-continuous.*

Proof. Let $x \in X$ and V be a θ-open set containing $f(x)$. Then, there exists an open set W such that $f(x) \in W \subset cl(W) \subset V$. Since f is weakly e-continuous, there exists an e-open set U containing x such that $f(U) \subset cl(W) \subset V$. Therefore, f is faintly e-continuous. $\square$

Corollary 3.9. *Every almost e-continuous function and also every almost δ-precontinuous and every almost δ-semicontinuous function is faintly e-continuous.*

Definition 11. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be faintly δ-precontinuous* [9] *if for each $x \in X$ and each θ-open set V of Y containing $f(x)$, there exists a δ-preopen set U containing x such that $f(U) \subset V$.*

Definition 12. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be faintly δ-semicontinuous* [10] *if for each $x \in X$ and each θ-open set V of Y containing $f(x)$, there exists a δ-semiopen set U containing x such that $f(U) \subset V$.*

Remark 3.10. Every faintly δ-precontinuous function and faintly δ-semicontinuous function is faintly e-continuous but not conversely as shown in the following example.

Example 3.11. Let X be the set of the real numbers, τ the indiscrete topology for X and σ the discrete topology for X. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is faintly e-continuous. But f is not faintly δ-semicontinuous.

Example 3.12. Let $X = \{a, b, c\}$ with topologies $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. Then the identity function $f : (X, \tau) \to (Y, \sigma)$ is faintly e-continuous. But f is not faintly δ-precontinuous.

Clearly the following diagram holds and none of its implications is reversible.

$$
\begin{array}{ccccc}
 & & \text{faintly continuous} & & \\
 & & \downarrow & & \\
 & & \text{faintly } \alpha\text{-continuous} & & \\
 & \swarrow & \downarrow & \searrow & \\
\text{faintly} & \rightarrow & \text{faintly } e\text{-continuous} & & \text{faintly} \\
\text{precontinuous} & \searrow & & \swarrow & \text{semicontinuous} \\
 & & \text{faintly } \gamma\text{-continuous} & & \\
 & & \downarrow & & \\
 & & \text{faintly } \beta\text{-continuous} & &
\end{array}
$$

The converses are not true in general, see ([15], Examples 4.5 and 4.6), ([13], Examples 2) and [14], Examples 4.4 and 4.5).

Example 3.13. Let X be the set of real numbers, τ the indiscrete topology for X and σ the discrete topology for X. Then the identity function $f : (X,\tau) \to (X,\sigma)$ is faintly e-continuous. But f is not faintly semi-continuous.

Example 3.14. Let $X = \{a,b,c\}$ with topologies $\tau = \{\emptyset, \{a\}, \{b\}, \{a,b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, X\}$. Then the identity function $f : (X,\tau) \to (Y,\sigma)$ is faintly e-continuous. But f is not faintly precontinuous.

Definition 13. *A topological space (X,τ) is said to be a T_e-space if every e-closed subset of (X,τ) is closed.*

Theorem 3.15. *Let (X,τ) be a T_e-space. The following are equivalent for a function $f : (X,\tau) \to (Y,\sigma)$:*

(i) *f is faintly continuous,*
(ii) *f is faintly precontinuous,*
(iii) *f is faintly e-continuous.*

Proof. Follows from the Definitions and Diagram above. □

Recall that, a space X is said to be submaximal if each dense subset of X is open in X and extremally disconnected (ED for short) if the closure of each open set of X is open in X.

Theorem 3.16. *Let (X,τ) be a submaximal ED, T_e-space. Then the following are equivalent for a function $f : (X,\tau) \to (Y,\sigma)$:*

(i) *f is faintly α-continuous,*
(ii) *f is faintly γ-continuous,*
(iii) *f is faintly semicontinuous,*
(iv) *f is faintly precontinuous,*
(v) *f is faintly β-continuous,*
(vi) *f is faintly continuous,*
(vii) *f is faintly e-continuous.*

Proof. By [14] we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (i) since that X is submaximal and ED then $\tau = \tau^{\alpha} = \gamma O(X,\tau) = SO(X,\tau) = PO(X,\tau) = \beta O(X,\tau)$.

(vi) $\Leftrightarrow$ (vii): This follows from the fact that if (X,τ) is T_e then $\tau = eO(X,\tau)$. □

A function $f : (X,\tau) \to (Y,\sigma)$ is said to be slightly e-continuous if for each $x \in X$ and each clopen set V of Y containing $f(x)$, there exists $U \in eO(X,x)$ such that $f(U) \subset V$.

Theorem 3.17. *If a function $f : (X,\tau) \to (Y,\sigma)$ is faintly e-continuous, then it is slightly e-continuous.*

Proof. Let $x \in X$ and V be any clopen subset of (Y, σ) containing $f(x)$. Then V is θ-open in Y. Since f is faintly e-continuous, there exists $U \in eO(X, x)$ such that $f(U) \subset V$. This shows that f is slightly e-continuous. $\square$

Recall that a function $f : (X, \tau) \to (Y, \sigma)$ is said to be:
(i) quasi θ-continuous [11], if $f^{-1}(V)$ is θ-open in X for every θ-open set V of Y.
(ii) strongly θ-continuous [16], if $f^{-1}(V) \in \tau_\theta$ for every open set V of Y.

Theorem 3.18. *The following statements hold for functions $f : X \to Y$ and $g : Y \to Z$:*

- (i) *If f is faintly e-continuous and g is quasi θ-continuous, then $g \circ f : X \to Z$ is faintly e-continuous,*
- (ii) *If f faintly e-continuous and g is a strongly θ-continuous, then $g \circ f : X \to Z$ is e-continuous,*
- (iii) *If f is e-continuous and g is quasi θ-continuous, then $g \circ f : X \to Z$ is faintly e-continuous.*

Definition 14. *An e-frontier of a subset A of (X, τ) is e-$Fr(A)$ = e-cl(A) $\cap$ e-cl$(X \backslash A)$* [4, 6].

Theorem 3.19. *The set of all points $x \in X$ at which a function $f : (X, \tau) \to (Y, \sigma)$ is not faintly e-continuous is identical with the union of the e-frontier of the inverse images of θ-open subsets of Y containing $f(x)$.*

Proof. Necessity. Suppose that f is not faintly e-continuous at $x \in X$. Then there exists a θ-open set V of Y containing $f(x)$ such that $f(U)$ is not a subset of V for each $U \in eO(X)$ containing x. Hence we have $U \cap (X \backslash f^{-1}(V)) \neq \emptyset$ for each $U \in eO(X, x)$. It follows that $x \in$ *e-cl*$(X \backslash f^{-1}(V))$. On the other hand, we have that $x \in f^{-1}(V) \subset$ *e-cl*$(f^{-1}(V))$. This means that $x \in$ *e-*$Fr(f^{-1}(U))$.
Sufficiency. Suppose that $x \in$ *e-*$Fr(f^{-1}(U))$ for some $V \in \theta O(Y, f(x))$. Now, we assume that f is faintly e-continuous at $x \in X$. Then there exists $U \in eO(X)$ containing x such that $f(U) \subset V$. Therefore, we have $U \subset f^{-1}(V)$ and hence $x \in$ *e-int*$(f^{-1}(V)) \subset X \backslash$*e-*$Fr(f^{-1}(V))$. This is a contradiction. This means that f is not faintly e-continuous. $\square$

Theorem 3.20. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and $g : (X, \tau) \to (X \times Y, \tau \times \sigma)$ the graph function of f, defined by $g(x) = (x, f(x))$ for every $x \in X$. If g is faintly e-continuous, then f is faintly e-continuous.*

Proof. Let $x \in X$ and V be a θ-open set in (Y, σ) containing $f(x)$. Then $X \times V$ is θ-open in $X \times Y$ ([13], Theorem 5) and contains $g(x) = (x, f(x))$. Therefore there exists $U \in eO(X, x)$ such that $g(U) \subset X \times V$. This implies that $f(U) \subset V$. Thus f is faintly e-continuous. $\square$

Definition 15. *A space (X, τ) is said to be:*

(i) *e-connected* [4, 6] *if X cannot be written as a disjoint union of two nonempty disjoint e-open sets,*
(ii) *e-compact* [4, 6] *(resp. θ-compact* [11]*) if each e-open (resp. θ-open) cover of X has a finite subcover.*

Theorem 3.21. *If $f : (X, \tau) \to (Y, \sigma)$ is a faintly e-continuous surjection function and (X, τ) is a e-connected space, then Y is a connected space.*

Proof. Assume that (Y, σ) is not connected. Then there exist nonempty open sets V_1 and V_2 such that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = Y$. Hence we have $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$ and $f^{-1}(V_1) \cup f^{-1}(V_2) = X$. Since f is surjective, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are nonempty subsets of X. Since V_i is open and closed, V_i is θ-open for each $i = 1, 2$. Since f is faintly e-continuous, $f^{-1}(V_i) \in eO(X)$. Therefore, (X, τ) is not e-connected. This is a contradiction and hence (Y, σ) is connected. □

Theorem 3.22. *The surjective faintly e-continuous image of an e-compact space is θ-compact.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a faintly e-continuous function from a e-compact space X onto a space Y. Let $\{G_\alpha\colon\ \alpha \in I\}$ be any θ-open cover of Y. Since f is faintly e-continuous, $\{f^{-1}(G_\alpha)\colon\ \alpha \in I\}$ is a e-open cover of X. Since X is e-compact, there exists a finite subcover $\{f^{-1}(G_i) : i = 1, 2, \ldots n\}$ of X. Then it follows that $\{G_i\colon\ i = 1, 2, \ldots n\}$ is a finite subfamily which covers Y. Hence Y is θ-compact. □

Theorem 3.23. *$f : (X, \tau) \to (Y, \sigma)$ is a faintly e-continuous if e-$cl(f^{-1}(V) \subset f^{-1}cl_\theta(V))$ for each θ-open subset V of Y.*

Proof. Let $V \in \theta O(Y)$. By the hypothesis $e\text{-}cl(f^{-1}(V) \subset f^{-1}cl_\theta(V)) = f^{-1}(V)$. Therefore $f^{-1}(V)$ is e-closed. By Theorem 3.1 f is faintly e-continuous. □

4. Separation axioms

Recall that a topological space (X, τ) is said to be:

(i) e-T_1 [4, 6] (resp. θ-T_1) if for each pair of distinct points x and y of X, there exist e-open (resp. θ-open) sets U and V containing x and y, respectively, such that $y \notin U$ and $x \notin V$,
(ii) e-T_2 [4, 6] (resp. θ-T_2 [17]) if for each pair of distinct points x and y in X, there exist disjoint e-open (resp. θ-open) sets U and V in X such that $x \in U$ and $y \in V$.

Remark 4.1. [1] Hausdorff $\Leftrightarrow$ θ-T_1 $\Leftrightarrow$ θ-T_0.

Theorem 4.2. *If $f : (X, \tau) \to (Y, \sigma)$ is a faintly e-continuous injection and Y is a θ-T_1 (or Hausdorff) space, then X is an e-T_1 space.*

Proof. Suppose that Y is θ-T_1. For any distinct points x and y in X, there exist $V, W \in \sigma_\theta$ such that $f(x) \in V$, $f(y) \notin V$, $f(x) \notin W$ and $f(y) \in W$. Since f is faintly e-continuous, $f^{-1}(V)$ and $f^{-1}(W)$ are e-open subsets of (X, τ) such that $x \in f^{-1}(V)$, $y \notin f^{-1}(V)$, $x \notin f^{-1}(W)$ and $y \in f^{-1}(W)$. This shows that X is e-T_1. □

Theorem 4.3. *If $f : (X, \tau) \to (Y, \sigma)$ is faintly e-continuous injection and Y is a θ-T_2 space, then X is an e-T_2 space.*

Proof. Suppose that Y is θ-T_2. For any pair of distinct points x and y in X, there exist disjoint θ-open sets U and V in Y such that $f(x) \in U$ and $f(y) \in V$. Since f is faintly e-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are e-open in X containing x and y, respectively. Therefore, $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ because $U \cap V = \emptyset$. This shows that X is e-T_2. □

Definition 16. *A space (X, τ) is said to be:*

(i) *θ-regular (resp. e-regular* [3]*) if for each θ-closed (resp. e-closed) set F and each point $x \notin F$, there exist disjoint θ-open (resp. e-open) sets U and V such that $F \subset U$ and $x \in V$,*

(ii) *θ-normal (resp. e-normal* [3]*) if for any pair of disjoint θ-closed (resp. e-closed) subsets F_1 and F_2 of X, there exist disjoint θ-open (resp. e-open) sets U and V such that $F_1 \subset U$ and $F_2 \subset V$.*

Definition 17. *A function $f : (X, \tau) \to (Y, \sigma)$ is called:*

(i) *eθ-open if $f(V) \in \sigma_\theta$ for each $V \in eO(X)$,*

(ii) *eθ-closed if $f(V)$ is θ-closed in Y for each $V \in eC(X)$.*

Theorem 4.4. *If f is a faintly e-continuous eθ-open bijective function from an e-regular space (X, τ) onto a space (Y, σ), then (Y, σ) is θ-regular.*

Proof. Let F be a θ-closed subset of Y and $y \notin F$. Take $y = f(x)$. Since f is faintly e-continuous, $f^{-1}(F)$ is e-closed in X so that $f^{-1}(y) = x \notin f^{-1}(F)$. Take $G = f^{-1}(F)$. We have $x \notin G$. Since X is e-regular, then there exist disjoint e-open sets U and V in X such that $G \subset U$ and $x \in V$. We obtain that $F = f(G) \subset f(U)$ and $y = f(x) \in f(V)$ such that $f(U)$ and $f(V)$ are disjoint θ-open sets. This shows that Y is θ-regular. □

Theorem 4.5. *If f is a faintly e-continuous eθ-open injective function from a e-normal space (X, τ) onto a space (Y, σ), then Y is θ-normal.*

Proof. Let F_1 and F_2 be disjoint θ-closed subsets of Y. Since f is faintly e-continuous, $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are e-closed sets. Take $U = f^{-1}(F_1)$ and $V = f^{-1}(F_2)$. We have $U \cap V = \emptyset$. Since X is e-normal, there exist disjoint

e-open sets A and B such that $U \subset A$ and $V \subset B$. We obtain that $F_1 = f(U) \subset f(A)$ and $F_2 = f(V) \subset f(B)$ such that $f(A)$ and $f(B)$ are disjoint θ-open sets. Thus, Y is θ-normal. □

Recall that for a function $f : (X, \tau) \to (Y, \sigma)$, the subset $\{(x, f(x)) : x \in X\} \subset X \times Y$ is called the graph of f and is denoted by $G(f)$.

Definition 18. *A graph $G(f)$ of a function $f : (X, \tau) \to (Y, \sigma)$ is said to be (θ, e)-closed if for each $(x, y) \in (X \times Y)\backslash G(f)$, there exist $U \in eO(X, x)$ and $V \in \sigma_\theta$ containing y such that $(U \times V) \cap G(f) = \emptyset$.*

Lemma 4.6. *A graph $G(f)$ of a function $f : (X, \tau) \to (Y, \sigma)$ is (θ, e)-closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in eO(X, x)$ and $V \in \sigma_\theta$ containing y such that $f(U) \cap V = \emptyset$.*

Proof. It is an immediate consequence of Definition 18. □

Theorem 4.7. *If $f : (X, \tau) \to (Y, \sigma)$ is a faintly e-continuous function and (Y, σ) is θ-T_2, then $G(f)$ is (θ, e)-closed.*

Proof. Let $(x, y) \in (X \times Y)\backslash G(f)$. Then $f(x) \neq y$. Since Y is θ-T_2, there exist θ-open sets V and W in Y such that $f(x) \in V$, $y \in W$ and $V \cap W = \emptyset$. Since f is faintly e-continuous, $f^{-1}(V) \in eO(X, x)$. Take $U = f^{-1}(V)$. We have $f(U) \subset V$. Therefore, we obtain $f(U) \cap W = \emptyset$. This shows that $G(f)$ is (θ, e)-closed. □

Theorem 4.8. *Let $f : (X, \tau) \to (Y, \sigma)$ have a (θ, e)-closed graph $G(f)$. If f is a faintly e-continuous injection, then (X, τ) is e-T_2.*

Proof. Let x and y be any two distinct points of X. Then since f is injective, we have $f(x) \neq f(y)$. Then, we have $(x, f(y)) \in (X \times Y)\backslash G(f)$. By Lemma 4.6, $U \in eO(X)$ and $V \in \sigma_\theta$ such that $(x, f(y)) \in U \times V$ and $f(U) \cap V = \emptyset$. Hence $U \cap f^{-1}(V) = \emptyset$ and $y \notin U$. Since f is faintly e-continuous, there exists $W \in eO(X, y)$ such that $f(W) \subset V$. Therefore, we have $f(U) \cap f(W) = \emptyset$. Since f is injective, we obtain $U \cap W = \emptyset$. This implies that (X, τ) is e-T_2. □

Definition 19. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to have an ec-closed graph if for each $(x, y) \in (X \times Y)\backslash G(f)$, there exists $U \in eO(X, x)$ and an open set V of Y containing y such that $(U \times cl(V)) \cap G(f) = \emptyset$.*

Lemma 4.9. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function. Then its graph $G(f)$ is ec-closed in $X \times Y$ if and only if for each point $(x, y) \in (X \times Y) \setminus G(f)$, there exist $U \in eO(\tau)$ and $V \in \sigma$ containing x and y, respectively, such that $f(U) \cap cl(V) = \emptyset$.*

Proof. It is an immediate consequence of Definition 19. □

Theorem 4.10. *If $f : (X, \tau) \to (Y, \sigma)$ is a surjective function with an ec-closed graph, then (Y, σ) is Hausdorff.*

Proof. Let y_1 and y_2 be any distinct points of Y. Then since f is surjective, there exists $x_1 \in X$ such that $f(x_1) = y_1$; hence $(x_1, y_2) \in (X \times Y) \setminus G(f)$. Since $G(f)$ is ec-closed, there exist $U \in eO(X, x_1)$ and an open set V of Y containing y_2 such that $f(U) \cap cl(V) = \emptyset$. Therefore, we have $y_1 = f(x_1) \in f(U) \subset Y \setminus cl(V)$. Then there exists an open set H of Y such that $y_1 \in H$ and $H \cap V = \emptyset$. Moreover, we have $y_2 \in V$ and V is open in Y. This shows that Y is Hausdorff. □

Theorem 4.11. *If $f : (X, \tau) \to (Y, \sigma)$ has an (θ, e)-closed graph, then it has an ec-closed graph.*

Proof. Let $x \in X$ and $y \neq f(x)$. Then $(x, y) \in (X \times Y) \backslash G(f)$. By Lemma 4,6, there exist $U \in eO(X, x)$ and a θ-open set V containing y such that $f(U) \cap V = \emptyset$. Since V is θ-open, there exists an open set V_0 such that $y \in V_0 \subset cl(V_0) \subset V$ so that $f(U) \cap cl(V_0) = \emptyset$. It follows from Lemma 4.9 that the graph of f is ec-closed. □

Corollary 4.12. *If $f : (X, \tau) \to (Y, \sigma)$ is faintly e-continuous and (Y, σ) is θ-T_2, then f has an ec-closed graph.*

Proof. The proof follows from Theorems 4.7 and 4.11. □

Theorem 4.13. *If $f : (X, \tau) \to (Y, \sigma)$ has the (θ, e)-closed graph , then $f(K)$ is closed in (Y, σ) for each subset K which is e-compact relative to X.*

Proof. Suppose that $y \notin f(K)$. Then $(x, y) \notin G(f)$ for each $x \in K$. Since $G(f)$ is (θ, e)-closed, there exist $U_x \in eO(X, x)$ and a θ-open set V_x of Y containing y such that $f(U_x) \cap V_x = \emptyset$ by Lemma 4.6. The family $\{U_x : x \in K\}$ is a cover of K by e-open sets. Since K is e-compact relative to (X, τ), there exists a finite subset K_0 of K such that $K \subset \bigcup\{U_x : x \in K_0\}$. Set $V = \bigcap\{V_x : x \in K_0\}$. Then V is a θ-open set, (hence open) in Y containing y. Therefore, we have $f(K) \cap V \subset [\bigcup_{x \in K_0} f(U_x)] \cap V \subset \bigcup_{x \in K_0} [f(U_x) \cap V] = \emptyset$. It follows that $y \notin cl(f(K))$. Therefore, $f(K)$ is closed in (Y, σ). □

Corollary 4.14. *If $f : (X, \tau) \to (Y, \sigma)$ is faintly e-continuous and (Y, σ) is θ-T_2, then $f(K)$ is closed in (Y, σ) for each subset K which is e-compact relative to (X, τ).*

Proof. The proof follows from Theorems 4.7 and 4.13. □

Theorem 4.15. *If $f, g : X \to Y$ are faintly e-continuous functions, X is a T_e-space and Y is θ-T_2, then $E = \{x \in X : f(x) = g(x)\}$ is closed in X.*

Proof. Suppose that $x \notin E$. Then $f(x) \neq g(x)$. Since Y is θ-T_2, there exist $V \in \theta O(Y, f(x))$ and $W \in \theta O(Y, g(x))$ such that $V \cap W = \emptyset$. Since f and g are faintly e-continuous, there exist $U \in eO(X, x)$ and $G \in eO(X, x)$ such that $f(U) \subset V$ and $g(G) \subset W$. Set $D = U \cap G$. Since X is a T_e-space, then $D \cap E = \emptyset$ with D a subset open such that $x \in D$. Then $x \notin cl(E)$ and thus E is closed in X. □

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