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SHEAVES OF PARAGRADED RINGS

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Dedicated to Professor Veselin Perić

ABSTRACT. In this paper we deal with sheaves of the category of paragraded rings with the same set of grades Δ, in short, the category of paragraded rings of type Δ, denoted by R_Δ^P. The definition of a sheaf of category R_Δ^P is the same as for any other category, but we are interested in the paragraded structure sheaf, as we named it here, and so, in some way we introduce the theory of paragraded structures into algebraic geometry. For this purpose we need to deal with homogeneous ideals, particularly, we need to introduce the prime spectrum of $R \in \mathrm{obj}(R_\Delta^P)$ and to analyze the localization of R.

1. Introduction

Since the notion of paragraded rings relies on the notion of paragraded groups, we will first say something about paragraded groups ([11]).

It is well known that graded groups are not closed with respect to direct product in the sense that the direct product of homogeneous parts is not the homogeneous part of that direct product. Therefore, paragraded groups ([11]) were introduced, as a generalization of graded groups in order to overcome this problem.

Definition 1.1 ([11]). *The map* $\pi : \Delta \to \mathrm{Sg}(G)$, $\pi(\delta) = G_\delta$ $(\delta \in \Delta)$, *of a partially ordered set* $(\Delta, <)$, *which is a complete semi-lattice from bellow and inductively ordered from beyond, into the set* $\mathrm{Sg}(G)$ *of subgroups of the group* G, *is called a* paragraduation *if it satisfies the following six-axiom system:*

i) $\pi(0) = G_0 = \{e\}$, *where* $0 = \inf \Delta$; $\delta < \delta' \Rightarrow G_\delta \subseteq G_{\delta'}$;

Remark 1.2. $H = \bigcup_{\delta\in\Delta} G_\delta$ is called *the homogeneous part* of G with respect to π.

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Remark 1.3. If $x \in H$, we say that $\delta(x) = \inf\{ \delta \in \Delta \mid x \in G_\delta \}$ is the grade of x. We have $\delta(x) = 0$ iff $x = e$. Elements $\delta(x)$, $x \in H$, are called *principal grades* and they form a set which we will denote by Δ_p.

ii) $\theta \subseteq \Delta \Rightarrow \bigcap_{\delta\in\theta} G_\delta = G_{\inf\theta}$;
iii) *If* $x, y \in H$ *and* $yx = zxy$, *then* $z \in H$ *and* $\delta(z) \leq \inf\{\delta(x), \delta(y)\}$;
iv) *The homogeneous part* H *is a generating set of* G;
v) *Let* $A \subseteq H$ *be a subset such that for all* $x, y \in A$ *there exists an upper bound for* $\delta(x), \delta(y)$. *Then there exists an upper bound for all* $\delta(x)$, $x \in A$;
vi) G *is generated by* H *with the set of* H*-internal and left commutation relations* (*see* [11]).

The group with paragraduation is called paragraded group.

Definition 1.4 ([11])**.** *The map* π *from Definition* 1.1 *which satisfies axioms* $i) - v)$ *and the following axiom:*

vi') *Let* $\delta_1, \dots, \delta_s \in \Delta_p$ *be pairwise incomparable and let* $x_i, x_i' \in H$ $(i = 1, \dots, s)$ *be such that* $x_1 \cdot \dots \cdot x_s = x_1' \cdot \dots \cdot x_s'$ *and* $x_i, x_i' \in G_{\delta_i}$ *for all* $i = 1, \dots, s$. *Then* $\delta(x_i^{-1}x_i') < \delta_i$.

is called extragraduation. *The group with extragraduation is called* extragraded group.

Remark 1.5 ([11])**.** It should be noticed that the fifth axiom is equivalent to the axiom:

v') Let $A \subseteq H$ be a subset such that for all $x, y \in A$ we have $xy \in H$. Then there exists $\delta \in \Delta$ such that $A \subseteq G_\delta$;

The following theorems are of great importance.

Theorem 1.6 ([11])**.** *Every extragraded group is a paragraded group.*

Theorem 1.7 ([11])**.** *The direct product of paragraded groups is a paragraded group and the homogeneous part of the product is the product of homogeneous parts of the components.*

We will make use of the maps between the paragraded groups as well. Let G_1 and G_2 be two paragraded groups with sets of grades Δ_1 and Δ_2, and homogeneous parts H_1 and H_2 respectively.

Definition 1.8 ([9])**.** *We say that a homomorphism* $f : G_1 \to G_2$ *is* quasi-homogeneous *if*

$$(\forall x \in H_1)\ f(x) \in H_2.$$

As we will see, we shall confine ourselves to the case $\Delta_1 = \Delta_2$.

Definition 1.9 ([9])**.** *The ring* $(R, +, \cdot)$ *is called paragraded if its additive group* $(R, +)$ *is a paragraded group, with set of grades* Δ*, and if*

$$(\forall \xi, \eta \in \Delta)(\exists \zeta \in \Delta)\ R_\xi R_\eta \subseteq R_\zeta.$$

Definition 1.10 ([9])**.** *If* R *is a paragraded ring, then the map* $(\xi, \eta) \to \xi\eta$ *from* $\Delta \times \Delta$ *to* Δ *is called a* Δ-multiplication of grades *if the following holds:*

a) $R_\xi R_\eta \subseteq R_{\xi\eta}$;
b) $(\forall \xi, \xi', \eta, \eta' \in \Delta)\ \xi \leq \xi' \wedge \eta \leq \eta' \Rightarrow \xi\eta \leq \xi'\eta'$.

If R is a paragraded ring, then there exists $\zeta = \sup\{\ \delta(z) \mid z \in R_\xi R_\eta\ \}$. If we put $\zeta = \xi\eta$, we will get Δ-multiplication and we call it the *minimal multiplication* ([9]).

Definition 1.11 ([9])**.** *Let* R_1 *and* R_2 *be two paragraded rings and* $f : R_1 \to R_2$ *a homomorphism. We say that this homomorphism is* quasihomogeneous *if it is a* quasihomogeneous *homomorphism from a paragraded group* $(R_1, +)$ *to a paragraded group* $(R_2, +)$.

According to Theorem 1.7, the following theorem holds.

Theorem 1.12 ([11])**.** *The direct product of paragraded rings is a paragraded ring and the homogeneous part of the product is the product of homogeneous parts of the components.*

Paragraded rings with the same set of grades Δ, together with morphisms

$$\{\ f \in \hom(R, R') \mid f(R_\delta) \subseteq R'_\delta,\ \delta \in \Delta\ \},$$

form the category R^P_Δ. We call this category the *category of paragraded rings of type* Δ ([4]).

We will need some basic facts concerning sheaves. We begin with the definition of a *protosheaf.*

Definition 1.13. *A continuous map* $p : E \to X$*, where* E *and* X *are topological spaces is called a* local homeomorphism *if for every* $e \in E$ *there exists a neighborhood* S *of the element* e*, which is called a* sheet*, such that* $p(S)$ *is an open set and* $p|_S : S \to p(S)$ *a homeomorphism. The triple* (E, p, X) *is called a* protosheaf *if the local homeomorphism* p *is surjective. The space* E *is called a* sheaf space*,* p *its* projection*, and* X *a* base space*. For every* $x \in X$*,* $E_x = p^{-1}(x)$ *is called the* stalk *over* x*.*

Definition 1.14. *Let* X *be a topological space. A* presheaf *of a category* R^P_Δ *over* X *is a function* $\mathcal{F}$ *which to every open set* $U \subseteq X$ *assigns* $\mathcal{F}(U) \in R^P_\Delta$ *and to every pair of open sets* $V \subseteq U$ *assigns a morphism, called a* restriction*,*

$$\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$$

for which

$$\rho_U^U = 1_U$$

and

$$\rho_W^U = \rho_W^V \rho_V^U$$

for all open $W \subseteq V \subseteq U$. In other words, a presheaf of category R_Δ^P is a contravariant functor $\mathcal{F} : \mathcal{U} \to R_\Delta^P$, where $\mathcal{U}$ is the category of open sets of X.

It is known that a presheaf $\mathcal{F}$ is a sheaf if $\mathcal{F}$ satisfies an equalizer condition. For more details, see [16] and [15].

2. The paragraded prime spectrum

Let R be a paragraded ring with set of grades Δ.

Definition 2.1. *A homogeneous ideal P of the ring R is called* prime homogeneous *if $P \neq R$ and if for every two homogeneous ideals I, J from $IJ \subseteq P$ follows that $I \subseteq P$ or $J \subseteq P$.*

Theorem 2.2. *A homogeneous ideal P of the ring R is prime homogeneous iff*

$$aRb \subseteq P \Rightarrow a \in P \vee b \in P,$$

for every $a, b \in H$.

Proof. Let us assume that the ideal P is prime homogeneous and that $aRb \subseteq P$, where $a, b \in H$. Let $A = RaR$ and $B = RbR$. Then A and B are homogeneous ideals, for a and b are homogeneous elements. Now, $AB \subseteq R(aRb)R \subseteq RPR \subseteq P$, so, since P is a prime homogeneous, $A \subseteq P$ or $B \subseteq P$. If $A \subseteq P$, then $a \in P$, and if $B \subseteq P$, then $b \in P$, which is easily verified as in the unparagraded case. The reverse statement is obvious. □

Definition 2.3. *The set of all prime homogeneous ideals of the ring R is called* the paragraded spectrum *and denoted by* $\mathrm{Spec}^{\mathrm{pgr}}(R)$.

3. Localization of paragraded rings

Let R be a paragraded ring with the set of grades Δ and with the homogeneous part H. We assume throughout this section that R is also a commutative ring with $1 \in H$.

We begin with a simple result.

Lemma 3.1. *If S is a multiplicative system of R, then $S \cap H$ is also a multiplicative system.*

Proof. If $a, b \in S \cap H$, then $a, b \in S$ and $a, b \in H$. Since S is a multiplicative system, we have $ab \in S$. But we also have $ab \in H$, because for the paragraded ring $H^2 \subseteq H$ holds. □

Let us now view R only as a ring and let S be its multiplicative system. Then, as we know, we may construct the ring of fractions $S^{-1}R$. According to Lemma 3.1, for a multiplicative system we may also take the set $S \cap H$. Denote it again by S.

Now, observe the map

$$\pi_{S^{-1}R} : \delta \to \{ \frac{a}{s} \mid a \in R_\xi, s \in R_\eta, \eta\delta = \xi \} \qquad (\delta \in \Delta).$$

Proposition 3.2. *If the minimal multiplication is associative and commutative, then the map $\pi_{S^{-1}R}$ is the paragraduation of the ring $S^{-1}R$.*

Proof. First we prove that $\pi_{S^{-1}R}(\delta)$ is a subgroup of $(S^{-1}R, +)$, $\delta \in \Delta$. Let $\frac{a}{s}, \frac{b}{t} \in \pi_{S^{-1}R}(\delta)$. Then there exist $\xi_1, \xi_2, \eta_1, \eta_2 \in \Delta$ such that $a \in R_{\xi_1}$, $b \in R_{\xi_2}$, $s \in R_{\eta_1}$, $t \in R_{\eta_2}$, and

$$\eta_1\delta = \xi_1 \tag{3.1}$$

$$\eta_2\delta = \xi_2. \tag{3.2}$$

Now we have $at \in R_{\xi_1}R_{\eta_2} \subseteq R_{\xi_1\eta_2}$, $bs \in R_{\xi_2}R_{\eta_1} \subseteq R_{\xi_2\eta_1}$, hence from the associativity and commutativity of minimal multiplication and from one of the identities (3.1) or (3.2) it follows that $at, bs \in R_{\eta_1\eta_2\delta}$, so $at+bs \in R_{\eta_1\eta_2\delta}$. Also, $st \in R_{\eta_1}R_{\eta_2} \subseteq R_{\eta_1\eta_2}$. Hence, $\frac{a}{s} + \frac{b}{t} \in \pi_{S^{-1}R}(\delta)$.

Axioms of paragraduation are easy to check if we take the properties of minimal multiplication and axioms of paragraduation which are satisfied for the ring R into account. For example, let us prove that from $\delta_1 < \delta_2$ it follows that $\pi_{S^{-1}R}(\delta_1) \subseteq \pi_{S^{-1}R}(\delta_2)$, for $\delta_1, \delta_2 \in \Delta$. Let $\frac{a}{s} \in \pi_{S^{-1}R}(\delta_1)$. Then there exist $\xi, \eta \in \Delta$ such that $a \in R_\xi$, $s \in R_\eta$, and $\eta\delta_1 = \xi$. Since $\delta_1 < \delta_2$, we have $\eta\delta_1 < \eta\delta_2$, i.e. $\xi < \eta\delta_2$. Hence, $R_\xi \subseteq R_{\eta\delta_2}$, so $a \in R_{\eta\delta_2}$, i.e. $\frac{a}{s} \in \pi_{S^{-1}R}(\delta_2)$.

Now let us prove that

$$(\forall \xi, \eta \in \Delta)(\exists \zeta \in \Delta)\ \pi_{S^{-1}R}(\xi)\pi_{S^{-1}R}(\eta) \subseteq \pi_{S^{-1}R}(\zeta)$$

holds. If $\frac{a}{s} \in \pi_{S^{-1}R}(\xi)$, and $\frac{b}{t} \in \pi_{S^{-1}R}(\eta)$, then there exist $\lambda_1, \lambda_2, \theta_1, \theta_2 \in \Delta$ such that $a \in R_{\lambda_1}$, $s \in R_{\theta_1}$, $b \in R_{\lambda_2}$, $t \in R_{\theta_2}$, $\theta_1\xi = \lambda_1$ and $\theta_2\eta = \lambda_2$. Now, $ab \in R_{\lambda_1}R_{\lambda_2} \subseteq R_{\lambda_1\lambda_2}$ and $st \in R_{\theta_1}R_{\theta_2} \subseteq R_{\theta_1\theta_2}$. Since

$$\theta_1\theta_2\xi\eta = (\theta_1\xi)(\theta_2\eta) = \lambda_1\lambda_2,$$

we have $\frac{ab}{st} \in \pi_{S^{-1}R}(\xi\eta)$. □

4. Sheaves of paragraded rings and the paragraded structure sheaf

Definition 4.1. *If X and E are topological spaces and $p : E \to X$ is a continuous map, then the triple $\mathcal{S} = (E, p, X)$ is called an* etale-sheaf of the category R^P_Δ *if:*

i) (E, p, X) *is protosheaf;*
ii) *For all* $x \in X$, $E_x = p^{-1}(x) \in R^P_\Delta$;
iii) *Inversion, addition and multiplication are continuous.*

Remark 4.2. Inversion is defined by $e \to -e$, $e \in E$. Addition and multiplication are operations defined by $(e, e') \to e+e'$ and $(e, e') \to ee'$, respectively, where $(e, e') \in \{ (e, e') \in E \times E \mid p(e) = p(e') \}$ and the product topology is considered.

Definition 4.3. *If* $\mathcal{S} = (E, p, X)$ *is an etale-sheaf of the category* R^P_Δ *and* $U \subseteq X$ *is an open set, then a* section over U *is a continuous map* $\sigma : U \to E$ *such that* $p\sigma = 1_U$. *If* $U = X$, *then a section over* U *is called a* global section. *Define* $\Gamma(\emptyset, \mathcal{S}) = \{0\}$ *and* $\Gamma(U, \mathcal{S})$ *to be the set of all the sections* $\sigma : U \to E$ *if* $U \neq \emptyset$.

Proposition 4.4. *Let* $\mathcal{S} = (E, p, X)$ *be etale-sheaf of the category* R^P_Δ *and let* $\mathcal{F} = \Gamma(-, \mathcal{S})$.

i) $\mathcal{F}(U) \in R^P_\Delta$, *for every open* $U \subseteq X$.
ii) $\mathcal{F} = \Gamma(-, \mathcal{S})$ *is a presheaf of the category* R^P_Δ, *which is called* the sheaf of sections *of* $\mathcal{S}$.

Proof. i) First we prove that $\mathcal{F}(U) \neq \emptyset$, for every open $U \subseteq X$. If $U = \emptyset$, then $\mathcal{F}(U) = \{0\}$, according to definition, and hence $\mathcal{F}(\emptyset) \neq \emptyset$. Now assume that $\emptyset \neq U \subseteq X$ is an open set. If $x \in U$, pick $e \in E_x$ and a sheet S which contains e. Since p is an open map, $p(S) \cap U$ is an open neighborhood of x. The map $p(S) \to S \subseteq E$ is a section, and we denote by σ_S its restriction on $p(S) \cap U$. The family of such sets $p(S) \cap U$ forms an open cover of U, hence the maps σ_S may be glued together to give a section from $\mathcal{F}(U)$. Hence, $\mathcal{F}(U) \neq \emptyset$, for every open $U \subseteq X$.

Let $\sigma, \tau \in \mathcal{F}(U)$. Then the map $\sigma+\tau$, defined by $(\sigma+\tau)(x) := \sigma(x)+\tau(x)$ is continuous, because, by definition, the maps σ and τ are continuous. Hence, $\sigma + \tau \in \mathcal{F}(U)$. For the same reason, from $\sigma \in \mathcal{F}(U)$ it follows that $-\sigma \in \mathcal{F}(U)$, which is defined in an obvious way. If we define $\sigma\tau(x) := \sigma(x)\tau(x)$, then $\sigma\tau$ will also belong to $\mathcal{F}(U)$.

Let $\sigma \in \mathcal{F}(U)$ be the section for which $\sigma(x)$ is a homogeneous element, for all $x \in U$. Hence, for all $x \in U$, there exist $\delta \in \Delta$ and $y \in X$ such that $\sigma(x) \in \pi_y(\delta)$, for $\sigma(x) \in E$, and $E = \bigcup_{y \in X} E_y$, where by π_y we denoted the paragraduation of E_y. These types of sections are called *homogeneous sections*. Let us define the map $\pi : \Delta \to \mathrm{Sg}(\mathcal{F}(U))$ with

$$\pi(\delta) = \{ \sigma \in \mathcal{F}(U) \mid (\forall x \in U)\ \sigma(x) \in \pi_x(\delta) \}.$$

Now, let us observe $\pi(0)$. Then $\pi(0)$ is the set of all homogeneous sections σ for which $\sigma(x) \in \pi_x(0)$, for all $x \in U$. Since $\pi_y(0) = \{0\}$, for all $y \in X$, we have $\sigma(x) = 0$, for all $x \in U$, hence $\sigma = 0$.

Let $\delta_1 < \delta_2$, $\delta_1, \delta_2 \in \Delta$. Let us assume that $\sigma_1 \in \pi(\delta_1)$. Then, $\sigma_1(x) \in \pi_x(\delta_1)$, for all $x \in U$. Since $\pi_x(\delta_1) \subseteq \pi_x(\delta_2)$, $\sigma_1(x) \in \pi_x(\delta_2)$, for all $x \in U$, so $\sigma_1 \in \pi(\delta_2)$. Hence, $\pi(\delta_1) \subseteq \pi(\delta_2)$.

If $\theta \subseteq \Delta$, observe $\bigcap_{\delta\in\theta} \pi(\delta)$. If $\sigma \in \bigcap_{\delta\in\theta} \pi(\delta)$, then for every $x \in U$, $\sigma(x) \in \pi_x(\delta)$, for all $\delta \in \theta$, i.e. $\sigma(x) \in \bigcap_{\delta\in\theta} \pi_x(\delta) = \pi_x(\inf\theta)$, and hence $\sigma \in \pi(\inf\theta)$.

The third axiom of paragraduation is easily verified and the same holds for the fifth axiom. If $\sigma \in \mathcal{F}(U)$, then for all $x \in U$, $\sigma(x) \in \bigcap_{x\in U} E_x$, and $E_x = [H_x]$, hence homogeneous sections generate $\mathcal{F}(U)$.

Since $\mathcal{F}(U)$ is generated by homogeneous sections and every image of a homogeneous section is generated via relations of paragraduation, the same holds for $\mathcal{F}(U)$.

Let $\sigma_1 \in \pi(\xi)$ and $\sigma_2 \in \pi(\eta)$, where $\xi, \eta \in \Delta$. Then for all $x \in U$, $\sigma_1(x) \in \pi_x(\xi)$ and $\sigma_2(x) \in \pi_x(\eta)$. Hence, $\sigma_1(x)\sigma_2(x) \in \pi_x(\xi)\pi_x(\eta)$, for all $x \in U$. Since $E_x \in R^P_\Delta$, $\sigma_1(x)\sigma_2(x) \in \pi_x(\xi\eta)$, where $\xi\eta \in \Delta$ is a product by the laws of the minimal multiplication.

ii) $\mathcal{F}$ is a contravariant functor $\mathcal{U} \to R^P_\Delta$. □

It is easy to see that all the presheaves of the category R^P_Δ, with morphisms defined in an obvious manner, form a category, denoted by $\mathrm{pSh}(X, R^P_\Delta)$. The category of all the sheaves of the category R^P_Δ, denoted by $\mathrm{Sh}(X, R^P_\Delta)$, is defined to be the full subcategory of $\mathrm{pSh}(X, R^P_\Delta)$.

Proposition 4.5. *Let* $\mathcal{F}, \mathcal{G} \in \mathrm{pSh}(X, R^P_\Delta)$.

i) *If* U *is an open subset of* X *and*

$$(\mathcal{F} \oplus \mathcal{G})(U) = \mathcal{F}(U) \oplus \mathcal{G}(U),$$

then $\mathcal{F} \oplus \mathcal{G}$ *is a product and a coproduct of the category* $\mathrm{pSh}(X, R^P_\Delta)$.

ii) *If* $\mathcal{F}$ *and* $\mathcal{G}$ *are sheaves, then so is* $\mathcal{F} \oplus \mathcal{G}$ *and it is a product and a coproduct of the category* $\mathrm{Sh}(X, R^P_\Delta)$.

Proof. i) If $\delta \in \Delta$, let us define the map $\pi : \Delta \to \mathrm{Sg}(\mathcal{F}(U) \oplus \mathcal{G}(U))$ with

$$\pi(\delta) = \pi_{\mathcal{F}(U)}(\delta) \oplus \pi_{\mathcal{G}(U)}(\delta),$$

where by $\pi_{\mathcal{F}(U)}$ and $\pi_{\mathcal{G}(U)}$ we denote the paragraduations of $\mathcal{F}(U)$ and $\mathcal{G}(U)$, respectively. Then π is a paragraduation of the ring $\mathcal{F}(U) \oplus \mathcal{G}(U)$, (Theorem 1.12) hence $\mathcal{F}(U) \oplus \mathcal{G}(U) \in R^P_\Delta$. It was proven in paper [5] that the category of paragraded modules of type Δ, M^P_Δ, has products and coproducts and, hence, the same holds true for R^P_Δ. The rest of the proof of this assertion relies upon the mentioned result in an obvious way and we shall omit it.

ii) If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then the equalizer condition is satisfied for both $\mathcal{F}$ and $\mathcal{G}$. Hence, the following sequence is exact.

$$0 \to (\mathcal{F} \oplus \mathcal{G})(U) \xrightarrow{\alpha} \prod_{i \in I} (\mathcal{F} \oplus \mathcal{G})(U_i) \overset{\beta', \beta''}{\rightrightarrows} \prod_{(i,j) \in I \times I} (\mathcal{F} \oplus \mathcal{G})(U_{(i,j)}),$$

where $\alpha : \sigma \to (\sigma|_{U_i})$, $\beta' : \sigma_i \to \sigma_i|_{U_{(i,j)}}$, $\beta'' : \sigma_i \to \sigma_i|_{U_{(j,i)}}$. $\square$

Now, let us assume that $R \in R^P_\Delta$ is a commutative ring with $1 \in H$ and that the minimal multiplication is associative and commutative. Let $X = \mathrm{Spec}^{\mathrm{pgr}}(R)$. The same way as in the category of commutative rings, one can prove that X is a topological space with a base consisting of all the sets

$$D(s) = \{ \ P \in \mathrm{Spec}^{\mathrm{pgr}}(R) \mid s \notin P \ \}.$$

To every set $D(s)$ we may assign $S^{-1}R$, the ring of fractions with respect to $S = \{1, s, s^2, \dots\} \cap H$, which is an object of the category R^P_Δ according to Proposition 3.2. We denote this ring shortly by R_s. The following statement is true (for proof, see [16]).

Proposition 4.6. *Suppose that X is a topological space and $\mathcal{B}$ a base for the topology. Let $\mathcal{F}$ be the data of a presheaf given only for open sets of the base, which satisfies the condition that whenever $U = \bigcup_{i \in I} U_i$ with $U, U_i \in \mathcal{B}$, the sequence*

$$\mathcal{F}(U) \xrightarrow{f} \prod_{i \in I} \mathcal{F}(U_i) \overset{g,h}{\rightrightarrows} \prod_{(i,j) \in I \times I} \mathcal{F}(U_i \cap U_j)$$

is an equalizer diagram, where

$$f(s) = (\rho^U_{U_i}(s))_{i \in I},$$
$$g((s_i)_{i \in I}) = (\rho^{U_i}_{U_i \cap U_j}(s_i))_{(i,j) \in I \times I},$$
$$h((s_i)_{i \in I}) = (\rho^{U_j}_{U_i \cap U_j}(s_j))_{(i,j) \in I \times I}.$$

Then there is a sheaf $\mathcal{G}$ on X, unique up to isomorphism, such that

$$(\forall U \in \mathcal{B}) \ \Gamma(U, \mathcal{G}) = \mathcal{F}(U)$$

and the restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ and $\Gamma(U, \mathcal{G}) \to \Gamma(V, \mathcal{G})$ agree for open $V \subseteq U$.

The map $\mathcal{F} : \mathcal{B} \to R^P_\Delta$, defined by

$$\mathcal{F}(D(s)) = R_s,$$

where $\mathcal{B}$ is the base of topology of X, satisfies the conditions of the previous Proposition, and hence, there exists a sheaf $\mathcal{O}$ of the category R^P_Δ over the space $X = \mathrm{Spec}^{\mathrm{pgr}}(R)$, which is called *the paragraded structure sheaf*, for which we have

$$\Gamma(D(s), \mathcal{O}) = R_s.$$

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