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TRANSLATES OF SEQUENCES FOR SOME SMALL SETS

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This paper is dedicated to G. Čupona, S. Kurepa, W. Rudin, V. Perić and B. Schweizer who have passed away recently

Abstract. D. Borwein and S. Z. Ditor have found a measurable subset A of the real line having positive Lebesgue measure and a decreasing sequence (d_n) of reals converging to 0 such that, for each x, $x + d_n \notin A$ for infinitely many n. The set they constructed is nowhere dense. This result prompted us to further explore the question of subsets of R and R^2 that are of "small size" and the existence of null sequences with the described property and hence attain some related results.

1. Introduction

D. Borwein and S.Z. Ditor [1] have proved the following theorem, answering a question of P. Erdos.

Theorem 1.1. (Borwein, Ditor 1978)

(1) *If A is a measurable set in R with $m(A) > 0$, and (d_n) is a sequence of reals converging to 0, then for almost all $x \in A$, $x + d_n \in A$ for infinitely many n.*

(2) *There exists a measurable set A in R with $m(A) > 0$, and a (decreasing) sequence (d_n) converging to 0, such that, for each x, $x + d_n \notin A$ for infinitely many n.*

We recollect that in the proof of (2), the set of natural numbers N is partitioned

$$N = \{1, 2, \dots N_1\} \bigcup \{N_1 + 1, N_1 + 2, \dots N_2\} \bigcup \dots \bigcup \{N_k + 1, N_k + 2, \dots N_{k+1}\} \bigcup \dots$$

with $N_1 < N_2 < N_3 < \cdots < N_k < \cdots$ and A is constructed as follows:

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$$x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}, \quad a_n \in \{0,1\}, \quad \text{is in } A$$

iff for every k, there exist $m \in I_k$ so that $a_m = 0$, and $a_n = 1$ for infinitely many n, where

$$I_1 = \{1, 2, \dots N_1\}, I_2 = \{N_1 + 1, \dots, N_2\}, \dots I_k = \{N_{k-1} + 1, \dots, N_k\}, \dots.$$

A is closed and $m(A)$ can be made arbitrarily close to 1.

The sequence (d_n) is constructed as follows: For $k \in N$,

$$D_k = \left\{ x = \sum_{m \in I_k} \frac{a_m}{2^m}, \quad a_m \in \{0,1\}, \text{ at least one } a_m = 1 \right\}$$

$x = 0.000\dots 0 a_{N_{k-1}+1} \dots a_{N_k} 00 \dots$ and $D = \bigcup_{k=1}^{\infty} D_k$ is countable. The elements of D converge to 0.

We see that $m(A) > 0$, but A is nowhere dense. In 2010, H. I. Miller [2] proved a generalization of part (2) of the Borwein-Ditor theorem for nowhere dense subsets of $[0,1]$ (and consequently of R).

Theorem 1.2. (H.I. Miller 2010) *Suppose A is a nowhere dense subset of $[0,1]$. There exists a (decreasing) sequence (d_n) converging to 0, such that, for each x, $x + d_n \notin A$ for infinitely many n.*

Here we show a two-dimensional version of the above theorem.

Theorem 1.3. *Suppose A is a nowhere dense subset of $[0,1]X[0,1]$. There exists a sequence $(d_n) \in R^2$ converging to 0, such that, for each x, $x + d_n \notin A$ for infinitely many n.*

Proof. Suppose $n \in N$ is arbitrarily fixed. Divide $[0,1]X[0,1]$ into nxn squares by partitioning the interval $[0,1]$ into n subintervals $0 < \frac{1}{n} < \frac{2}{n} < \dots < 1$. In each square, fix the disk of radius $\frac{1}{4n}$ centered around the center of the square. Inside that disk, fix a disk that is disjoint with A. There exists $\delta_n \in N$ and $\epsilon_n > 0$ such that: for each $x \in [0,1]X[0,1]$, one of the rays $x + \lambda e^{i\frac{\pi}{\delta_n}}, x + \lambda e^{i\frac{2\pi}{\delta_n}}, x + \lambda e^{i\frac{3\pi}{\delta_n}}, \dots x + \lambda e^{i\frac{2\delta_n \pi}{\delta_n}}$ ($\lambda > 0$) crosses one of the fixed nxn disks disjoint with A with a segment of length $\epsilon_n > 0$.

Fix the maximal $k_n \in N$ with $k_n \epsilon_n < \frac{2}{n}$. Let

$$D_n = \Big\{ \epsilon_n \cdot e^{i\frac{\pi}{\delta_n}}, 2\epsilon_n \cdot e^{i\frac{\pi}{\delta_n}}, \dots, k_n\epsilon_n \cdot e^{i\frac{\pi}{\delta_n}}, \epsilon_n \cdot e^{i\frac{2\pi}{\delta_n}}, 2\epsilon_n \cdot e^{i\frac{2\pi}{\delta_n}}, \dots, k_n\epsilon_n \cdot e^{i\frac{2\pi}{\delta_n}}, \dots$$
$$\epsilon_n \cdot e^{i\frac{2\delta_n\pi}{\delta_n}}, 2\epsilon_n \cdot e^{i\frac{2\delta_n\pi}{\delta_n}}, \dots, k_n\epsilon_n \cdot e^{i\frac{2\delta_n\pi}{\delta_n}} \Big\}.$$

For each $x \in A$, there exists $d \in D_n$, $x + d \notin A$. Let $D = \bigcup_{n=1}^{\infty} D_n$. D is countable. The elements of D converge to 0. If they are arranged as a sequence, the proof is complete. $\square$

The question naturaly arises whether an analogous statement can be proved for other types of small sets in R, (for example sets of measure 0). We show that there exists a set of outer measure 0 in R for which the opposite of the statement in part (2) of the Borwein-Ditor theorem holds. First we show a more general result.

Theorem 1.4. *Suppose $A = [0,1] \setminus X$, where X is a set of first category. Then for every sequence (d_n) converging to 0, there exists $x \in A$, $x + d_n \in A$ for n large enough.*

Proof. Since X is a set of first category, $X = \bigcup_{i=1}^{\infty} X_i$ where X_i is nowhere dense in $[0,1]$ for $i \in N$. Then $A = \bigcap_{i=1}^{\infty} A_i$ where $A_i = [0,1] \setminus X_i$ for $i \in N$.

Suppose (d_n) is a sequence of reals converging to 0. Let $n_0 \in N$ be fixed so that $|d_n| < \frac{1}{4}$ for $n \geq n_0$. We need to find $x \in R$ that satisfies:

(A) $x \in A_i$ for $i \in N$,
(B) $x + d_n \in A_i$ for $i \in N$, $n \geq n_0$.

There are countably many conditions in (A) as well as in (B), so we can order them together as a sequence of conditions $C_1, C_2, \ldots C_k \ldots$.
Claim:
The set of $x \in [\frac{1}{4}, \frac{3}{4}]$ that satisfies condition C_k has a complement that is nowhere dense in $[\frac{1}{4}, \frac{3}{4}]$, for $k \in N$.
Proof of claim: If C_k is the condition that $x \in A_i$ for some i, then since X_i is nowhere dense in $[0,1]$, the claim is true. Suppose C_k is the condition that $x + d_n \in A_i$ for some $n \geq n_0$, and some $i \in N$. Since A_i has a complement that is nowhere dense in $[0,1]$, $-d_n + A_i$ has a complement that is nowhere dense in $[-d_n, 1 - d_n]$ and consequently nowhere dense in $[\frac{1}{4}, \frac{3}{4}]$ (since $|d_n| < \frac{1}{4}$). So the set of x for which $x + d_n \in A_i$ has a complement that is nowhere dense in $[\frac{1}{4}, \frac{3}{4}]$.
The claim is proved.

From the above claim, we see that the set of x in $[\frac{1}{4}, \frac{3}{4}]$ satisfying conditions $C_1, C_2, \ldots C_k \ldots$, is an intersection of countably many sets that have nowhere dense complements. This set has a complement of first category and is therefore nonempty. This completes the proof. $\square$

Remark 1.5. In the proof of the above theorem, it has been shown that the set of x satisfying the conclusion is actually large (has a complement of first category).

The next theorem is a corollary of Theorem 1.4.

Theorem 1.6. *There exists a set* $A \subset [0,1]$ *with outer measure* 0*, such that for every sequence* (d_n) *converging to* 0*, there exists* $x \in A$*,* $x + d_n \in A$ *for* n *large enough.*

Proof. By the proof of Borwein-Ditor (2) (see the first page), for each n we can construct $X_n \subset [0,1]$, X_n nowhere dense, $m(X_n) = 1 - \dfrac{1}{n}$ (by choosing $N_1, N_2, \ldots N_k \ldots$ appropriately). Let

$$A = [0,1] \setminus \bigcup_{n=1}^{\infty} X_n = \bigcap_{n=1}^{\infty} X_n^c.$$

Then A has outer measure 0 and the conclusion follows from Theorem 1.4. □

We add a natural generalization of Theorem 1.4 in two dimensions.

Theorem 1.7. *Suppose* $A = [0,1]X[0,1] \setminus X$*, where* X *is a set of first category. Then for every sequence* $(d_n) \in R^2$ *converging to* 0*, there exists* $x \in A$*,* $x + d_n \in A$ *for* n *large enough.*

Proof. Since X is a set of first category, $X = \bigcup_{i=1}^{\infty} X_i$ where X_i is nowhere dense in $[0,1]X[0,1]$ for $i \in N$. Then $A = \bigcap_{i=1}^{\infty} A_i$ where $A_i = [0,1]X[0,1] \setminus X_i$ for $i \in N$.

Suppose (d_n) is a sequence of vectors in R^2 converging to 0. Let $n_0 \in N$ be fixed so that $|d_n| < \frac{1}{4}$ for $n \geq n_0$. We need to find $x \in R^2$ that satisfies:

(A) $x \in A_i$ for $i \in N$,

(B) $x + d_n \in A_i$ for $i \in N$, $n \geq n_0$.

There are countably many conditions in (A) as well as in (B), so we can order them together as a sequence of conditions $C_1, C_2, \ldots C_k \ldots$.

Let B denote the closed disk of radius $\frac{1}{4}$ centered around $(\frac{1}{2}, \frac{1}{2})$. The following claim can be verified by the same reasoning that was used in the proof of Theorem 1.4 :

The set of $x \in B$ that satisfies condition C_k has a complement that is nowhere dense in B, for $k \in N$.

From the above claim, we see that the set of x satisfying conditions $C_1, C_2, \ldots C_k \ldots$, is nonempty. This completes the proof. □

References

[1] D. Borwein and S. Z. Ditor, *Translates of sequences in sets of positive measure*, Canad. Math. Bull., 21 (1978), 497–498.

[2] H.I. Miller and A.J. Ostaszewski, *Group action, shift compactness and the KBD theorem*, submitted for publication.

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