SARAJEVO JOURNAL OF MATHEMATICS
Vol.7 (20) (2011), 163–175

ON A REFINEMENT OF HARDY'S INEQUALITIES VIA SUPERQUADRATIC AND SUBQUADRATIC FUNCTIONS

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ABSTRACT. Let A_k be an integral operator defined by

$$A_k f(x) := \frac{1}{K(x)} \int_{\Omega_2} k(x,y) f(y) d\mu_2(y),$$

where $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is a general nonnegative kernel,and $(\Omega_1, \Sigma_1, \mu_1)$, $(\Omega_2, \Sigma_2, \mu_2)$ are measure spaces with σ-finite measures and

$$K(x) := \int_{\Omega_2} k(x,y) d\mu_2(y), \quad x \in \Omega_1.$$

In this paper we define a functional as a difference between the right-hand side and the left-hand side of the refined Hardy type inequality with general measures and kernels using the notation of superquadratic and subquadratic functions inequality and study its properties, such as exponential and logarithmic convexity.

1. INTRODUCTION

In 1920 G.H. Hardy announced in [6] and proved in [7] the following result: Let $p > 1$ and $f \in L^p(0, \infty)$ be a nonnegative function, then

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(x)dx, \tag{1.1}$$

holds. This result is referred to as the classical Hardy's integral inequality. Since Hardy established inequality (1.1) it has been investigated and generalized in several directions. Recent results concerning refinements of multidimensional Hardy-type and Hardy's inequalities via superquadratic functions are given in [12] and [13]. Another important inequality is the following.

If $p > 1$ and f is a nonnegative function such that $f \in L^p(\mathbb{R}_+)$, then

$$\int_0^\infty \left(\int_0^\infty \frac{f(x)}{x+y} \, dx \right)^p dy \leq \left(\frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \right)^p \int_0^\infty f^p(y) \, dy. \tag{1.2}$$

2000 *Mathematics Subject Classification.* Primary 26D10, Secondary 26D15.

Key words and phrases. Kernels, measures, Hardy type operators, superquadratic function, subquadratic function, integral identities.

Inequality (1.2) is sometimes referred to as Hilbert's or Hardy-Hilbert's inequality. Here we focus on a class of superquadratic functions to obtain new results concerning Hardy-type inequalities. Our main tool in this paper is to use the notation of superquadratic and subquadratic functions introduced by Abramovich, Jameson and Sinnamon in [2] (see also[1] and [3]).

Definition 1. *A function $\varphi : [0,\infty) \to \mathbb{R}$ is superquadratic provided that for all $x \geq 0$ there exists a constant $C_x \in \mathbb{R}$ such that*

$$\varphi(y) - \varphi(x) - \varphi(|y-x|) \geq C_x (y-x) \quad \text{for all } y \geq 0.$$

We say that f is subquadratic if $-f$ is superquadratic.

Definition 2. *A function $f : [0,\infty) \to \mathbb{R}$ is superadditive provided $f(x+y) \geq f(x) + f(y)$ for all $x, y \geq 0$. If the reverse inequality holds, then f is said to be subadditive.*

Lemma 1.1. *Suppose $\varphi : [0,\infty) \to \mathbb{R}$ is continuously differentiable and $\varphi(0) \leq 0$. If φ' is superadditive or $\frac{\varphi'(x)}{x}$ is non-decreasing, then φ is superquadratic.*

In [8] K. Krulić et al. study some new weighted Hardy type inequalities on $(\Omega_1, \Sigma_1, \mu_1)$, $(\Omega_2, \Sigma_2, \mu_2)$, measure spaces with σ-finite measures with an integral operator A_k defined by

$$A_k f(x) := \frac{1}{K(x)} \int_{\Omega_2} k(x,y) f(y) d\mu_2(y), \tag{1.3}$$

where $f : \Omega_2 \to \mathbb{R}$ is a measurable function, $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is measurable and nonnegative kernel and

$$0 < K(x) := \int_{\Omega_2} k(x,y) d\mu_2(y) < \infty, \text{ for all } x \in \Omega_1. \tag{1.4}$$

In [4] the following refined Hardy type inequality is given:

Theorem 1.1. *Let u be a weight function, $k(x,y) \geq 0$. Assume that $\frac{k(x,y)}{K(x)} u(x)$ is locally integrable on Ω_1 for each fixed $y \in \Omega_2$. Define v by*

$$v(y) := \int_{\Omega_1} \frac{k(x,y)}{K(x)} u(x) d\mu_1(x) < \infty. \tag{1.5}$$

Suppose $I = [0,c)$, $c \leq \infty$, $\varphi : I \to \mathbb{R}$. If φ is a superquadratic function, then the inequality

$$\int_{\Omega_1} \varphi(A_k f(x)) u(x) d\mu_1(x) + \int_{\Omega_2} \int_{\Omega_1} \varphi\left(|f(y) - A_k f(x)|\right) \cdot \frac{u(x) k(x,y)}{K(x)} d\mu_1(x)\, d\mu_2(y) \leq \int_{\Omega_2} \varphi(f(y)) v(y) d\mu_2(y) \tag{1.6}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$*, such that* $Imf \subseteq I$*, where* A_k *is defined by* $(1.3) - (1.4)$.
If φ *is subquadratic, then the inequality sign in* (1.6) *is reversed.*

Now, we introduce some necessary notation and recall some basic facts about convex, log-convex functions (see e.g. [9]) as well as exponentially convex functions. This is a sub-class of convex functions introduced by Bernstein in [5] (see also [10], [11]).

Definition 3. *A function* $h\colon (a,b) \to \mathbb{R}$ *is* exponentially convex *if it is continuous and*

$$\sum_{i,j=1}^{n} t_i t_j h(x_i + x_j) \geq 0, \tag{1.7}$$

holds for every $n \in \mathbb{N}$ *and all sequences* $(t_n)_{n\in\mathbb{N}}$ *and* $(x_n)_{n\in\mathbb{N}}$ *of real numbers, such that* $x_i + x_j \in (a,b)$*,* $1 \leq i,j \leq n$*.*

Proposition 1.1. *Let* $h\colon (a,b) \to \mathbb{R}$*. The following are equivalent*

(i) *h is exponentially convex,*
(ii) *h is continuous and*

$$\sum_{i,j=1}^{n} t_i t_j h\left(\frac{x_i + x_j}{2}\right) \geq 0$$

for all $n \in \mathbb{N}$*, all sequences* $(t_n)_{n\in\mathbb{N}}$ *of real numbers, and all sequences* $(x_n)_{n\in\mathbb{N}}$ *in* (a,b)*,*
(iii) *h is continuous and*

$$\det\left[h\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{n} \geq 0,$$

for every $x_i \in (a,b)$*,* $i = 1,2,\ldots,n$*.*

Condition (iii) for $n = 2$ means that it holds

$$h(x_1)h(x_2) - h^2\left(\frac{x_1 + x_2}{2}\right) \geq 0,$$

hence, exponentially convex function is log-convex in the Jensen sense, and, being continuous, it is also a log-convex function.

Now, let us recall the Galvani theorem for log-convex functions.

Lemma 1.2. *Let positive function* $f\colon I \to \mathbb{R}$ *be log-convex and let* $a_1, a_2, b_1, b_2 \in I$ *be such that* $a_1 \leq b_1$*,* $a_2 \leq b_2$ *and* $a_1 \neq a_2$*,* $b_1 \neq b_2$*. Then the following inequality is valid*

$$\left[\frac{f(a_2)}{f(a_1)}\right]^{\frac{1}{a_2 - a_1}} \leq \left[\frac{f(b_2)}{f(b_1)}\right]^{\frac{1}{b_2 - b_1}}.$$

The following lemma gives us a characterization of log-convex functions.

Lemma 1.3. *The function Φ is log-convex on an interval I, if and only if for all $a, b, c \in I$, $a < b < c$,*

$$[\Phi(b)]^{c-a} \leq [\Phi(a)]^{c-b}[\Phi(c)]^{b-a} \text{ holds.}$$

The paper is organized as follows. After this introduction, in Section 2 we prove the Lagrange and the Cauchy-type mean value theorems and in Section 3 we study the exponential and logarithmic convexity of the difference between the left-hand and the right-hand side of the generalized Hardy type inequality (1.6).

Notations and Conventions. Throughout this paper we use bold letters to denote $n-$tuples of real numbers, e.g. $\mathbf{x} = (x_1, \ldots, x_n)$, or $\mathbf{y} = (y_1, \ldots, y_n)$. Also, we set $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^n$ and $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n$. Furthermore, the relations $<$, $\leq$, $>$, and $\geq$ are, as usual, defined componentwise. For example, for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we write $\mathbf{x} < \mathbf{y}$ if $x_i < y_i$, $i = 1, \ldots, n$. Furthermore, all functions are assumed to be measurable and expressions of the form $0 \cdot \infty$, $\frac{\infty}{\infty}$, and $\frac{0}{0}$ are taken to be equal to zero. Moreover, $u(\boldsymbol{x})$ denotes a weight function, i.e. a nonnegative and measurable function on the actual interval or more general set.

2. MEAN VALUE THEOREMS

Let us continue by defining a linear functional as a difference between the right-hand side and the left-hand side of the refined Hardy type inequality (1.6):

$$\begin{aligned} A(\varphi) = & \int_{\Omega_2} \varphi(f(y)) v(y) d\mu_2(y) - \int_{\Omega_1} \varphi(A_k f(x)) u(x) d\mu_1(x) \\ & - \int_{\Omega_2} \int_{\Omega_1} \varphi\left(|f(y) - A_k f(x)|\right) \frac{u(x)k(x,y)}{K(x)} d\mu_1(x)\, d\mu_2(y) \end{aligned} \tag{2.1}$$

It is clear, that if φ is superquadratic function, then $A(\varphi) \geq 0$.

Now, we give a mean value theorem. First, we state and prove the Lagrange-type mean value theorem.

Lemma 2.1. *Let $\varphi \in C^2(I)$, $I = (0, \infty)$ such that*

$$m \leq \frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi^2} \leq M, \text{ for all } \xi \in I$$

Consider the functions φ_1, φ_2 defined by

$$\varphi_1(x) = \frac{Mx^3}{3} - \varphi(x)\,,\ \varphi_2(x) = \varphi(x) - \frac{mx^3}{3}.$$

Then $\frac{\varphi_1'}{x}$ and $\frac{\varphi_2'}{x}$ are increasing functions . If $\varphi_i(0) = 0$, $i = 1, 2$, then they are superquadratic functions.

Theorem 2.1. *If* $\frac{\varphi'}{x} \in C^1(I)$ *and* $\varphi(0) = 0$, *then the following equality holds*

$$A(\varphi)=\frac{1}{3}\frac{\xi\varphi''(\xi)-\varphi'(\xi)}{\xi^2}\left(\int_{\Omega_2} f^3(y)v(y)d\mu_2(y)-\int_{\Omega_1}(A_kf(x))^3u(x)d\mu_1(x)\right.$$
$$\left.-\int_{\Omega_2}\int_{\Omega_1}|f(y)-A_kf(x)|^3\,\frac{u(x)k(x,y)}{K(x)}d\mu_1(x)\,d\mu_2(y)\right), \quad (2.2)$$

where A_kf, K *are defined by* $(1.3)-(1.4)$, *respectively.*

Proof. Suppose $\frac{\varphi'}{x}$ is bounded, that is $\min(\frac{\varphi'}{x}) = m$ and $\max(\frac{\varphi'}{x}) = M$. Then by applying Theorem 1.1 on functions φ_2, φ_2 from Lemma 2.1 the following two inequalities hold:

$$A(\varphi)\le\frac{M}{3}\left(\int_{\Omega_2} f^3(y)v(y)d\mu_2(y)-\int_{\Omega_1}(A_kf(x))^3u(x)d\mu_1(x)\right.$$
$$\left.-\int_{\Omega_2}\int_{\Omega_1}|f(y)-A_kf(x)|^3\,\frac{u(x)k(x,y)}{K(x)}d\mu_1(x)\,d\mu_2(y)\right) \quad (2.3)$$

and

$$A(\varphi)\ge\frac{m}{3}\left(\int_{\Omega_2} f^3(y)v(y)d\mu_2(y)-\int_{\Omega_1}(A_kf(x))^3u(x)d\mu_1(x)\right.$$
$$\left.-\int_{\Omega_2}\int_{\Omega_1}|f(y)-A_kf(x)|^3\,\frac{u(x)k(x,y)}{K(x)}d\mu_1(x)\,d\mu_2(y)\right).$$

By combining the above two inequalities we have that there exist $\xi \in (0,\infty)$ such that we get (2.2). Moreover if (for example) $\frac{\varphi'}{x}$ is bounded from above we have that (2.3) is valid. Of course (2.3) holds if $\frac{\varphi'}{x}$ is not bounded. □

Theorem 2.2. *If* $\frac{\varphi'}{x}, \frac{\psi'}{x} \in C^1(I)$, $\varphi(0) = \psi(0) = 0$, *then we have that*

$$\frac{A(\varphi)}{A(\psi)}=\frac{\xi\varphi''(\xi)-\varphi'(\xi)}{\xi\psi''(\xi)-\psi'(\xi)}, \qquad \xi \in I$$

provided the denominators are not equal to zero.

Proof. We consider a function k defined as $k = c_1\varphi - c_2\psi$, where c_1, c_2 are defined by

$$c_1 = A(\psi),\ c_2 = A(\varphi).$$

Then

$$\frac{k'}{x}=c_1\frac{\varphi'}{x}-c_2\frac{\psi'}{x}\in C^1(I),$$

after a short calculation we obtain that $A(k) = 0$ and

$$(c_1(\xi\varphi''(\xi) - \varphi'(\xi)) - c_2(\xi\psi''(\xi) - \psi'(\xi))) \\ \times \left(\int_{\Omega_2} f^3(y)v(y)d\mu_2(y) - \int_{\Omega_1} (A_k f(x))^3 u(x)d\mu_1(x) \right. \\ \left. - \int_{\Omega_2}\int_{\Omega_1} |f(y) - A_k f(x)|^3 \frac{u(x)k(x,y)}{K(x)} d\mu_1(x)d\mu_2(y) \right) = 0. \quad (2.4)$$

Since the right expression in (2.4) is different from 0, we conclude that

$$\frac{c_2}{c_1} = \frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi\psi''(\xi) - \psi'(\xi)} = \frac{A(\varphi)}{A(\psi)},$$

provided that the denominator is not zero. This completes the proof. □

As a special case of Theorems 2.1 and 2.2 we obtain the following results:

Example 2.1. Let $\Omega_1 = \Omega_2 = (\mathbf{0}, \boldsymbol{b})$, $\mathbf{0} < \boldsymbol{b} \leq \boldsymbol{\infty}$, replace $d\mu_1(x)$ and $d\mu_2(y)$ by the Lebesgue measures $d\mathbf{x}$ and $d\mathbf{y}$, respectively, and $k(\mathbf{x}, \mathbf{y}) = \mathbf{1}$, $\mathbf{0} \leq \mathbf{y} \leq \mathbf{x}$, $k(\mathbf{x}, \mathbf{y}) = \mathbf{0}$, $\mathbf{y} > \mathbf{x}$. Then $K(\mathbf{x}) = x_1 \cdots x_n$ and

$$A_k f(\mathbf{x}) = \frac{1}{x_1 \cdots x_n} \int_0^{x_1} \cdots \int_0^{x_n} f(\mathbf{y})d\mathbf{y}.$$

Moreover, replace $u(\mathbf{x})$ by $u(\mathbf{x})/x_1 \cdots x_n$ and $v(\mathbf{y})$ by $v(\mathbf{y})/y_1 \cdots y_n$, then v coincides with

$$v(\mathbf{t}) = t_1 \ldots t_n \int_{t_1}^{b_1} \cdots \int_{t_n}^{b_n} \frac{u(\boldsymbol{x})}{x_1^2 \ldots x_n^2} d\boldsymbol{x}, \boldsymbol{t} \in (\mathbf{0}, \mathbf{b})$$

and A which we now denote by $\widetilde{A}$ becomes

$$\widetilde{A}(\varphi) = \int_0^{b_1} \cdots \int_0^{b_n} v(\boldsymbol{x})\varphi(f(\boldsymbol{x}))\frac{d\boldsymbol{x}}{x_1 \cdots x_n} \\ - \int_0^{b_1} \cdots \int_0^{b_n} u(\boldsymbol{x})\varphi\left(\frac{1}{x_1 \cdots x_n}\int_0^{x_1} \cdots \int_0^{x_n} f(\boldsymbol{t})d\boldsymbol{t}\right)\frac{d\boldsymbol{x}}{x_1 \cdots x_n} \\ - \int_0^{b_1} \cdots \int_0^{b_n} \int_{t_1}^{b_1} \cdots \int_{t_n}^{b_n} \varphi\left(\left|f(\boldsymbol{t}) - \frac{1}{x_1 \cdots x_n}\int_0^{x_1} \cdots \int_0^{x_n} f(\boldsymbol{t})d\boldsymbol{t}\right|\right) \\ \times \frac{u(\boldsymbol{x})}{x_1^2 \cdots x_n^2} d\boldsymbol{x}d\boldsymbol{t}$$

and (2.2) takes the form

$$\begin{aligned}\widetilde{A}(\varphi) =& \frac{1}{3}\frac{\xi\varphi''(\xi)-\varphi'(\xi)}{\xi^2}\left(\int_0^{b_1}\cdots\int_0^{b_n} v(\boldsymbol{x})f^3(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}\right.\\ &-\int_0^{b_1}\cdots\int_0^{b_n} u(\boldsymbol{x})\left(\frac{1}{x_1\cdots x_n}\int_0^{x_1}\cdots\int_0^{x_n} f(\boldsymbol{t})d\boldsymbol{t}\right)^3\frac{d\boldsymbol{x}}{x_1\cdots x_n}\\ &-\int_0^{b_1}\cdots\int_0^{b_n}\int_{t_1}^{b_1}\cdots\int_{t_n}^{b_n}\left|f(\boldsymbol{t})-\frac{1}{x_1\cdots x_n}\int_0^{x_1}\cdots\int_0^{x_n} f(\boldsymbol{t})d\boldsymbol{t}\right|^3\\ &\left.\times\frac{u(\boldsymbol{x})}{x_1^2\cdots x_n^2}d\boldsymbol{x}d\boldsymbol{t}\right).\end{aligned}$$

Example 2.2. Let $\Omega_1=\Omega_2=(\boldsymbol{b},\boldsymbol{\infty})$, $\boldsymbol{0}\le\boldsymbol{b}<\boldsymbol{\infty}$, replace $d\mu_1(x)$ and $d\mu_2(y)$ by the Lebesgue measures $d\mathbf{x}$ and $d\mathbf{y}$, respectively and $k(\mathbf{x},\mathbf{y})=\frac{1}{y_1^2\cdots y_n^2}$, $\mathbf{y}\ge\mathbf{x}$, $k(\mathbf{x},\mathbf{y})=\mathbf{0}$, $\boldsymbol{b}\le\mathbf{y}<\mathbf{x}$. Then $K(\mathbf{x})=\frac{1}{x_1\cdots x_n}$ and

$$A_k f(\mathbf{x})=x_1\cdots x_n\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty}\frac{f(\mathbf{y})}{y_1^2\cdots y_n^2}d\mathbf{y}.$$

Replacing $u(\mathbf{x})$ by $u(\mathbf{x})/x_1\ldots x_n$ and $v(\mathbf{y})$ by $v(\mathbf{y})/y_1\ldots y_n$, we obtain

$$v(\mathbf{t})=\frac{1}{t_1\cdots t_n}\int_{b_1}^{t_1}\cdots\int_{b_n}^{t_n}u(\boldsymbol{x})d\boldsymbol{x}\ <\infty,\ \boldsymbol{t}\in(\mathbf{b},\boldsymbol{\infty})$$

and A which we now denote by $\widehat{A}$ becomes

$$\begin{aligned}\widehat{A}(\varphi) =& \int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty} v(\boldsymbol{x})\varphi(f(\boldsymbol{x}))\frac{d\boldsymbol{x}}{x_1\cdots x_n}\\ &-\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty} u(\boldsymbol{x})\varphi\left(x_1\cdots x_n\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty} f(\boldsymbol{t})\frac{d\boldsymbol{t}}{t_1^2\cdots t_n^2}\right)\frac{d\boldsymbol{x}}{x_1\cdots x_n}\\ &-\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty}\int_{b_1}^{t_1}\cdots\int_{b_n}^{t_n}\varphi\left(\left|f(\boldsymbol{t})-x_1\cdots x_n\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty} f(\boldsymbol{t})\frac{d\boldsymbol{t}}{t_1^2\cdots t_n^2}\right|\right)\\ &\times\ u(\boldsymbol{x})d\boldsymbol{x}\frac{d\boldsymbol{t}}{t_1^2\cdots t_n^2}\end{aligned}$$

and (2.2) takes the form

$$\begin{aligned}\widehat{A}(\varphi) =& \frac{1}{3}\frac{\xi\varphi''(\xi)-\varphi'(\xi)}{\xi^2}\left(\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty} v(\boldsymbol{x})f^3(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}\right.\\ &-\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty} u(\boldsymbol{x})\left(x_1\cdots x_n\int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty} f(\boldsymbol{t})\frac{d\boldsymbol{t}}{t_1^2\cdots t_n^2}\right)^3\frac{d\boldsymbol{x}}{x_1\cdots x_n}\\ &-\int_{b_1}^{\infty}\cdots\int_{b_n}^{\infty}\int_{b_1}^{t_1}\cdots\int_{b_n}^{t_n}\left|f(\boldsymbol{t})-x_1\cdots x_n\int_0^{x_1}\cdots\int_0^{x_n} f(\boldsymbol{t})\frac{d\boldsymbol{t}}{t_1^2\cdots t_n^2}\right|^3\end{aligned}$$

$$\times u(\boldsymbol{x})d\boldsymbol{x}\frac{d\boldsymbol{t}}{t_1^2\cdots t_n^2}\Bigg).$$

We state the following result concerning inequality (1.2) by applying Theorem 2.2 with $\varphi(u)=u^p,\ p\geq 2$.

Example 2.3. Let $\Omega_1=\Omega_2=(0,\infty)$, $\varphi(u)=u^p,\ p\geq 2$ and replace $d\mu_1(x)$ and $d\mu_2(y)$ by the Lebesque measures dx and dy, respectively, let $k(x,y)=\frac{(\frac{y}{x})^{-1/p}}{x+y}$ and $u(x)=\frac{1}{x}$. Then we find that $K(x)=\frac{\pi}{\sin(\pi/p)}$ and $v(y)=\frac{1}{y}$. Replace $f(x)$ by $f(x)x^{1/p}$, so A which is now denoted by H_f becomes

$$H_f=\int_0^\infty f^p(y)\,dy-\left(\frac{\sin\left(\frac{\pi}{p}\right)}{\pi}\right)^p\int_0^\infty\left(\int_0^\infty\frac{f(x)}{x+y}\,dx\right)^p dy$$
$$-\frac{\sin\left(\frac{\pi}{p}\right)}{\pi}\int_0^\infty\int_0^\infty\left|f(y)-\frac{\sin\left(\frac{\pi}{p}\right)}{\pi}\left(\frac{x}{y}\right)^{\frac{1}{p}}\int_0^\infty\frac{f(y)}{x+y}dy\right|^p\frac{x^{\frac{1}{p}-1}}{x+y}dx\,dy$$

and (2.2) takes the form

$$H_f=\frac{p(p-2)\xi^{p-3}}{3}$$
$$\times\left(\int_0^\infty f^3(y)y^{\frac{3}{p}-1}dy-\left(\frac{\sin\left(\frac{\pi}{p}\right)}{\pi}\right)^3\int_0^\infty\left(\int_0^\infty\frac{f(y)}{x+y}dy\right)^3x^{\frac{3}{p}-1}dx\right.$$
$$\left.-\frac{\sin\left(\frac{\pi}{p}\right)}{\pi}\int_0^\infty\int_0^\infty\left|f(y)-\frac{\sin\left(\frac{\pi}{p}\right)}{\pi}\left(\frac{x}{y}\right)^{\frac{1}{p}}\int_0^\infty\frac{f(y)}{x+y}dy\right|^3\frac{x^{\frac{1}{p}-1}}{x+y}dxdy\right).$$

3. Exponential convexity

Lemma 3.1. *Consider the function φ_p for $p>0$ defined as*

$$\varphi_p(x)=\begin{cases}\frac{x^p}{p(p-2)}, & p\neq 2\\ \frac{x^2}{2}\log x, & p=2\end{cases}\tag{3.1}$$

Then, with the convention $0\,log\,0=0$, it is superquadratic.

For linear functional A defined by (2.1) we have $A(\varphi_p)\geq 0$ for all $p>0$.

Lemma 3.2. *Let us define the function*

$$\phi_p(x)=\begin{cases}\frac{pxe^{px}-e^{px}+1}{p^3}, & p\neq 0\\ \frac{x^3}{3}, & p=0.\end{cases}\tag{3.2}$$

Then $\left(\frac{\phi_p'(x)}{x}\right)'=e^{px}>0$, and $\phi_p(0)=0$, therefore ϕ_p is superquadratic.

Properties of the mapping $p\mapsto A(\varphi_p)$ are given in the following theorem:

Theorem 3.1. *For A as in* (2.1) *and φ_p as in* (3.2) *we have the following:*

(i) *the mapping $p \mapsto A(\varphi_p)$ is continuous for $p > 0$,*

(ii) *for every $n \in \mathbb{N}$ and $p_i \in \mathbb{R}_+, p_{ij} = \frac{p_i+p_j}{2}$, $i, j = 1, 2, \dots, n$, the matrix $[A(\varphi_{p_{ij}})]_{i,j=1}^n$ is a positive semi-definite, that is*

$$\det[A(\varphi_{p_{ij}})]_{i,j=1}^n \geq 0,$$

(iii) *the mapping $p \mapsto A(\varphi_p)$ is exponentially convex,*

(iv) *the mapping $p \mapsto A(\varphi_p)$ is log-convex,*

(v) *for $p_i \in \mathbb{R}_+$, $i = 1, 2, 3$, $p_1 < p_2 < p_3$,*

$$[A(\varphi_{p_2})]^{p_3-p_1} \leq [A(\varphi_{p_1})]^{p_3-p_2} [A(\varphi_{p_3})]^{p_2-p_1}.$$

Proof. (i) Notice that

$$A(\varphi_p) = \begin{cases} \frac{1}{p(p-2)} \Big[\int_{\Omega_2} f^p(y) v(y) d\mu_2(y) - \int_{\Omega_1} (A_k f(x))^p u(x) d\mu_1(x) \\ \quad - \int_{\Omega_2} \int_{\Omega_1} |f(y) - A_k f(x)|^p \frac{u(x)k(x,y)}{K(x)} d\mu_1(x)\, d\mu_2(y) \Big], \ \ p \neq 2; \\ \frac{1}{2} \Big[\int_{\Omega_2} f^2(y) \log(f(y)) v(y) d\mu_2(y) \\ \ - \int_{\Omega_1} (A_k f(x))^2 \log(A_k f(x))) u(x) d\mu_1(x) - \int_{\Omega_2} \int_{\Omega_1} |f(y) \\ \ - A_k f(x)|^2 \log|f(y) - A_k f(x)| \frac{u(x)k(x,y)}{K(x)} d\mu_1(x)\, d\mu_2(y) \Big], p = 2. \end{cases}$$

It is obviously continuous for $p > 0$, $p \neq 2$. Suppose $p \to 2$:

$$\lim_{p \to 2} A(\varphi_p) = \lim_{p \to 2} \frac{1}{p(p-2)} \left[\int_{\Omega_2} f^p(y) v(y) d\mu_2(y) - \int_{\Omega_1} (A_k f(x))^p u(x) d\mu_1(x) \right.$$
$$\left. - \int_{\Omega_2} \int_{\Omega_1} |f(y) - A_k f(x)|^p \frac{u(x)k(x,y)}{K(x)} d\mu_1(x)\, d\mu_2(y) \right].$$

Since

$$\int_{\Omega_2} f^2(y) v(y) d\mu_2(y) - \int_{\Omega_1} (A_k f(x))^2 u(x) d\mu_1(x)$$
$$- \int_{\Omega_2} \int_{\Omega_1} |f(y) - A_k f(x)|^2 \frac{u(x)k(x,y)}{K(x)} d\mu_1(x)\, d\mu_2(y) = 0$$

applying L'Hospital rule we obtain after a simple calculation that

$$\lim_{p \to 2} A(\varphi_p) = A(\varphi_2).$$

Hence, the mapping $p \mapsto A(\varphi_p)$ is continuous for $p > 0$.

(ii) Define the function $F(x) = \sum_{i,j=1}^n u_i u_j \varphi_{p_ij}(x)$, where $p_{ij} = \frac{p_i+p_j}{2}$ then

$$\left(\frac{F'(x)}{x} \right)' = \sum_{i,j=1}^n u_i u_j \left(\frac{\varphi'_{p_ij}(x)}{x} \right)' = \left(\sum_{i=1}^n u_i x^{\frac{p_i-3}{2}} \right)^2 \geq 0$$

and $F(0) = 0$. This implies F is superquadratic so using this F in the place of φ in (1.6) we have

$$A(F) = \sum_{i,j=1}^{n} u_i u_j A(\varphi_{p_{ij}}) \geq 0 \tag{3.3}$$

and from this we have that the matrix $B = [A(\varphi_{\frac{p_i+p_j}{2}})]_{i,j=1}^{n}$, is positive-semidefinite $i.e.$ $\det B \geq 0$.
(iii), (iv) and (v) are trivial consequences of (i), (ii) and the definition of exponentially convex and log-convex functions. □

Using the function ϕ_p instead of φ_p, the following result follows.

Theorem 3.2. *For A as in (2.1) and ϕ_p as in (3.2) we have the following:*

(i) *the mapping $p \mapsto A(\phi_p)$ is continuous on $\mathbb{R}$,*
(ii) *for every $n \in \mathbb{N}$ and $p_i \in \mathbb{R}, p_{ij} = \frac{p_i+p_j}{2}$, $i, j = 1, 2, \ldots, n$, the matrix $[A(\phi_{p_{ij}})]_{i,j=1}^{n}$ is a positive semi-definite, that is*

$$\det[A(\phi_{p_{ij}})]_{i,j=1}^{n} \geq 0,$$

(iii) *the mapping $p \mapsto A(\phi_p)$ is exponentially convex,*
(iv) *the mapping $p \mapsto A(\phi_p)$ is log-convex,*
(v) *for $p_i \in \mathbb{R}$, $i = 1, 2, 3$, $p_1 < p_2 < p_3$,*

$$[A(\phi_{p_2})]^{p_3-p_1} \leq [A(\phi_{p_1})]^{p_3-p_2} [A(\phi_{p_3})]^{p_2-p_1}.$$

4. Cauchy means

Theorem 2.2 enables us to define new means, because if on the right hand side the function of ξ is denoted by $K(\xi)$ and is invertible, then by Theorem 2.2 we have

$$\xi = K^{-1}\left(\frac{A(\varphi)}{A(\psi)}\right),$$

which presents a new Cauchy's mean.

Specially, if we choose $\varphi = \varphi_s, \psi = \varphi_r$, where $r, s \in \mathbb{R}_+$, $r \neq s$, $r, s \neq 2$, we obtain

$$\xi^{s-r} = \frac{r(r-2)}{s(s-2)} \times$$
$$\frac{\int_{\Omega_2} f^s(y)v(y)d\mu_2(y) - \int_{\Omega_1}(A_k f(x))^s u(x)d\mu_1(x) - \int_{\Omega_2}\int_{\Omega_1} C^s(x,y)g(x,y)d\mu_1(x)d\mu_2(y)}{\int_{\Omega_2} f^r(y)v(y)d\mu_2(y) - \int_{\Omega_1}(A_k f(x))^r u(x)d\mu_1(x) - \int_{\Omega_2}\int_{\Omega_1} C^r(x,y)g(x,y)d\mu_1(x)d\mu_2(y)},$$

where $C(x, y) = |f(y) - A_k f(x)|$ and $g(x, y) = \frac{u(x)k(x,y)}{K(x)}$.

Now we can give the following definition.

Definition 4. *For $r, s \in \mathbb{R}_+$ we define the new mean $M_{r,s}$ as follows*

$$M_{s,r} = \Bigg(\frac{r(r-2)}{s(s-2)} \times \frac{\int_{\Omega_2} A_{s,0}(y)d\mu_2(y) - \int_{\Omega_1} B_{s,0}(x)d\mu_1(x) - \int_{\Omega_2}\int_{\Omega_1} C_{s,0}(x,y)d\mu_1(x)d\mu_2(y)}{\int_{\Omega_2} A_{r,0}(y)d\mu_2(y) - \int_{\Omega_1} B_{r,0}(x)d\mu_1(x) - \int_{\Omega_2}\int_{\Omega_1} C_{r,0}(x,y)d\mu_1(x)d\mu_2(y)}\Bigg)^{\frac{1}{s-r}},$$

Taking a limit we define exluded cases. For $r \neq 2$ we have

$$M_{r,2} = M_{2,r} = \Bigg(\frac{r(r-2)}{2} \times \frac{\int_{\Omega_2} A_{2,1}(y)d\mu_2(y) - \int_{\Omega_1} B_{2,1}(x)d\mu_1(x) - \int_{\Omega_2}\int_{\Omega_1} C_{2,1}(x,y)d\mu_1(x)d\mu_2(y)}{\int_{\Omega_2} A_{r,0}(y)d\mu_2(y) - \int_{\Omega_1} B_{r,0}(x)d\mu_1(x) - \int_{\Omega_2}\int_{\Omega_1} C_{r,0}(x,y)d\mu_1(x)d\mu_2(y)}\Bigg)^{\frac{1}{2-r}}$$

$$M_{r,r} = \exp\Bigg(\frac{\int_{\Omega_2} A_{r,1}(y)d\mu_2(y) - \int_{\Omega_1} B_{r,1}(x)d\mu_1(x) - \int_{\Omega_2}\int_{\Omega_1} C_{r,1}(x,y)d\mu_1(x)d\mu_2(y)}{\int_{\Omega_2} A_{r,0}(y)d\mu_2(y) - \int_{\Omega_1} B_{r,0}(x)d\mu_1(x) - \int_{\Omega_2}\int_{\Omega_1} C_{r,0}(x,y)d\mu_1(x)d\mu_2(y)} - \frac{2r-2}{r(r-2)}\Bigg),$$

and for $r = 2$

$$M_{2,2} = \exp\Bigg(\frac{\int_{\Omega_2} A_{2,2}(y)d\mu_2(y) - \int_{\Omega_1} B_{2,2}(x)d\mu_1(x) - \int_{\Omega_2}\int_{\Omega_1} C_{2,2}(x,y)d\mu_1(x)d\mu_2(y)}{\int_{\Omega_2} A_{2,1}(y)d\mu_2(y) - \int_{\Omega_1} B_{2,1}(x)d\mu_1(x) - \int_{\Omega_2}\int_{\Omega_1} C_{2,1}(x,y)d\mu_1(x)d\mu_2(y)} - \frac{1}{2}\Bigg),$$

where

$A_{p,n}(y) = f^p(y)(\log(f(y))^n v(y)$,
$B_{p,n}(x) = (A_k f(x))^p(\log(A_k f(x))^n u(x)$,
$C_{p,n}(x,y) = |f(y) - A_k f(x)|^p \log^n |f(y) - A_k f(x)| g(x,y)$, $n = 0, 1, 2$, $p > 0$.

Note that these means are symmetric and we can easily check that the special cases in the above definition are limits of the general case. That is,

$$M_{r,r} = \lim_{s\to r} M_{s,r}$$
$$M_{2,r} = M_{r,2} = \lim_{s\to 2} M_{s,r} = \lim_{s\to 2} M_{r,s},$$
$$M_{2,2} = \lim_{r\to 2} M_{r,r}.$$

The monotonicity of the above defined means is given in the following theorem.

Theorem 4.1. *Let $s, t, u, v \in \mathbb{R}_+$ be such that $s \le u$, $t \le v$, $s \neq t$, $u \neq v$. Then*

$$M_{t,s} \le M_{v,u}. \tag{4.1}$$

Proof. Since the function $s \mapsto A(\varphi_s)$ is log-convex, then by Lemma 1.2 for any $s,t,u,v \in \mathbb{R}_+$, such that $s \le u$, $t \le v$, $s \ne t$, $u \ne v$, we have

$$\left(\frac{A(\varphi_t)}{A(\varphi_s)}\right)^{\frac{1}{t-s}} \le \left(\frac{A(\varphi_v)}{A(\varphi_u)}\right)^{\frac{1}{v-u}}$$

which is equivalent to (4.1).
For $s = t$ and $u = v$ we can consider the limiting case. □

Acknowledgements. The research of the authors was supported by the Croatian Ministry of Science, Education and Sports, under the Research Grant 117-1170889-0888.

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(Received: October 15, 2010)
(Revised: February 17, 2011)

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