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SUMMATION PROCESSES AND GAUSSIAN QUADRATURES

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This paper is dedicated to Professors Veselin Perić and Svetozar Kurepa

ABSTRACT. In this survey paper we present two classes of summation procedures based on ideas related to Gaussian quadratures. Such summation/integration procedures can be applied to the summation of slowly convergent series. Numerical examples are included.

1. INTRODUCTION

In this survey paper we consider some summation processes for series

$$\sum_{k=1}^{+\infty} f(k), \tag{1.1}$$

with a given function f with certain properties, based on ideas related to Gauss-Christoffel quadratures. The series (1.1) appears very often in mathematics, physics and other sciences. In particular, slowly convergent series appear in many problems in applied and computation sciences. There are several numerical methods based on linear and nonlinear transformations. In general, starting from the sequence of partial sums of the series, these transformations give other sequences with faster convergence to the same limit (the sum of the series). Some summation methods can be found in the books of Henrici [14], Lindelöf [17], Levin [16], Wimp [30], Mitrinović and Kečkić [26], Brezinski and Redivo Zaglia [1], Sidi [28], and Mastroianni and Milovanović [18] (see also Jolley [15] for a collection of explicit expressions of some sums).

The basic idea in our methods is to replace the sum (1.1) by a finite quadrature sum

$$\sum_{k=1}^{+\infty} f(k) \approx \sum_{\nu=1}^{n} A_\nu g(x_\nu), \tag{1.2}$$

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where the function g is connected with f in some way, and the weights $A_\nu \equiv A_\nu^{(n)}$ and abscissae $x_\nu \equiv x_\nu^{(n)}$, $\nu = 1, \dots, n$, are chosen in such a way as to approximate closely the sum (1.1) for a large class of functions with a relatively small number n. In our approach we take a Gaussian quadrature sum as the sum on the right-hand side in (1.2). Regarding the relationship between the functions f and g, two different methods are analyzed:

- methods of integral transforms;
- direct methods.

In the first kind of methods there is an integral transform between f and g, for example, f is the Laplace transform of g, etc. A characterization of direct methods is that $g \equiv f$.

The paper is organized as follows. Section 2 is devoted to methods of integral transforms (the Laplace transform method and a method of contour integration over a rectangle). Also, some series with irrational terms are treated. Some direct methods are analyzed in Section 3.

2. Methods of integral transforms

Besides the series (1.1) we also consider the corresponding "alternating" series. Namely, let

$$T = \sum_{k=1}^{+\infty} f(k) \qquad \text{and} \qquad S = \sum_{k=1}^{+\infty} (-1)^k f(k) \tag{2.1}$$

be convergent series. Methods of summation of slowly convergent series based on integral representations of series and an application of the Gaussian quadratures have been recently developed in [5, 6, 7], [9, 10], [11, pp. 239–253], [12], [13], [20, 21, 22].

Two methods will be discussed. The first of them is related to an application of the Laplace transform and the second one is connected with a contour integration over a rectangle. At the end of this section we consider some series with irrational terms.

2.1. **The Laplace transform method.** Suppose that the general term of T (and S) is expressible in terms of the Laplace transform, or its derivative, of a known function. Here, we consider two cases:

(a) Let

$$f(s) = \int_0^{+\infty} \mathrm{e}^{-st} g(t)\, \mathrm{d}t, \qquad \mathrm{Re}\, s \geq 1. \tag{2.2}$$

Then

$$T = \sum_{k=1}^{+\infty} f(k) = \sum_{k=1}^{+\infty} \int_0^{+\infty} \mathrm{e}^{-kt} g(t)\, \mathrm{d}t = \int_0^{+\infty} \left(\sum_{k=1}^{+\infty} \mathrm{e}^{-kt} \right) g(t)\, \mathrm{d}t,$$

i.e.,

$$T = \int_0^{+\infty} \frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}}\, g(t)\,\mathrm{d}t = \int_0^{+\infty} \frac{t}{\mathrm{e}^t - 1}\frac{g(t)}{t}\,\mathrm{d}t. \tag{2.3}$$

Thus, the summation of series is now transformed to an integration problem.

Similarly, for the "alternating" series, we obtain

$$S = \sum_{k=1}^{+\infty} (-1)^k f(k) = \int_0^{+\infty} \frac{1}{\mathrm{e}^t + 1}(-g(t))\,\mathrm{d}t. \tag{2.4}$$

The first idea for numerical calculation of the integrals (2.3) and (2.4) is an application of the Gauss-Laguerre quadrature rule [18, p. 325], with the weight $w(t) = \mathrm{e}^{-t}$, but the convergence of these Gauss-Laguerre rules can be very slow because of the presence of poles on the imaginary axis at the points $2k\pi i$ ($k = \pm 1, \pm 2, \ldots$) and $(2k+1)\pi i$ ($k = 0, \pm 1, \pm 2, \ldots$), respectively.

Another approach was taken in [13] using Gaussian quadrature formulas on $(0, +\infty)$,

$$\int_0^{+\infty} w(t)h(t)\,\mathrm{d}t = \sum_{\nu=1}^{n} A_\nu h(x_\nu) + R_n(h), \tag{2.5}$$

with respect to the weight functions $w(t) = \varepsilon(t) = t/(\mathrm{e}^t - 1)$ (*Einstein weight*) and $w(t) = \varphi(t) = 1/(\mathrm{e}^t + 1)$ (*Fermi weight*) and applying them to (2.3) and (2.4), respectively. These functions are widely used in solid state physics.

For example, applying the Gauss-Einstein formula (2.5) to (2.3), with $w(t) = \varepsilon(t)$ and $h(t) = g(t)/t$, we obtain

$$T = \sum_{k=1}^{+\infty} f(k) = \sum_{\nu=1}^{n} A_\nu \frac{g(x_\nu)}{x_\nu} + R_n(h).$$

If h is a smooth function, this quadrature formula converges rapidly.

Example 2.1. We consider two simple series

$$T = \sum_{k=1}^{+\infty} \frac{1}{(k+1)^2} = \frac{\pi^2}{6} - 1 \qquad \text{and} \qquad S = \sum_{k=1}^{+\infty} \frac{(-1)^k}{(k+1)^2} = \frac{\pi^2}{12} - 1,$$

with the function $f(s) = (s+1)^{-2}$. According to (2.2) we have $g(t) = t\,\mathrm{e}^{-t}$. Then, (2.3) and (2.4) reduce to

$$T = \int_0^{+\infty} \varepsilon(t)\,\mathrm{e}^{-t}\,\mathrm{d}t \quad \text{and} \quad S = \int_0^{+\infty} \varphi(t)(-t\,\mathrm{e}^{-t})\,\mathrm{d}t,$$

respectively. Gauss-Einstein and Gauss-Fermi quadrature in n points can be used for calculating these integrals, respectively. Table 2.1 shows the n-point approximations $T(n)$ and $S(n)$ together with the relative errors $r_n(T)$ and $r_n(S)$. The first digit in error is underlined and numbers in parentheses indicate decimal exponents.

TABLE 2.1. Gaussian approximations of the sums T and S, with the corresponding relative errors

q.f.	Gauss-Einstein		Gauss-Fermi	
n	$T(n)$	$r_n(T)$	$S(n)$	$r_n(S)$
5	$.644\underline{7}42$	$3.0(-4)$	$-.177\underline{7}53$	$1.2(-3)$
10	$.6449340\underline{5}94$	$1.1(-8)$	$-.1775329\underline{7}80$	$6.5(-8)$
15	$.644934066848\underline{0}17$	$3.2(-13)$	$-.17753296657\underline{6}25$	$2.1(-12)$

As we can see the corresponding Gaussian rules converge rapidly. For example, 15-point Gauss-Einstein quadrature yields 12 correct decimal digits of the series T. In contrast, $10\,000$ terms of this series would give only 3-digit accuracy. Also, on the basis of Leibniz' convergence criterion for the series S, notice that the same accuracy as the one achieved for $n = 15$ would require the summation of approximately 690 000 terms.

(b) Also we can put

$$f(s) = \int_0^{+\infty} t\,\mathrm{e}^{-st}\,g(t)\,\mathrm{d}t, \qquad \operatorname{Re} s \geq 1. \tag{2.6}$$

Then, after a short calculation, we obtain

$$T = \int_0^{+\infty} \varepsilon(t) g(t)\,\mathrm{d}t \quad \text{and} \quad S = \int_0^{+\infty} \varphi(t)(-tg(t))\,\mathrm{d}t.$$

Example 2.2. Let $f(s) = s^{-1}\exp(-1/s)$. According to (2.6) we have that $g(t) = J_0(2\sqrt{t})$, where J_0 is the Bessel function of the first kind and order zero. It can be used for finding sums of the following series

$$\text{(a)} \qquad \sum_{k=1}^{+\infty} (k-1)k^{-3}\exp(-1/k),$$

$$\text{(b)} \qquad \sum_{k=1}^{+\infty} (-1)^{k-1}(k-1)k^{-3}\exp(-1/k),$$

$$\text{(c)} \qquad \sum_{k=1}^{+\infty} (-1)^{k-1}k^{-1}\exp(-1/k).$$

The first $10\,000$ terms of these series yield, respectively, 3, 7 and 4 correct decimal digits.

Since $J_0(2\sqrt{t})$ is an entire function, we expect Gaussian quadratures (with respect to Einstein and Fermi weights) to converge rapidly. This is confirmed in Table 2.2, which shows the relative errors for the n-point rule, $n = 2(2)12$. The exact sums (to 24 significant digits), as determined by Gaussian quadratures, are $0.342918943844609780961838$, $-0.0441559381340836052736928$, and $0.197107936397950656955672$, respectively,

TABLE 2.2. Relative errors in Gaussian approximation of the sums (a), (b) and (c)

n	(a)	(b)	(c)
2	$4.48(-2)$	$8.95(-1)$	$1.77(-2)$
4	$3.80(-6)$	$2.39(-4)$	$9.65(-7)$
6	$3.35(-11)$	$3.71(-9)$	$6.32(-12)$
8	$7.01(-17)$	$1.13(-14)$	$1.05(-17)$
10	$6.86(-23)$	$1.08(-20)$	$1.27(-23)$
12		$1.93(-23)$	

Thus, if the series T and S are slowly convergent and the respective functions in the integral representations are smooth, then low-order Gaussian quadrature (2.5) applied to the integrals on the right provides a possible summation procedure. Several numerical examples were analyzed in [13, §4]. In particular, Gautschi [9] analyzed examples with the general term $f(k) = k^{-1/2}/(k+a)^m$, where $\operatorname{Re} a \geq 0$ and $m \geq 1$. The series T with $a = m = 1$ appeared in a study of spirals given by Davis [4].

A problem which arises with this procedure is the determination of the original function g for a given series. In the next subsection we give a simpler method.

2.2. **Contour integration over a rectangle.** We give an alternative summation/integration procedure for the series (2.1), when for $k \geq m$, the function f is analytic in the region

$$\left\{z \in \mathbb{C} \;\middle|\; \operatorname{Re} z \geq \alpha,\ m-1 < \alpha < m\right\}. \tag{2.7}$$

In fact, we consider the series

$$T_m = \sum_{k=m}^{+\infty} f(k) \quad \text{and} \quad S_m = \sum_{k=m}^{+\infty} (-1)^k f(k),$$

where $m \in \mathbb{Z}$.

The method requires the indefinite integral F of f chosen so as to satisfy certain decay properties ((C1) – (C3) below). Using contour integration over a rectangle in the complex plane we are able to reduce T_m and S_m to a problem of Gaussian quadrature rules on $(0, +\infty)$ with respect to the hyperbolic weight functions

$$w_1(t) = \frac{1}{\cosh^2 t} \qquad \text{and} \qquad w_2(t) = \frac{\sinh t}{\cosh^2 t}, \tag{2.8}$$

respectively (see Milovanović [20]).

Assume that f and g are analytic functions in a certain domain D of the complex plane with singularities $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$, respectively, in a region $G = \operatorname{int}\Gamma\,(\subset D)$, where Γ is a closed contour. Then by Cauchy's residue theorem, we have

$$\frac{1}{2\pi i}\oint_\Gamma f(z)g(z)\,dz = \sum_\nu \operatorname*{Res}_{z=a_\nu}\Big(f(z)g(z)\Big) + \sum_\nu \operatorname*{Res}_{z=b_\nu}\Big(f(z)g(z)\Big). \tag{2.9}$$

Let

$$G = \Big\{z \in \mathbb{C} \mid \alpha \le \operatorname{Re} z \le \beta,\ |\operatorname{Im} z| \le \frac{\delta}{\pi}\Big\},$$

where $m-1 < \alpha < m$, $n < \beta < n+1$ $(m, n \in \mathbb{Z}, m \le n)$, $\Gamma = \partial G$, and $g(z) = \pi/\tan \pi z$. Then from (2.9) it immediately follows that

$$T_{m,n} = \sum_{\nu=m}^{n} f(\nu) = \frac{1}{2\pi i}\oint_\Gamma f(z)\frac{\pi}{\tan \pi z}\,\mathrm{d}z - \sum_\nu \operatorname*{Res}_{z=a_\nu}\Big(f(z)\frac{\pi}{\tan \pi z}\Big).$$

Similarly, for $g(z) = \pi/\sin \pi z$ we have

$$S_{m,n} = \sum_{\nu=m}^{n} (-1)^\nu f(\nu) = \frac{1}{2\pi i}\oint_\Gamma f(z)\frac{\pi}{\sin \pi z}\,dz - \sum_\nu \operatorname*{Res}_{z=a_\nu}\Big(f(z)\frac{\pi}{\sin \pi z}\Big).$$

For a holomorphic function f in G, the last formulas become

$$T_{m,n} = \frac{1}{2\pi i}\oint_\Gamma f(z)\frac{\pi}{\tan \pi z}\,\mathrm{d}z \quad \text{and} \quad S_{m,n} = \frac{1}{2\pi i}\oint_\Gamma f(z)\frac{\pi}{\sin \pi z}\,\mathrm{d}z.$$

After integration by parts, these formulas reduce to

$$T_{m,n} = \frac{1}{2\pi i}\oint_\Gamma \Big(\frac{\pi}{\sin \pi z}\Big)^2 F(z)\,\mathrm{d}z \tag{2.10}$$

and

$$S_{m,n} = \frac{1}{2\pi i}\oint_\Gamma \Big(\frac{\pi}{\sin \pi z}\Big)^2 \cos \pi z\, F(z)\,\mathrm{d}z, \tag{2.11}$$

where F is an integral of f.

Assume now the following conditions for the function F (cf. [17, p. 57]):

(C1) *F is a holomorphic function in the region* (2.7);

(C2) $\lim\limits_{|t|\to+\infty} e^{-c|t|} F\left(x + it/\pi\right) = 0$, *uniformly for* $x \ge \alpha$;

(C3) $\lim\limits_{x\to+\infty} \int_{-\infty}^{+\infty} e^{-c|t|} \left|F\left(x + it/\pi\right)\right| \,\mathrm{d}t = 0$,

where $c = 2$ *or* $c = 1$, *when we consider* $T_{m,n}$ *or* $S_{n,m}$, *respectively.*

Setting $\alpha = m - 1/2$, $\beta = n + 1/2$, and letting $\delta \to +\infty$ and $n \to +\infty$ (i.e., $\beta \to +\infty$), we prove that the integrals in (2.10) and (2.11) over Γ reduce to integrals along the line $z = \alpha + iy$ $(-\infty < y < +\infty)$, so that

$$T_m = T_{m,\infty} = -\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\left(\frac{\pi}{\sin\pi z}\right)^2 F(z)\,\mathrm{d}z \tag{2.12}$$

and

$$S_m = S_{m,\infty} = -\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\left(\frac{\pi}{\sin\pi z}\right)^2 \cos\pi z\, F(z)\,\mathrm{d}z. \tag{2.13}$$

Equality (2.12) can be reduced to

$$T_m = -\frac{1}{2}\int_{-\infty}^{+\infty} w_1(t)F\left(\alpha + it/\pi\right)\,\mathrm{d}t = \int_{0}^{+\infty} w_1(t)\Phi\left(\alpha, t/\pi\right)]\,\mathrm{d}t, \tag{2.14}$$

where w_1 is defined in (2.8) and

$$\Phi(x,y) = -\frac{1}{2}\left[F(x+iy) + F(x-iy)\right].$$

Similarly, (2.13) reduces to

$$S_m = \int_{0}^{+\infty} w_2(t)\Psi\left(\alpha, t/\pi\right)\,\mathrm{d}t, \tag{2.15}$$

where w_2 is also defined in (2.8) and

$$\Psi(x,y) = \frac{(-1)^m}{2i}\left[F(x+iy) - F(x-iy)\right].$$

Here, $\alpha = m - 1/2$. Formulas (2.14) and (2.15) suggest that the Gaussian quadrature is applied to the integrals on the right, using the weight functions w_1 and w_2, respectively (see [20]).

Example 2.3. Consider again the series from Example 2.1, denoted now as T_1 and S_1, respectively. Here, $f(z) = (z+1)^{-2}$, and $F(z) = -(z+1)^{-1}$, the integration constant being zero on account of condition (C3). Thus,

$$\Phi(x,y) = \mathrm{Re}\,\frac{1}{z+1} = \frac{x+1}{(x+1)^2+y^2}, \quad \Psi(x,y) = \mathrm{Im}\,\frac{1}{z+1} = \frac{-y}{(x+1)^2+y^2}.$$

Now, we apply Gaussian quadrature formulas with respect to the hyperbolic weights w_1 and w_2 given in (2.8) to T_1 and S_1, respectively. Table 2.3 shows the corresponding n-point Gaussian approximations $T_1(n)$ and $S_1(n)$ to T_1 and S_1, respectively, together with the relative errors $r_n(T_1)$ and $r_n(S_1)$, for $n = 5(5)25$.

These results can be significantly improved if we apply this method to sum the series T_m, $m > 1$. That is, we use

$$T_1 = \sum_{k=1}^{m-1}\frac{1}{(k+1)^2} + T_m, \qquad T_m = \sum_{k=m}^{+\infty}\frac{1}{(k+1)^2}. \tag{2.16}$$

TABLE 2.3. Gaussian approximation of the sums T_1 and S_1 and relative errors

n	$T_1(n)$	$r_n(T_1)$	$S_1(n)$	$r_n(S_1)$
5	$.644934\underline{1}49$	$1.3(-7)$	$-.1775\underline{5}20$	$1.1(-4)$
10	$.644934066\underline{7}76$	$1.1(-10)$	$-.17753\underline{3}03$	$3.5(-7)$
15	$.644934066848\underline{1}58$	$1.1(-13)$	$-.17753296\underline{5}69$	$5.0(-9)$
20	$.64493406684822\underline{7}33$	$1.4(-15)$	$-.1775329665\underline{9}17$	$8.9(-11)$
25	$.6449340668482264\underline{4}05$	$6.2(-18)$	$-.177532966575\underline{2}86$	$3.4(-12)$

Then, for $m = 2(1)5$ we obtain results whose relative errors are presented in Table 2.4.

TABLE 2.4. Relative errors in Gaussian approximation of the sum T_1 expressed in the form (2.16) for $m = 2(1)5$

n	$m=2$	$m=3$	$m=4$	$m=5$
5	$5.4(-9)$	$1.9(-10)$	$8.6(-12)$	$3.7(-13)$
10	$1.1(-13)$	$1.7(-16)$	$7.9(-18)$	$2.0(-19)$
15	$3.8(-17)$	$3.7(-20)$	$1.1(-22)$	$3.8(-25)$
20	$4.0(-20)$	$1.2(-24)$	$1.9(-27)$	$2.3(-29)$
25	$1.1(-22)$	$2.0(-27)$	$2.6(-30)$	$2.5(-33)$
30	$1.4(-25)$	$1.1(-31)$	$2.2(-33)$	
35	$3.2(-27)$	$2.4(-32)$		
40	$3.6(-30)$			

The rapid speed of convergence of the summation process as m increases is due to the poles $\pm i\big(m+1/2\big)\pi$ of $\Phi\big(m-1/2, t/\pi\big)$ moving away from the real line. It is interesting to note that a similar approach with the *Laplace transform method* does not lead to acceleration of convergence. For example, in the case of (2.16), we have that

$$T_m = \sum_{k=1}^{+\infty} \frac{1}{(k+m)^2} = \int_0^{+\infty} \varepsilon(t) e^{-mt}\, dt.$$

Example 2.4. The application of the *Laplace transform method* to the series

$$\sum_{k=1}^{+\infty} (k-1)k^{-3} \exp(-1/k) = .3429189438446097809618376779 02 \tag{2.17}$$

leads to an integration of the Bessel function $J_0(2\sqrt{t})$ (see Example 2.2). Here, however, we work with the exponential function $F(z) = -e^{-1/z}/z$, i.e.,

$$\Phi(x,y) = \frac{1}{r^2} e^{-x/r^2} \Big(x \cos \frac{y}{r^2} + y \sin \frac{y}{r^2} \Big), \qquad r^2 = x^2 + y^2.$$

As for accuracy, a similar situation prevails as in the previous example. Table 2.5 shows the relative errors in Gaussian approximations for $n = 2(4)18$ and $m = 1(1)3$.

TABLE 2.5. Relative errors in Gaussian approximation of the sum (2.17)

n	$m=1$	$m=2$	$m=3$
2	$2.9(-3)$	$1.2(-5)$	$2.1(-8)$
6	$1.3(-4)$	$3.7(-8)$	$1.2(-10)$
10	$1.8(-5)$	$3.7(-11)$	$9.9(-14)$
14	$1.2(-6)$	$1.2(-12)$	$1.2(-16)$
18	$1.3(-7)$	$8.5(-15)$	$6.6(-19)$

Example 2.5. Consider now

$$T_1(a) = \sum_{k=1}^{+\infty} \frac{1}{\sqrt{k}(k+a)}\,. \tag{2.18}$$

This series with $a = 1$ appeared in a study of spirals (see Davis [4]) and defines the "Theodorus constant." The first $1\,000\,000$ terms of the series $T_1(1)$ give the result $1.8580\ldots$, i.e., $T_1(1) \approx 1.86$ (only 3-digit accuracy). Using the *method of the Laplace transform*, Gautschi (see [9, Example 5.1]) calculated (2.18) for $a = .5$, 1, 2, 4, 8, 16, and 32. As a increases, the convergence of the Gauss quadrature formula slows down considerably. For example, when $a = 8$, we have results with relative errors presented in Table 2.6.

In order to achieve better accuracy, when a is large, Gautschi [9] used "stratified" summation by letting $k = \lambda + \kappa a_0$ and summing over all $\kappa \geq 0$ for $\lambda = 1, 2, \ldots, a_0$, where $a_0 = \lfloor a \rfloor$ denotes the largest integer $\leq a$ ($a = a_0 + a_1$, $a_0 \geq 1$, $0 \leq a_1 < 1$). Now, we directly apply the *method of contour integration*

TABLE 2.6. Relative errors in the *method of Laplace transform* for the series (2.18) with $a = 8$.

$n=5$	$n=10$	$n=15$	$n=20$	$n=25$	$n=30$	$n=35$	$n=40$
$1.4(-1)$	$2.3(-2)$	$1.5(-3)$	$1.9(-4)$	$2.5(-5)$	$2.1(-6)$	$2.5(-7)$	$2.6(-8)$

over the rectangle to (2.18) with

$$F(z) = \frac{2}{\sqrt{a}} \left(\arctan \sqrt{\frac{z}{a}} - \frac{\pi}{2} \right),$$

where the integration constant is taken so that $F(\infty) = 0$. For computing the arctan function in the complex plane ($z^2 \neq -1$) we use the formula

$$\arctan z = \frac{1}{2} \arg(u + iv) + \frac{i}{4} \log \frac{x^2 + (y+1)^2}{x^2 + (y-1)^2},$$

where $z = x + iy$, $u = 1 - x^2 - y^2$, $v = 2x$.

TABLE 2.7. Relative errors in Gaussian approximation of the sum (2.19) for $m = 4$

n	$a = .5$	$a = 1.$	$a = 2.$	$a = 4.$
5	1.4(−11)	8.4(−12)	4.5(−12)	2.6(−12)
10	6.8(−18)	4.4(−18)	2.2(−18)	1.2(−18)
15	5.4(−22)	2.7(−22)	1.6(−22)	1.0(−22)
20	1.2(−25)	5.9(−26)	3.3(−26)	2.0(−26)
25	1.0(−28)	5.2(−29)	3.0(−29)	1.9(−29)
30	1.1(−31)	5.7(−32)	3.3(−32)	2.0(−32)
n	$a = 8.$	$a = 16.$	$a = 32.$	$a = 64.$
5	1.7(−12)	1.1(−12)	7.6(−13)	5.2(−13)
10	7.7(−19)	5.1(−19)	3.4(−19)	2.4(−19)
15	6.7(−23)	4.5(−23)	3.0(−23)	2.1(−23)
20	1.3(−26)	8.7(−27)	5.9(−27)	4.1(−27)
25	1.2(−29)	8.1(−30)	5.5(−30)	3.8(−30)
30	1.3(−32)	9.0(−33)	6.2(−33)	3.6(−33)

TABLE 2.8. The exact sums $T_1(a)$

a	$T_1(a)$
1/2	2.1344166429862372611014895280 4
1	1.8600250792211903071806959157 2
2	1.5396805123533020128750184199 8
4	1.2182740146698908458291597629 1
8	9.3137293400310387168575138966 5(−1)
16	6.9493171464104559016304607166 9(−1)
32	5.0992651702721134803613196760 2(−1)
64	3.6993169824967113220994236490 7(−1)

As before, we can represent (2.18) in the form

$$T_1(a) = \sum_{k=1}^{m-1} \frac{1}{\sqrt{k}(k+a)} + T_m(a), \qquad T_m(a) = \sum_{k=m}^{+\infty} \frac{1}{\sqrt{k}(k+a)}, \tag{2.19}$$

and then use Gaussian quadrature formula to calculate $T_m(a)$. Relative errors in approximations for $T_1(a)$, when $m = 4$ and $a = p_\nu$, $\nu = 0(1)7$, where $p_0 = .5$ and $p_{\nu+1} = 2p_\nu$, are displayed in Table 2.7.

As we can see from Table 2.7, the method presented is very efficient. Moreover, its convergence is slightly faster if the parameter a is larger. The exact sums $T_1(a)$ (to 30 significant digits), as determined by Gaussian quadrature, are presented in Table 2.8.

Numerical experiments show that it is enough to use only the quadrature with respect to the first weight $w_1(t) = 1/\cosh^2 t$. Namely, in the series S_m we can include the hyperbolic sine as a factor in the corresponding integrand so that

$$S_m = \int_0^{+\infty} w_1(t)[\Psi(m-1/2, t/\pi)\sinh(t)]\, \mathrm{d}t.$$

Such an application was given in [22] to summation of slowly convergent series

$$T_m(\nu,a,p) = \sum_{k=m}^{+\infty} \frac{k^{\nu-1}}{(k+a)^p} \quad \text{and} \quad S_m(\nu,a,p) = \sum_{k=m}^{+\infty} (-1)^k \frac{k^{\nu-1}}{(k+a)^p}\,,$$

where $m \in \mathbb{Z}$, $0 < \nu \le 1$, and a and p are such as to ensure convergence of these series.

Numerical methods for summation of certain slowly convergent power series were considered by Gautschi [10], [11, pp. 249–253], Dassiè, Vianello, and Zanovello [2, 3], Dahlquist [7], etc.

2.3. **Series with irrational terms.** In this subsection we consider some series of the form

$$U_\pm(a,\nu) = \sum_{k=1}^{+\infty} \frac{(\pm 1)^{k-1}}{(k^2+a^2)^{\nu+1/2}}\,.$$

In 1916 Kapteyn (see [29, p. 386]) proved the formula

$$U_+(a,\nu) = \sum_{k=1}^{+\infty} \frac{1}{(k^2+a^2)^{\nu+1/2}} = \frac{\sqrt{\pi}}{(2a)^\nu \Gamma(\nu+1/2)} \int_0^{+\infty} \frac{t^\nu}{e^t-1} J_\nu(at)\, dt$$

which is valid when $\operatorname{Re}\nu > 0$ and $|\operatorname{Im} a| < 1$. Here, J_ν is the Bessel function of the order ν. Since for $F(p) = 1/(p^2+a^2)^{\nu+1/2}$ $\big(\operatorname{Re}\nu > -1/2,\ \operatorname{Re} p > |\operatorname{Im} a|\big)$ the original function is

$$f(t) = \frac{\sqrt{\pi}}{(2a)^\nu \Gamma(\nu+1/2)}\, t^\nu J_\nu(at),$$

using the Laplace transform method we find

$$U_-(a,\nu) = \sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{(k^2+a^2)^{\nu+1/2}} = \frac{\sqrt{\pi}}{(2a)^\nu \Gamma(\nu+1/2)} \int_0^{+\infty} \frac{t^\nu}{e^t+1} J_\nu(at)\, dt.$$

Thus, this method leads to an integration of the Bessel function $J_\nu(at)$ with Einstein's weight $\varepsilon(t)$ or Fermi's weight $\varphi(t)$. For some special values of ν, we can use also quadratures with respect to the weights $t^{\pm 1/2}\varepsilon(t)$ and $t^{\pm 1/2}\varphi(t)$ (see [13] and [9]).

In order to sum the series $U_-(a,0)$, $a>0$, we can integrate the function $F(z) = g(z)/\sqrt{z^2+a^2}$, with $g(z) = \pi/\sin \pi z$, over the circle

$$C_n = \Big\{ z \in \mathbb{C} \;\Big|\; |z| = n + \frac{1}{2} \Big\}, \qquad n > a,$$

with cuts along the imaginary axis, so that the critical singularities ia and $-ia$ are eliminated (cf. [26, p. 217]). Precisely, the contour of integration Γ is given by $\Gamma = C_n^1 \cup l_1 \cup \gamma_1 \cup l_2 \cup C_n^2 \cup l_3 \cup \gamma_2 \cup l_4$, where C_n^1 and C_n^2 are parts of the circle C_n, γ_1 and γ_2 are small circular parts of radius ε and centres at $\pm ia$, and l_k $(k=1,2,3,4)$ are the corresponding line segments.

Let $F^*(z)$ be the branch of $F(z)$ which corresponds to the value of the square root which is positive for $z=1$. Since

$$\oint_\Gamma F^*(z)\, dz = 2\pi i \sum_{k=-n}^{n} \frac{(-1)^k}{\sqrt{k^2+a^2}},$$

and $\int_{\gamma_1} \to 0$, $\int_{\gamma_2} \to 0$, when $\varepsilon \to +0$, and $\int_{C_n^1 \cup C_n^2} \to 0$, when $n \to +\infty$, we obtain

$$\sum_{k=1}^{+\infty} \frac{(-1)^k}{\sqrt{k^2+a^2}} = -\frac{1}{2a} + \int_a^{+\infty} \frac{du}{\sinh \pi u \sqrt{u^2-a^2}},$$

i.e.,

$$\sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{\sqrt{k^2+a^2}} = \frac{1}{2a} - \frac{1}{2} \int_{-1}^{+1} \Big(t \sinh \frac{\pi a}{t} \Big)^{-1} \frac{dt}{\sqrt{1-t^2}}\,.$$

Thus, we reduced $U_-(a,0)$ to a problem of Gauss-Chebyshev quadrature. Since $t \mapsto \big(t \sinh(\pi a/t)\big)^{-1}$ is an even function we can apply the $(2n)$-point Gaussian approximations with only n functional evaluations, so that we have

$$U_-(a,0) \approx GC(n) = \frac{1}{2a} - \frac{\pi}{2n} \sum_{k=1}^{n} \Big(\tau_k \sinh \frac{\pi a}{\tau_k} \Big)^{-1},$$

where $\tau_k = \cos\big((2k-1)\pi/(4n)\big)$, $k = 1, \dots, n$.

Remark 2.1. The same method can be applied to the summation of the series

$$\sum_{k=-\infty}^{+\infty} f(k, \sqrt{k^2+a^2}) \quad \text{and} \quad \sum_{k=-\infty}^{+\infty} (-1)^k f(k, \sqrt{k^2+a^2}) \quad (a>0),$$

where f is a rational function. Then we integrate the function $z \mapsto F(z) = f(z, \sqrt{z^2+a^2})g(z)$, with $g(z) = \pi/\tan \pi z$ and $g(z) = \pi/\sin \pi z$, respectively, over the circle C_n with the cuts.

Some examples of series with irrational terms were given in [21].

3. Direct methods

These kinds of methods have been recently introduced by Milovanović and Cvetković [23]. Namely, they were primarily interested in a linear functional of the form

$$L_a^{p,q}(f) = \sum_{k=0}^{+\infty} \frac{1}{p^k} f\left(\frac{a}{q^k}\right) = \int_{\mathbb{R}} f(t)\,\mathrm{d}\mu(t), \tag{3.1}$$

which is a direct generalization of the q-integral defined by

$$\int_0^a f(x) d_{1/q}x := a(1-1/q) \sum_{k=0}^{+\infty} \frac{1}{q^k} f\left(\frac{a}{q^k}\right).$$

There is a simple connection between the q-integral and $L_a^{p,q}$,

$$\int_0^a f(x) d_{1/q}x = a(1-1/q) L_a^{q,q}(f).$$

For arbitrary (possibly complex values of p and q) it can be proved that polynomials orthogonal with respect to $L_a^{p,q}$ exist under the conditions

$$|pq^k| > 1 \quad (k \in \mathbb{N}_0) \quad \text{and} \quad q^k \neq 1 \quad (k \in \mathbb{N}). \tag{3.2}$$

Theorem 3.1. *The polynomials $\{p_k\}_{k\in\mathbb{N}_0}$, orthonormal with respect to $L_a^{p,q}$, exist under the conditions given in* (3.2)*, and they satisfy the following three-term recurrence relation*

$$xp_k(x) = \beta_{k+1}p_{k+1} + \alpha_k p_k(x) + \beta_k p_{k-1}(x),$$

where

$$\begin{aligned}
\alpha_k &= aq^k \frac{p+q-2pq^k(1+q)+pq^{2k}(p+q)}{(pq^{2k-1}-1)(pq^{2k+1}-1)}, \quad k \geq 0,\\
\beta_0^2 &= \frac{p}{p-1},\\
\beta_k^2 &= a^2 p\, q^{2k} \frac{(q^k-1)^2(pq^{k-1}-1)^2}{(pq^{2k-2}-1)(pq^{2k-1}-1)^2(pq^{2k}-1)}, \quad k \geq 1.
\end{aligned} \tag{3.3}$$

To any two uniformly bounded sequences of complex numbers α_k, β_k, $k \in \mathbb{N}_0$, we associate the infinite (possibly) complex Jacobi matrix

$$J = \begin{bmatrix} \alpha_0 & \beta_1 & 0 & \dots \\ \beta_1 & \alpha_1 & \beta_2 & \dots \\ 0 & \beta_2 & \alpha_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{3.4}$$

This Jacobi matrix can be interpreted to be a linear operator acting on the Hilbert space ℓ^2 of all complex square-summable sequences with the usual scalar product $\langle u, v\rangle = \sum_{i\in\mathbb{N}_0} u_i\overline{v_i}$. The value of the operator can be defined as the result of matrix multiplication of the infinite matrix given in (3.4) with an infinite vector representing an element from ℓ^2. We refer to the Jacobi matrix when we mean to refer to the associated linear operator and vice versa.

For the functional $L_a^{p,q}$, the symbol $J_a^{p,q}$ represents a complex Jacobi matrix and/or a linear operator acting on ℓ^2 related to it and constructed with the sequences given in (3.3).

It is known that all zeros of orthogonal polynomials lie in the closure of the numerical range of the operator $J_a^{p,q}$. We recall that the numerical range of an operator J is defined by

$$\Theta(J) = \{\langle Jx, x\rangle \mid x \in \ell^2,\ ||x|| = 1\}$$

and its closure is denoted by $\Gamma(J) = \overline{\Theta(J)}$.

In [23] we proved the following result:

Theorem 3.2. *Under the condition $|q| > 1$, the linear operator $J_a^{p,q}$ is compact, indeed even of trace class. In the case when $|q| = 1$, with $q^n \neq 1$, $n \in \mathbb{N}$, and $|p| > 1$, the linear operator $J_a^{p,q}$ is bounded but not compact. All zeros of the related orthogonal polynomials lie in the set*

$$\Gamma(J_a^{p,q}) \subset \{z||z| \le |\beta_1| + |\alpha_0|\},$$

when $|q| > 1$. In the second case, all zeros of the related orthogonal polynomials lie in the set

$$\Gamma(J_a^{p,q}) \subset \left\{ z \,\middle|\, |z| \le |a|\frac{|p|^2 + 6|p| + 1 + 4\sqrt{|p|}(|p|-1)}{(|p|-1)^2} \right\}.$$

The corresponding quadratures of Gaussian type for (3.1) and a few numerical examples can be also found in [23]. The convergence of Gaussian quadrature rules for approximating certain series was investigated by Milovanović and Cvetković [24].

Very recently Monien [27] has considered the linear functional given by

$$L(f) = \sum_{k=1}^{+\infty} \frac{1}{k^2} f\left(\frac{1}{k^2}\right).$$

The corresponding (monic) orthogonal polynomials $\{\pi_n\}_{n=0}^{+\infty}$ satisfy the three-term recurrence relation $\pi_{n+1}(t) = (t - a_n)\pi_n(t) - b_n\pi_{n-1}(t)$, $p_0(t) = 1$, $p_{-1}(t) = 0$, with

$$a_n = \frac{2\pi^2}{(4n+1)(4n+5)}, \quad b_n = \frac{\pi^4}{(4n-1)(4n+1)^2(4n+3)}, \quad n \in \mathbb{N},$$

and $a_0 = \pi^2/15$. These polynomials are orthogonal with respect to the following scalar product

$$\langle \pi_n, \pi_m \rangle = \sum_{k=1}^{+\infty} \frac{1}{k^2}\pi_n\left(\frac{1}{k^2}\right)\pi_m\left(\frac{1}{k^2}\right) = \frac{(4n+3)\pi^3}{2^{4n+5}\Gamma^2(2n+5/2)}\delta_{nm}$$

and they can be expressed in terms of Bessel polynomials with an imaginary argument. It is interesting to mention that these polynomials appear in some extremal problems of Markov type in the L^2-norm on $(-1, 1)$ (see Milovanović [19], as well as the monograph [25, pp. 574–582]).

Using available software tools the corresponding Gaussian quadrature rules have been constructed [27]. Also, two nice examples have been presented.

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REFERENCES

[1] C. Brezinski, M. Redivo Zaglia, *Extrapolation Methods: Theory and Practice,* North-Holland, Amsterdam, 1991.

[2] S. Dassiè, M. Vianello, R. Zanovello, *Asymptotic summation of power series*, Numer. Math., 80 (1998), 61–73.

[3] S. Dassiè, M. Vianello, R. Zanovello, *A new summation method for power series with rational coefficients*, Math. Comp., 69 (2000), 749–756.

[4] P. J. Davis, *Spirals: from Theodorus to Chaos* (With contributions by Walter Gautschi and Arieh Iserles), A.K. Peters, Wellesley, MA, 1993.

[5] G. Dahlquist, *On summation formulas due to Plana, Lindelöf and Abel, and related Gauss-Christoffel rules, I*, BIT, 37 (1997), 256–295.

[6] G. Dahlquist, *On summation formulas due to Plana, Lindelöf and Abel, and related Gauss-Christoffel rules, II,* BIT, 37 (1997), 804–832.

[7] G. Dahlquist, *On summation formulas due to Plana, Lindelöf and Abel, and related Gauss-Christoffel rules, III*, BIT, 39 (1999), 51–78.

[8] S. Engblom, *Gaussian quadratures with respect to discrete measures*, Uppsala University, Technical Report 2006–007 (2006), 1–17.

[9] W. Gautschi, *A class of slowly convergent series and their summation by Gaussian quadrature*, Math. Comp., 57 (1991), 309–324.

[10] W. Gautschi, *On certain slowly convergent series occurring in plate contact problems*, Math. Comp., 57 (1991), 325–338.

[11] W. Gautschi, *Orthogonal Polynomials: Computation and Approximation,* Clarendon Press, Oxford 2004.

[12] W. Gautschi, *The Hardy–Littlewood function: an exercise in slowly convergent series*, J. Comput. Appl. Math., 179 (2005), 249–254.

[13] W. Gautschi, G.V. Milovanović, *Gaussian quadrature involving Einstein and Fermi functions with an application to summation of series*, Math. Comp., 44 (1985), 177–190.

[14] P. Henrici, *Applied and Computational Complex Analysis,* Vol. 1, Wiley, New York, 1984.

[15] L. B. W. Jolley, *Summation of Series,* Dover Publications, Inc., New York, 1961.

[16] D. Levin, *Methods for accelerating convergence of infinite series and integrals,* Ph.D. thesis, Tel Aviv University, 1975.

[17] E. Lindelöf, *Le Calcul des Résidus,* Gauthier–Villars, Paris, 1905.

[18] G. Mastroianni, G.V. Milovanović, *Interpolation Processes – Basic Theory and Applications,* Springer Verlag, Berlin – Heidelberg, 2008.

[19] G. V. Milovanović, *Various extremal problems of Markov's type for algebraic polynomials*, Facta Univ. Ser. Math. Inform., 2 (1987), 7–28.

[20] G. V. Milovanović, *Summation of Series and Gaussian Quadratures, In: Approximation and Computation* (R.V.M. Zahar, ed.), ISNM Vol. 119, Birkhäuser Verlag, Basel–Boston–Berlin, 1994, pp. 459–475.

[21] G. V. Milovanović, *Summation of Slowly Convergent Series via Quadratures, In: Advances in Numerical Methods and Applications* (I.T. Dimov, Bl. Sendov, and P.S. Vassilevski, eds.), World Scientific, Singapore, 1994, pp. 154–161.

[22] G. V. Milovanović, *Summation of series and Gaussian quadratures, II*, Numer. Algorithms, 10 (1995), 127–136.

[23] G. V. Milovanović, A. S. Cvetković, *An application of little* $1/q$*-Jacobi polynomials to summation of certain series*, Facta Univ. Ser. Math. Inform., 18 (2003), 31–46.

[24] G. V. Milovanović, A. S. Cvetković, *Convergence of Gaussian quadrature rules for approximation of certain series*, East J. Approx., 10 (2004), 171–187.

[25] G. V. Milovanović, D. S. Mitrinović, Th. M. Rassias, *Topics in Polynomials: Extremal Problems, Inequalities, Zeros,* World Scientific, Singapore – New Jersey – London – Hong Kong, 1994.

[26] D. S. Mitrinović, J. D. Kečkić, *The Cauchy Method of Residues – Theory and Applications,* Reidel, Dordrecht, 1984.

[27] H. Monien, *Gaussian quadrature for sums: a rapidly convergent summation scheme*, Math. Comp., 79 (2010), 857–869.

[28] A. Sidi, *Practical Extrapolation Methods: Theory and Applications,* Cambridge University Press, 2003.

[29] G. N. Watson, *Theory of Bessel Functions*, Cambridge, 1944.

[30] J. Wimp, *Sequence transformations and their applications,* Academic Press, New York, 1981.

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