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RIGHT CAPUTO FRACTIONAL LANDAU INEQUALITIES

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Abstract. Right Caputo fractional $\|\cdot\|_p$-Landau type inequalities, $p \in (1, \infty]$ are obtained with applications on $\mathbb{R}_-$.

1. Introduction

Let $p \in [1, \infty]$, $I = \mathbb{R}_+$ or $I = \mathbb{R}$, and $f : I \to \mathbb{R}$ be twice differentiable with $f, f'' \in L_p(I)$, then $f' \in L_p(I)$. Moreover, there exists a constant $C_p(I) > 0$ independent of f, such that

$$\left\|f'\right\|_{p,I} \leq C_p(I) \|f\|_{p,I}^{\frac{1}{2}} \left\|f''\right\|_{p,I}^{\frac{1}{2}}, \tag{1}$$

where $\|\cdot\|_{p,I}$ is the p-norm on the interval I, see [1], [4].

Development of these inequalities was initiated by E. Landau [12] in 1914. For the case of $p = \infty$ he proved that

$$C_\infty(\mathbb{R}_+) = 2 \text{ and } C_\infty(\mathbb{R}) = \sqrt{2}, \tag{2}$$

are the best constants in (1).

In 1932, G.H. Hardy and J.E. Littlewood [9] proved (1) for $p = 2$, with the best constants

$$C_2(\mathbb{R}_+) = \sqrt{2}, \text{ and } C_2(\mathbb{R}) = 1. \tag{3}$$

In 1935, G.H. Hardy, E. Landau and J.E. Littlewood [10] showed that the best constant $C_p(\mathbb{R}_+)$ in (1) satisfies

$$C_p(\mathbb{R}_+) \leq 2, \text{ for } p \in [1, \infty), \tag{4}$$

which yields $C_p(\mathbb{R}) \leq 2$ for $p \in [1, \infty)$. In fact in [5] and [11], it was shown that

$$C_p(\mathbb{R}) \leq \sqrt{2}.$$

In this article we obtain fractional Landau inequalities with respect to $\|\cdot\|_p$, $p \in (1, \infty]$, involving the right Caputo fractional derivative.

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2. Main results

We need

Definition 1. ([6], [7], [8], [13]) *Let $f \in L_1([a,b])$, $\alpha > 0$. The right Riemann-Liouville fractional operator of order α, is given by*

$$I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (J-x)^{\alpha-1} f(J)\, dJ, \tag{5}$$

$\forall\ x \in [a,b]$, where Γ is the gamma function. We set $I_{b-}^{0} := I$ (the identity operator).

Definition 2. ([6], [7], [8], [13]) *Let $f \in AC^m([a,b])$ ($f^{(m-1)}$ is in $AC([a,b])$), $m \in \mathbb{N}$, $m = \lceil \alpha \rceil$, $\alpha > 0$ ($\lceil \cdot \rceil$ the ceiling of the number). We define the right Caputo fractional derivative of order $\alpha > 0$, by*

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (J-x)^{m-\alpha-1} f^{(m)}(J)\, dJ, \quad \forall\ x \leq b. \tag{6}$$

If $\alpha = m \in \mathbb{N}$, then

$$D_{b-}^{m} f(x) = (-1)^m f^{(m)}(x), \quad \forall\ x \in [a,b].$$

If $x > b$ we define $D_{b-}^{\alpha} f(x) = 0$.

In particular we have

Definition 3. *Let $0 < \alpha \leq 1$, $f \in AC([A,B])$ and define*

$$D_{B-}^{\alpha} f(x) := \frac{-1}{\Gamma(1-\alpha)} \int_x^B (J-x)^{-\alpha} f'(J)\, dJ. \tag{7}$$

Remark 4. Let $f' \in AC([A,B])$, $0 < \alpha \leq 1$, then $\lceil \alpha + 1 \rceil = 2$ and

$$D_{B-}^{\alpha} f'(x) = \frac{-1}{\Gamma(1-\alpha)} \int_x^B (J-x)^{-\alpha} f''(J)\, dJ$$

$$= \frac{-(-1)^2}{\Gamma(2-(\alpha+1))} \int_x^B (J-x)^{2-(\alpha+1)-1} f''(J)\, dJ = -D_{B-}^{\alpha+1} f(x)$$

i.e.

$$D_{B-}^{\alpha} f'(x) = -D_{B-}^{\alpha+1} f(x) \tag{8}$$

and hence

$$\left\| D_{B-}^{\alpha} f' \right\|_{\infty,[A,B]} = \left\| D_{B-}^{\alpha+1} f \right\|_{\infty,[A,B]}. \tag{9}$$

The focus is now on $(-\infty, B]$, $B \in \mathbb{R}$.

Let $a < b < B$, $a, b \in \mathbb{R}$. If $f''(J) \geq 0$, a.e., then

$$\frac{1}{\Gamma(1-\alpha)} \int_x^B (J-x)^{-\alpha} f''(J)\, dJ$$
$$\geq \frac{1}{\Gamma(1-\alpha)} \int_x^b (J-x)^{-\alpha} f''(J)\, dJ \geq 0, \text{ a.e.,}$$

i.e.

$$D_{B-}^{\alpha+1} f(x) \geq D_{b-}^{\alpha+1} f(x) \geq 0, \text{ a.e., for } x \leq b. \tag{10}$$

It holds therefore, that

$$\infty > \left\| D_{B-}^{\alpha+1} f \right\|_{\infty,(-\infty,B]} \geq \left\| D_{b-}^{\alpha+1} f \right\|_{\infty,(-\infty,b]}. \tag{11}$$

It is reasonable then to assume that

$$\left\| D_{b-}^{\alpha+1} f \right\|_{\infty,(-\infty,b]} \leq \left\| D_{B-}^{\alpha+1} f \right\|_{\infty,(-\infty,B]}. \tag{12}$$

(it is obvious when $\alpha = 1$), $\forall\, b \leq B$.

Remark 5. Let $0 < \alpha \leq 1$; $B, a \in \mathbb{R}$, with $f \in AC^2([a,B])$, $\forall a < B$. Assume $D_{B-}^{\alpha+1} f \in L_\infty((-\infty, B])$, thus $D_{B-}^{\alpha+1} f \in L_\infty([a,B])$. Let $a < b < B$, then $f \in AC^2([a,b])$, and $D_{B-}^{\alpha+1} f \in L_\infty((-\infty,b])$ and $D_{B-}^{\alpha+1} f \in L_\infty([a,b])$.

By Theorem 6 of [2], applied on f', we get that

$$\left| f'(b) \right| \leq \frac{1}{b-a} \left| f(b) - f(a) \right| + \frac{\left\| D_{b-}^{\alpha+1} f \right\|_{\infty,[a,b]}}{\Gamma(\alpha+2)} (b-a)^\alpha \tag{13}$$

$$\leq \frac{2 \|f\|_{\infty,(-\infty,B]}}{b-a} + \frac{\left\| D_{B-}^{\alpha+1} f \right\|_{\infty,(-\infty,B]} (b-a)^\alpha}{\Gamma(\alpha+2)}, \tag{14}$$

where we also assume that $\|f\|_{\infty,(-\infty,B]} < \infty$.

Setting $t = b - a$,

$$\left\| f' \right\|_{\infty,(-\infty,B]} \leq \frac{2 \|f\|_{\infty,(-\infty,B]}}{t} + \frac{\left\| D_{B-}^{\alpha+1} f \right\|_{\infty,(-\infty,B]} \cdot t^\alpha}{\Gamma(\alpha+2)}, \tag{15}$$

$\forall\, t \in (0, \infty)$.

One now can prove easily that the $R.H.S.$(15) has a global minimum and find it (see also [3]) with a usual calculus method.

We have proved the following

Theorem 6. *Let* $0 < \alpha \leq 1$; $B, a \in \mathbb{R}$, *with* $f \in AC^2([a,B])$, $\forall a < B$, *where* B *is fixed. Assume* $\|f\|_{\infty,(-\infty,B]} < \infty$, $D_{B-}^{\alpha+1} f \in L_\infty((-\infty,B])$, *and*

$$\left\| D_{b-}^{\alpha+1} f \right\|_{\infty,(-\infty,b]} \leq \left\| D_{B-}^{\alpha+1} f \right\|_{\infty,(-\infty,B]}, \tag{16}$$

$\forall\ b < B$, *then*

$$\left\|f'\right\|_{\infty,(-\infty,B]} \leq (\alpha+1)\cdot\left(\frac{2}{\alpha}\right)^{\left(\frac{\alpha}{\alpha+1}\right)}\cdot\left(\Gamma\left(\alpha+2\right)\right)^{-\frac{1}{(\alpha+1)}}$$
$$\cdot\left(\|f\|_{\infty,(-\infty,B]}\right)^{\left(\frac{\alpha}{\alpha+1}\right)}\cdot\left(\left\|D_{B-}^{\alpha+1}f\right\|_{\infty,(-\infty,B]}\right)^{\frac{1}{(\alpha+1)}}. \tag{17}$$

The case of $B = 0$ *is given by the following Corollary.*

Corollary 7. *Let* $0 < \alpha \leq 1$; $f \in AC^2([a,0])$, $\forall\ a < 0$. *Assume that* $\|f\|_{\infty,\mathbb{R}_-} < \infty$, $D_{0-}^{\alpha+1}f \in L_\infty(\mathbb{R}_-)$, *and*

$$\left\|D_{b-}^{\alpha+1}f\right\|_{\infty,(-\infty,b]} \leq \left\|D_{0-}^{\alpha+1}f\right\|_{\infty,\mathbb{R}_-}, \tag{18}$$

$\forall\ b < 0$. *Then*

$$\left\|f'\right\|_{\infty,\mathbb{R}_-} \leq (\alpha+1)\cdot\left(\frac{2}{\alpha}\right)^{\left(\frac{\alpha}{\alpha+1}\right)}\cdot\left(\Gamma\left(\alpha+2\right)\right)^{-\frac{1}{(\alpha+1)}}$$
$$\cdot\left(\|f\|_{\infty,\mathbb{R}_-}\right)^{\left(\frac{\alpha}{\alpha+1}\right)}\cdot\left(\left\|D_{0-}^{\alpha+1}f\right\|_{\infty,\mathbb{R}_-}\right)^{\frac{1}{(\alpha+1)}}. \tag{19}$$

The case of $\alpha = 1$ *is given by the following Corollary.*

Corollary 8. *Let* $B, a \in \mathbb{R}$, *with* $f \in AC^2([a,B])$, $\forall a < B$, B *is fixed. Assume* $\|f\|_{\infty,(-\infty,B]} < \infty$, $f'' \in L_\infty((-\infty,B])$, *then*

$$\left\|f'\right\|_{\infty,(-\infty,B]} \leq 2\cdot\left(\|f\|_{\infty,(-\infty,B]}\right)^{\frac{1}{2}}\cdot\left(\left\|f''\right\|_{\infty,(-\infty,B]}\right)^{\frac{1}{2}}. \tag{20}$$

Corollary 9. *Let* $f \in AC^2([a,0])$, $\forall a < 0$. *Assume* $f, f'' \in L_\infty(\mathbb{R}_-)$. *Then*

$$\left\|f'\right\|_{\infty,\mathbb{R}_-} \leq 2\cdot\left(\|f\|_{\infty,\mathbb{R}_-}\right)^{\frac{1}{2}}\cdot\left(\left\|f''\right\|_{\infty,\mathbb{R}_-}\right)^{\frac{1}{2}}. \tag{21}$$

Remark 10. Let any $a, b \in (-\infty, B]$, $a < b$. Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $1 - \frac{1}{p} < \alpha \leq 1$, and $f \in AC^2([a,b])$. Assume also $D_{b-}^{\alpha+1}f \in L_q([a,b])$, then by [2], Theorem 8, for f' we get that

$$\left|f'(b)\right| \leq \frac{1}{b-a}\left|f(b)-f(a)\right| + \frac{\left\|D_{b-}^{\alpha+1}f\right\|_{L_q([a,b])}}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)}(b-a)^{\alpha-1+\frac{1}{p}}. \tag{22}$$

Assume here

$$\left\|D_{b-}^{\alpha+1}f\right\|_{q,(-\infty,b]} \leq \left\|D_{B-}^{\alpha+1}f\right\|_{q,(-\infty,B]} < \infty, \tag{23}$$

$\forall\ b \leq B$.

Therefore,

$$\left|f'(b)\right| \leq \frac{2\|f\|_{\infty,(-\infty,B]}}{b-a} + \frac{\left\|D_{B-}^{\alpha+1}f\right\|_{q,(-\infty,B]}}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)}(b-a)^{\alpha-1+\frac{1}{p}}, \tag{24}$$

$\forall\ a,b : a < b \leq B$, where B is fixed.

Let $t := b-a > 0$, then

$$\left\|f'\right\|_{\infty,(-\infty,B]} \leq \frac{2\|f\|_{\infty,(-\infty,B]}}{t} + \frac{\left\|D_{B-}^{\alpha+1}f\right\|_{q,(-\infty,B]}}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)}t^{\alpha-1+\frac{1}{p}}, \tag{25}$$

$\forall\ t \in (0,\infty)$, where it is assumed that $\|f\|_{\infty,(-\infty,B]} < \infty$.

As before we prove:

Theorem 11. *Let* $p,q > 1 : \frac{1}{p}+\frac{1}{q} = 1$, $1-\frac{1}{p} < \alpha \leq 1$, $f \in AC^2([a,B])$, $\forall a < B$. *Assume* $D_{B-}^{\alpha+1}f \in L_q((-\infty,B])$, $\|f\|_{\infty,(-\infty,B]} < \infty$, *and*

$$\left\|D_{b-}^{\alpha+1}f\right\|_{q,(-\infty,b]} \leq \left\|D_{B-}^{\alpha+1}f\right\|_{q,(-\infty,B]}, \tag{26}$$

$\forall\ b \leq B$, *then*

$$\left\|f'\right\|_{\infty,(-\infty,B]} \leq \left(\frac{2\left(\alpha+\frac{1}{p}\right)}{\alpha-1+\frac{1}{p}}\right)^{\left(\frac{\alpha-1+\frac{1}{p}}{\alpha+\frac{1}{p}}\right)} \cdot \frac{1}{\left(\Gamma(\alpha)\right)^{\frac{1}{\left(\alpha+\frac{1}{p}\right)}}\left(p(\alpha-1)+1\right)^{\frac{1}{(p\alpha+1)}}}$$

$$\cdot\left(\|f\|_{\infty,(-\infty,B]}\right)^{\left(\frac{a-1+\frac{1}{p}}{\alpha+\frac{1}{p}}\right)} \cdot \left(\left\|D_{B-}^{\alpha+1}f\right\|_{q,(-\infty,B]}\right)^{\frac{1}{\left(\alpha+\frac{1}{p}\right)}}. \tag{27}$$

The case of $B = 0$ *is given by Corollary* 12.

Corollary 12. *Let* $p,q > 1 : \frac{1}{p}+\frac{1}{q} = 1$, $1-\frac{1}{p} < \alpha \leq 1$, $f \in AC^2([a,0])$, $\forall a < 0$. *Assume* $D_{0-}^{\alpha+1}f \in L_q(\mathbb{R}_-)$, $\|f\|_{\infty,\mathbb{R}_-} < \infty$, *and*

$$\left\|D_{b-}^{\alpha+1}f\right\|_{q,(-\infty,b]} \leq \left\|D_{0-}^{\alpha+1}f\right\|_{q,\mathbb{R}_-}, \tag{28}$$

$\forall\ b \leq 0$. *Then*

$$\left\|f'\right\|_{\infty,\mathbb{R}_-} \leq \left(\frac{2\left(\alpha+\frac{1}{p}\right)}{\alpha-1+\frac{1}{p}}\right)^{\left(\frac{\alpha-1+\frac{1}{p}}{\alpha+\frac{1}{p}}\right)} \cdot \frac{1}{\left(\Gamma(\alpha)\right)^{\frac{1}{\left(\alpha+\frac{1}{p}\right)}}\left(p(\alpha-1)+1\right)^{\frac{1}{(p\alpha+1)}}}$$

$$\cdot\left(\|f\|_{\infty,\mathbb{R}_-}\right)^{\left(\frac{\alpha-1+\frac{1}{p}}{\alpha+\frac{1}{p}}\right)} \cdot \left(\left\|D_{0-}^{\alpha+1}f\right\|_{q,\mathbb{R}_-}\right)^{\frac{1}{\left(\alpha+\frac{1}{p}\right)}}. \tag{29}$$

The case for $p = q = 2$ follows.

Corollary 13. *Let* $\frac{1}{2} < \alpha \leq 1$, $f \in AC^2([a,0])$, $\forall a < 0$. *Assume* $D_{0-}^{\alpha+1} f \in L_2(\mathbb{R}_-)$, $\|f\|_{\infty,\mathbb{R}_-} < \infty$, *and*

$$\left\| D_{b-}^{\alpha+1} f \right\|_{2,(-\infty,b]} \leq \left\| D_{0-}^{\alpha+1} f \right\|_{2,\mathbb{R}_-}, \tag{30}$$

$\forall\ b \leq 0$, *then*

$$\left\| f' \right\|_{\infty,\mathbb{R}_-} \leq \left(\frac{2\alpha+1}{\alpha - \frac{1}{2}} \right)^{\left(\frac{\alpha-\frac{1}{2}}{\alpha+\frac{1}{2}} \right)} \cdot \frac{1}{(\Gamma(\alpha))^{\frac{1}{\left(\alpha+\frac{1}{2}\right)}} (2\alpha-1)^{\frac{1}{(2\alpha+1)}}}$$

$$\cdot \left(\|f\|_{\infty,\mathbb{R}_-} \right)^{\left(\frac{\alpha-\frac{1}{2}}{\alpha+\frac{1}{2}} \right)} \cdot \left(\left\| D_{0-}^{\alpha+1} f \right\|_{2,\mathbb{R}_-} \right)^{\frac{1}{\left(\alpha+\frac{1}{2}\right)}}. \tag{31}$$

The case $\alpha = 1$ gives:

Corollary 14. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $f \in AC^2([a,0])$, $\forall a < 0$. *Assume* $f'' \in L_q(\mathbb{R}_-)$, $f \in L_\infty(\mathbb{R}_-)$, *then*

$$\left\| f' \right\|_{\infty,\mathbb{R}_-} \leq (2(p+1))^{\frac{1}{p+1}} \cdot \left(\|f\|_{\infty,\mathbb{R}_-} \right)^{\frac{1}{p+1}} \cdot \left(\left\| f'' \right\|_{q,\mathbb{R}_-} \right)^{\frac{p}{p+1}}. \tag{32}$$

Finally, for $p = q = 2$ we have:

Corollary 15. *Let* $f \in AC^2([a,0])$, $\forall a < 0$. *Assume* $f'' \in L_2(\mathbb{R}_-)$, $f \in L_\infty(\mathbb{R}_-)$, *then*

$$\left\| f' \right\|_{\infty,\mathbb{R}_-} \leq \sqrt[3]{6} \cdot \left(\|f\|_{\infty,\mathbb{R}_-} \right)^{\frac{1}{3}} \cdot \left(\left\| f'' \right\|_{2,\mathbb{R}_-} \right)^{\frac{2}{3}}. \tag{33}$$

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