SARAJEVO JOURNAL OF MATHEMATICS
Vol.7 (20) (2011), 149–152

THE NUMBER OF IDEMPOTENTS IN COMMUTATIVE GROUP RINGS OF PRIME CHARACTERISTIC

P. V. DANCHEV

Abstract. Suppose R is a commutative unitary ring of prime characteristic p and G is a multiplicative abelian group. The cardinality of the set $\mathrm{id}(RG)$ consisting of all idempotent elements in the group ring RG, is explicitly calculated only in terms associated with R and G or their sections.

1. Introduction

Throughout this brief paper, we will assume that R is a commutative unitary ring (i.e., a commutative ring containing an identity element 1) of prime characteristic, for instance p, and G is an abelian group written multiplicatively as is customary when studying group rings. As usual, RG denotes the group ring of G over R, $G_0 = \coprod_p G_p$ denotes the torsion subgroup of G with p-primary component G_p and, for any natural number k, ζ_k denotes the primitive kth root of unity. Likewise, under the ordinary algebraic operations, $R[\zeta_k]$ designates the free R-module over R, generated algebraically as a ring (namely as an overring of R) by ζ_k, with dimension equal to $[R[\zeta_k] : R]$. All other unexplained notions and notations are standard and follow those from [3].

Traditionally, we put $\mathrm{id}(R)$ and $\mathrm{id}(RG)$ to be the sets of all idempotents in R and RG, respectively. Since 0 and 1 are trivial examples of such elements, the inequalities $|\,\mathrm{id}(RG)| \geq |\,\mathrm{id}(R)| \geq 2$ hold taking into account that $\mathrm{id}(R) \subseteq \mathrm{id}(RG)$. A question which naturally arises in some aspects of commutative group rings theory (see, e.g., [1] and [2]) is to compute in an explicit form the cardinality $|\,\mathrm{id}(RG)|$ (in other words, the number of all idempotents being finite or infinite) in a commutative group ring RG.

For an arbitrary commutative unitary ring L, it was proved in [4] that $|\,\mathrm{id}(LG)| = 2$ if, and only if, $|\,\mathrm{id}(L)| = 2$ and $\mathrm{supp}(G) \cap \mathrm{inv}(L) = \emptyset$, where

2010 *Mathematics Subject Classification.* 16S34, 16U60, 20K20, 20K21.
Key words and phrases. Groups, rings, idempotents, indecomposable rings, sets, cardinalities.

$\operatorname{supp}(G) = \{p : G_p \neq 1\}$ and $\operatorname{inv}(L) = \{p : p.1 \in L^*\}$ and L^* denotes the unit group of L (that is the set of all invertible elements in L). However, the mentioned paper does not give any strategy for computing $|\operatorname{id}(LG)|$ in the non-trivial case.

So, the purpose of the present short article is to do that but only for rings of prime characteristic. Our computations will mainly depend upon on $\operatorname{id}(R)$, of course, as well as on G_0 and its sections.

2. The main result

We begin with some preliminaries. The first statement appeared in both [1] and [2].

Lemma 2.1. ([1], [2]) *Let L be a commutative unitary ring. Then $L = \oplus_{1\leq i\leq n} L_i$, where each L_i is an indecomposable unitary subring of L, if and only if $|\operatorname{id}(L)| = 2^n$.*

Theorem 2.2. ([5]) *Let P be a commutative indecomposable unitary ring and let F be a finite abelian group of $exp(F) \in P^*$. Then*

$$PF \cong \oplus_{d/exp(F)} \oplus_{a(d)} P[\zeta_d],$$

where $a(d) = \frac{|\{a\in F: order(a)=d\}|}{[P[\zeta_d]:P]}$, and $\sum_{d/exp(F)} a(d)[P[\zeta_d] : P] = |F|$.

Proposition 2.3. ([6]) *Let P be a commutative indecomposable unitary ring and $k \geq 1$. Then $P[\zeta_k]$ is also a commutative indecomposable unitary ring.*

We now have all the machinery needed to prove the following.

Theorem 2.4. *Suppose R is a commutative unitary ring of prime characteristic p and G is an abelian group. Then the following hold:*

(1) $|\operatorname{id}(RG)| = |\operatorname{id}(R)|$ *if* $G_0 = G_p$;

(2) $|\operatorname{id}(RG)| = |\operatorname{id}(R)|.|G_0/G_p|$ *if either* $|\operatorname{id}(R)| \geq \aleph_0$ *or* $|G_0/G_p| \geq \aleph_0$;

(3) $|\operatorname{id}(RG)| = 2^{\sum_{d/exp(G_0/G_p)} \sum_{1\leq i\leq log_2|id(R)|} a_i(d)}$ *if both* $|id(R)| < \aleph_0$ *and* $|G_0/G_p| < \aleph_0$,

where $a_i(d) = \frac{|\{g\in G_0/G_p: order(g)=d\}|}{[R_i[\zeta_d]:R_i]}$ with $R_i = Re_i$ and $\{e_i\}_{1\leq i\leq n}$ the system of primitive idempotents of R; $n = log_2|id(R)|$.

Proof. Letting $e \in \operatorname{id}(RG)$, we have $e \in KG$ for some finitely generated subring K of R. Thus $K = R_1 \times \cdots \times R_n$ for some indecomposable subrings R_i of R with $1 \leq i \leq n$, and hence $KG = R_1G \times \cdots \times R_nG$. One may observe that $\operatorname{id}(KG) = \operatorname{id}(R_1G) \times \cdots \times \operatorname{id}(R_nG)$ in a set-theoretic sense.

That is why, furthermore, we may assume that R is finitely generated and even indecomposable.

Now, taking into account [4], every idempotent e from RG is either an idempotent from R, i.e. belongs to $\mathrm{id}(R)$, or is non-trivial and lies in $R(\coprod_{q\neq p} G_q)$ provided $\mathrm{id}(R) = \{0,1\}$. In fact, there are idempotents of the form $e = \frac{1}{|C|}\sum_{c\in C} c$, where $C \leq \coprod_{q\neq p} G_q \leq G_0$ is a finite subgroup such that $|C|$ inverts in R.

If now G is p-mixed, that is $G_0 = G_p$, it is readily seen that $\mathrm{supp}(G) \cap \mathrm{inv}(R) = \emptyset$ since $\mathrm{inv}(R)$ contains all primes but p. Consequently, applying the aforementioned result of [4], the only idempotents in RG are those from R and we are finished in that case.

Next, if one of $\mathrm{id}(R)$ or $G_0/G_p \cong \coprod_{q\neq p} G_q$ is infinite, we observe via the above that $|\,\mathrm{id}(RG)| \geq \aleph_0$. Notice that $\mathrm{id}(RG) = \mathrm{id}(RG_0)$ since $\mathrm{supp}(G) = \mathrm{supp}(G_0)$. But $RG_0 = R(\coprod_{q\neq p} G_q)G_p$ and hence as in the previous paragraph $\mathrm{id}(RG_0) = \mathrm{id}(R(\coprod_{q\neq p} G_q)) = \mathrm{id}(R(G_0/G_p))$. Therefore, we have $|\,\mathrm{id}(RG)| = \max(|\,\mathrm{id}(R)|, |M|)$, where M is the set of all finite subgroups F of $\coprod_{q\neq p} G_q$. But $\coprod_{q\neq p} G_q = \cup_{F\in M} F$ and, if G_0/G_p is infinite, this ensures that $|\coprod_{q\neq p} G_q| = |M|$. Moreover, if $\mathrm{id}(R)$ is infinite, R contains an infinite number of indecomposable subrings which number equals to $|\,\mathrm{id}(R)|$. So, we are done in this case.

Let us assume now that both $\mathrm{id}(R)$ and $G_0/G_p \cong \coprod_{q\neq p} G_q$ are finite. Since $\coprod_{q\neq p} G_q$ is pure in G_0 being its direct factor and G_0 is pure in G, it follows that $\coprod_{q\neq p} G_q$ is pure in G. So, one may write $G = (\coprod_{q\neq p} G_q) \times M \cong (G_0/G_p) \times M$ for some subgroup $M \cong G/\coprod_{q\neq p} G_q$ of G which is obviously p-mixed because $M_0 = (G/\coprod_{q\neq p} G_q)_0 = G_0/\coprod_{q\neq p} G_q \cong G_p$. Therefore $RG \cong (R(G_0/G_p))M$ and from point (a) it is easily verified that $\mathrm{id}(RG) = \mathrm{id}(R(G_0/G_p)) = \mathrm{id}(R(\coprod_{q\neq p} G_q))$.

On the other hand, since $\mathrm{id}(R)$ is finite, R can be decomposed as follows:

$$R = \oplus_{1\leq i\leq n} R_i,$$

where each R_i is indecomposable with the same characteristic p and $1 \leq i \leq n = \log_2 |\,\mathrm{id}(R)|$ by Lemma 2.1. Furthermore, we deduce that

$$R\Big(\coprod_{q\neq p} G_q\Big) \cong \oplus_{1\leq i\leq n} R_i\Big(\coprod_{q\neq p} G_q\Big).$$

Since $\coprod_{q\neq p} G_q$ is of finite exponent which inverts in R, according to Theorem 2.2, we have

$$R_i\Big(\coprod_{q\neq p} G_q\Big) \cong \oplus_{d/exp(G_0/G_p)} \oplus_{a_i(d)} R_i[\zeta_d]$$

where $a_i(d) = \frac{|\{g\in\coprod_{q\neq p} G_q : order(g)=d\}|}{[R_i[\zeta_d]:R_i]}$. But due to Proposition 2.3, the ring extensions $R_i[\zeta_d]$ are also indecomposable and their number is $\sum_{d/exp(G_0/G_p)} a_i(d)$. That is why

$$R\Big(\coprod_{q\neq p} G_q\Big) \cong \oplus_{1\leq i\leq n} \oplus_{d/exp(G_0/G_p)} \oplus_{a_i(d)} R_i[\zeta_d]$$
$$= \oplus_{d/exp(G_0/G_p)} \oplus_{1\leq i\leq n} \oplus_{a_i(d)} R_i[\zeta_d].$$

Thus we conclude that the number of all irreducible summands equals to $\sum_{d/exp(G_0/G_p)} \sum_{1\leq i\leq log_2|\,\mathrm{id}(R)|} a_i(d)$. Finally, we apply Lemma 2.1 again to obtain the desired equality, as asserted. □

A question which immediately arises is the following.

Problem. For any commutative unitary ring L and any abelian group G calculate $\mathrm{id}(LG)$.

References

[1] P. Danchev, *Warfield invariants in commutative group rings,* J. Algebra Appl., 8 (6) (2009), 829–836.

[2] P. Danchev, *Maximal divisible subgroups in modular group rings of p-mixed abelian groups,* Bull. Braz. Math. Soc., 41 (1) (2010), 63–72.

[3] G. Karpilovsky, *Commutative Group Algebras*, Marcel Dekker, New York, 1983.

[4] W. May, *Group algebras over finitely generated rings,* J. Algebra, 39 (1976), 483–511.

[5] T. Mollov, and N. Nachev, *Unit groups of commutative group rings,* Commun. Algebra, 34 (10) (2006), 3835–3857.

[6] N. Nachev, *Nilpotent elements and idempotents in commutative group rings,* Commun. Algebra, 33 (10) (2005), 3631–3637.

(Received: November 15, 2010)
(Revised: April 7, 2011)

Department of Mathematics
Plovdiv University
Plovdiv 4000, Bulgaria
E–mail: pvdanchev@yahoo.com