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ANALYTIC, REAL ANALYTIC AND HARMONIC GENERALIZED FUNCTIONS

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The paper is dedicated to Professor Veselin Perić and Svetozar Kurepa

Abstract. We recall definitions and assertions concerning the spaces noted in the title. Various classes of nonlinear problems can be studied within these spaces appropriate for the analysis of different kinds of singularities. Especially, we explain in this paper the notion of the generalized analytic wave front set.

1. Introduction

Generalized function algebras of Colombeau type contain Schwartz's distribution spaces so that all the linear operations for distributions hold; moreover, the product of smooth functions is preserved. The main advantage of generalized function algebras is that various classes of nonlinear problems can be studied in these frames as well as linear problems with different kinds of singularities. As one can expect, for the purpose of local and microlocal analysis, one needs to study classical function spaces within these algebras. Our aim in this note is to recall basic definitions and assertions concerning the spaces noted in the title. These spaces, in the frame of Colombeau algebra of generalized functions are studied by M. Oberguggenberger, D. Scarpalezos, V. Valmorin and the author. Especially we explain the notion of the generalized analytic wave front set.

All the presented results are published or will be published in the papers quoted at the beginning of the paragraphs.

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2. Definitions

We refer to [4], [5], [6], [12] and [28], [35] for the theory of Colombeau generalized functions and their use in various classes of equations. We refer to [9], [10], [11] and [16] for the local and microlocal $\mathcal{G}^\infty$-properties of Colombeau-type generalized functions.

2.1. **Definitions.** We recall the main definitions ([6], [12], [28]). Let ω be an open set in $\mathbb{R}^d$ and $\mathcal{E}(\omega)$ be the space of smooth functions with the sequence of seminorms $\mu_\nu(\phi) = \sup\{|\phi^{(\alpha)}(x)|; \alpha \leq \nu,\ x \in K_\nu\},\ \nu \in \mathbb{N}_0$, where $(K_\nu)_\nu$ is an increasing sequence of compact sets exhausting ω. Then the set of moderate nets $\mathcal{E}_M(\omega)$, respectively of null nets $\mathcal{N}(\omega)$, consists of nets $(f_\varepsilon)_\varepsilon \in \mathcal{E}(\omega)^{(0,1)}$ with the properties

$$(\forall n \in \mathbb{N})\ (\exists a \in \mathbb{R})\ (\mu_n(f_\varepsilon) = O(\varepsilon^a)),$$

$$\text{respectively,}\ \ (\forall n \in \mathbb{N})\ (\forall b \in \mathbb{R})\ (\mu_n(f_\varepsilon) = O(\varepsilon^b))$$

(O is the Landau symbol "big O"). Both spaces are algebras and the latter is an ideal of the former. Putting $v_n(r_\varepsilon) = \sup\{a;\ \mu_n(r_\varepsilon) = O(\varepsilon^a)\}$ and $e_n((r_\varepsilon)_\varepsilon, (s_\varepsilon)_\varepsilon) = \exp(-v_n(r_\varepsilon - s_\varepsilon)),\ n \in \mathbb{N}$, we obtain $(e_n)_n$, a sequence of ultra-pseudometrics on $\mathcal{E}_M(E)$ defining the ultra-metric topology (sharp topology) on $\mathcal{G}(E)$. The simplified Colombeau algebra $\mathcal{G}(\omega)$ is defined as the quotient $\mathcal{G}(\omega) = \mathcal{E}_M(\omega)/\mathcal{N}(\omega)$. This is also a differential algebra. If the nets $(f_\varepsilon)_\varepsilon$ consist of constant functions on ω (i.e. the seminorms μ_n reduce to the absolute value), then one obtains the corresponding spaces of nets of complex (or real) numbers $\mathcal{E}_M$ and $\mathcal{N}_0$. They are algebras, $\mathcal{N}_0$ is an ideal in $\mathcal{E}_{\mathcal{M}}$ and, as a quotient, one obtains the Colombeau algebra of generalized complex numbers $\bar{\mathbb{C}} = \mathcal{E}_M/\mathcal{N}_0$ (or $\bar{\mathbb{R}}$). It is a ring, not a field. The sharp topology in $\bar{\mathbb{C}}$ is defined as above. Note, a ball $\tilde{B}(x_0, r)$, where $x_0 = [(x_{0,\varepsilon})_\varepsilon] \in \bar{\mathbb{C}}$ and $r > 0$, in the sharp topology is given by $\tilde{B}(x_0, r) = \{[(x_\varepsilon)_\varepsilon]; |x_{0,\varepsilon} - x_\varepsilon| = O(\varepsilon^{-\ln r})\}$.

The embedding of Schwartz distributions in $\mathcal{E}'(\omega)$ is realized through the sheaf homomorphism $\mathcal{E}'(\omega) \ni f \mapsto [(f * \phi_\varepsilon|_\omega)_\varepsilon] \in \mathcal{G}(\omega)$, where the fixed net of mollifiers $(\phi_\varepsilon)_\varepsilon$ is defined by $\phi_\varepsilon = \varepsilon^{-d}\phi(\cdot/\varepsilon),\ \varepsilon < 1$, where $\phi \in \mathcal{S}(\mathbb{R}^d)$ satisfies

$$\int \phi(t)dt = 1,\ \int t^m \phi(t)dt = 0, m \in \mathbb{N}_0^n, |m| > 0.$$

($t^m = t_1^{m_1}, \ldots, t_n^{m_n}$ and $|m| = m_1 + \cdots + m_n$). $\mathcal{E}'(\omega)$ is embedded into $\mathcal{G}_c(\Omega)$ of compactly supported generalized functions and the sheaf homomorphism, extended onto $\mathcal{D}'$, gives the embedding of $\mathcal{D}'(\omega)$ into $\mathcal{G}(\omega)$.

The generalized algebra $\mathcal{G}_c(\omega)$ consists of elements in $\mathcal{G}(\omega)$ which are compactly supported while $\mathcal{G}^\infty(\omega)$ is defined in [28] as the quotient of $\mathcal{E}_M^\infty(\omega)$

and $\mathcal{N}(\omega)$, where $\mathcal{E}_M^\infty(\omega)$ consists of nets $(f_\varepsilon)_\varepsilon \in \mathcal{E}(\omega)^{(0,1)}$ with the property

$$(\forall K \subset\subset \omega)(\exists a \in \mathbb{R})(\forall n \in \mathbb{N})(|\sup_{x\in K} f_\varepsilon^{(n)}(x)| = O(\varepsilon^a)),$$

Note that $\mathcal{G}^\infty$ is a subsheaf of $\mathcal{G}$.

Concerning generalized points ([29], [12]), Colombeau extensions can be carried out for any metric space (A, d):

A net $(x_\varepsilon)_\varepsilon$ in A is called moderate, if

$$(\exists N \in \mathbb{N})(\exists x \in A)(d(x, x_\varepsilon) = O(\varepsilon^{-N}))$$

and an equivalence relation in $A^{(0,1]}$ is introduced by

$$(x_\varepsilon)_\varepsilon \sim (y_\varepsilon)_\varepsilon \Leftrightarrow (\forall p \geq 0)(d(x_\varepsilon, y_\varepsilon) = O(\varepsilon^p)).$$

$\widetilde{A} = A/\sim$ is called the set of generalized points in A, and as before,with valuations, we can define an ultra-metric in $\widetilde{A}$ (one usually calls the topology determined in that way the sharp topology). If $A = \Omega$ is an opens subset of $\mathbb{R}^d$, then $\widetilde{\Omega} = \Omega/\sim$ is the set of generalized points. Note that $\tilde{\mathbb{C}} = \bar{\mathbb{C}}$ ($\tilde{\mathbb{R}} = \bar{\mathbb{R}}$).

An element $\widetilde{x} \in \widetilde{\Omega}$ is called compactly supported if x_ε lies in a compact set for $\varepsilon < \varepsilon_0$ for some $\varepsilon_0 \in (0,1)$. The set of compactly supported points $\widetilde{x}$ ($\in \widetilde{\Omega}$) is denoted by $\widetilde{\Omega}_c$. By the functoriality of the Colombeau extension it follows that the evaluation map

$$(\mathcal{G}(\Omega) \times \widetilde{\Omega}_c) \to \widetilde{\mathbb{K}},\ (f, \widetilde{x}) \mapsto f(\widetilde{x}) := [(f_\varepsilon(x_\varepsilon))_\varepsilon]$$

is well defined and that $\widetilde{x} \mapsto f(\widetilde{x})$ as well as $f \mapsto f(\widetilde{x})$ are both continuous with respect to sharp topologies.

Compactly supported generalized points have been introduced by Oberguggenberger and Kunzinger in [29], where it is shown that elements of the special algebra can be determined uniquely by evaluation at these generalized points (whereas evaluation at standard points does not suffice for the determination of elements in $\mathcal{G}(\Omega)$). Moreover, it has been shown in [21] that near standard points are also sufficient for the point-value description in the special algebra. We recall that near standard points are elements $\widetilde{x} \in \widetilde{\Omega}_c$ with limit in Ω that is, that there exists $x \in \Omega$ such that for a representative $(x_\varepsilon)_\varepsilon$, $x_\varepsilon \to x$, as $\varepsilon \to 0$, holds. We denote by $\widetilde{\Omega}_{ns}$ the set of near standard points of Ω.

3. Holomorphic generalized functions [31]

We denote by $\mathcal{O}(\Omega)$ the space of holomorphic functions on Ω, where Ω is an open subset of $\mathbb{R}^2 = \mathbb{C}$; $D(z_0, r)$ denotes a disc with the center z_0 and radius $r > 0$.

Definition 1. *A generalized function* $f = [(f_\varepsilon)_\varepsilon] \in \mathcal{G}(\Omega)$ *is said to be holomorphic if* $\frac{\partial f}{\partial \bar{z}} = 0$ *in* $\mathcal{G}(\Omega)$.

The set of holomorphic generalized functions is denoted by $\mathcal{G}_H(\Omega)$. We recall the following local result, due to Colombeau-Galé [7]:

Theorem 1. $f \in \mathcal{G}_H(\Omega)$ *if and only if for every relatively compact open set* Ω' *in* Ω, f *admits a representative* $(f_\varepsilon)_\varepsilon \in \mathcal{E}_M(\Omega')$ *with* $f_\varepsilon \in \mathcal{O}(\Omega')$, $\varepsilon \in (0,1]$.

We have the following characterizations.

Theorem 2. *Let* $(f_\varepsilon)_\varepsilon \in \mathcal{E}(\Omega)^{(0,1]}$ *and suppose that for every point* $z_0 \in \Omega$ *there exist* $r_\varepsilon > 0, 0 < \varepsilon \leq 1$ *such that*

(i) $\bar{\partial} f_\varepsilon|_{D(z_0,r_\varepsilon)} = 0,\ 0 < \varepsilon \leq 1$, *i.e. the restrictions to the discs vanish.*

(ii) $\exists \eta > 0, \exists a > 0, \exists \varepsilon_0 \in (0,1],\ |f_\varepsilon^{(n)}(z_0)| \leq \eta^{n+1} n! \varepsilon^{-a},\ n \in \mathbb{N}, \varepsilon \in (0,\varepsilon_0)$.

Then $(f_\varepsilon)_\varepsilon \in \mathcal{E}_M(\Omega)$ *and* $[(f_\varepsilon)_\varepsilon] \in \mathcal{G}_H(\Omega)$.

This leads to the straightforward result that $\mathcal{G}_H(\Omega) \subset \mathcal{G}^\infty(\Omega)$.

We refer to [28] for the definition of $\mathcal{G}^\infty(\Omega)$. Note that Colombeau and Galé had mentioned in [7] that $\mathcal{G}_H(\Omega) \cap \mathcal{D}'(\Omega) = \mathcal{O}(\Omega)$. We collect properties of holomorphic generalized functions in the next theorem.

Theorem 3. *Let* $g \in \mathcal{G}_H(\Omega)$.

(i) g *admits a representative* $(g_\varepsilon)_\varepsilon$ *such that* $g_\varepsilon \in \mathcal{O}(\Omega)$, $\varepsilon \in (0,1]$.

(ii) $g = 0$ *if and only if for any open set* $\Omega' \subset\subset \Omega$ *there exists a representative* $(g_\varepsilon)_\varepsilon \in \mathcal{O}(\Omega')^{(0,1)}$ *such that*

$$\begin{aligned} &\forall z_0 \in \Omega', \forall a > 0, \exists \eta > 0, \exists \varepsilon_0 \in (0,1], \exists C > 0, \\ &|g_\varepsilon^{(n)}(z_0)| \leq \eta^{n+1} n! \varepsilon^a,\ n \in \mathbb{N},\ \varepsilon \in (0,\varepsilon_0). \end{aligned} \tag{3.1}$$

(iii) *Let* $z_0 \in \Omega$, $\eta > 0$. *Then* f *vanishes in the disc* $V = D(z_0, \frac{1}{\eta})$ *if and only if there exists a representative* $(g_\varepsilon)_\varepsilon$ *of* $f|V$ *in* $\mathcal{O}(V)^{(0,1]}$ *such that*

$$\forall a > 0, \exists \varepsilon_0 \in (0,1] : |g_\varepsilon^{(n)}(z_0)| \leq \eta^{n+1} n! \varepsilon^a;\ n \in \mathbb{N},\ \varepsilon \in (0,\varepsilon_0).$$

As a consequence, we obtain a simple proof of a theorem of Colombeau-Galé [8].

Theorem 4. *Let* Ω *be connected and let* $f \in \mathcal{G}_H(\Omega)$. *If* f *vanishes on a non-void open subset of* Ω, *then* f *vanishes on* Ω.

Note that the part (i) of Theorem 3 is a special case of the following theorem on elliptic partial differential operators whose proof, however, relies on a number of results from operator theory [24], [25].

Theorem 5. *Let Ω be an open subset of $\mathbb{R}^n$ and let P be an elliptic partial differential operator with constant coefficients. Let $U \in \mathcal{G}(\Omega)$ be a solution to $PU = 0$ in $\mathcal{G}(\Omega)$. Then U has a representative $(U_\varepsilon)_\varepsilon \in \mathcal{E}_M[\Omega]$ such that $PU_\varepsilon = 0,\ \varepsilon \leq 1$.*

The existence of global representatives has also been proved for other types of partial differential operators, for example, for the system of infinitesimal generators of the rotation group, see [27].

4. Real analytic generalized functions [33]

Definition 2. *Let ω denote an open set in $\mathbb{R}^d$ and x_0 a point of ω. A function $f \in \mathcal{G}(\omega)$ is said to be real analytic at x_0 if there exist an open ball $B = B(x_0, r)$ in ω containing x_0 and $(g_\varepsilon)_\varepsilon \in \mathcal{E}_M(B)$ such that*

(i)
$$f|B = [(g_\varepsilon)_\varepsilon] \text{ in } \mathcal{G}(B);$$

(ii)
$$(\exists \eta > 0)(\exists a > 0)(\exists \varepsilon_0 \in (0,1))$$
$$\sup_{x \in B} |\partial^\alpha g_\varepsilon(x)| \leq \eta^{|\alpha|+1} \alpha! \varepsilon^{-a},\ 0 < \varepsilon < \varepsilon_0,\ \alpha \in \mathbb{N}^d.$$

It is said that f is real analytic in ω if f is real analytic at each point of ω. The space of all generalized functions which are real analytic in ω is denoted by $\mathcal{G}_A(\omega)$.

The analytic singular support, singsupp $_{ga}f$, is the complement of the set of points $x \in \omega$ where f is real analytic.

It follows from the definition that $\mathcal{G}_A$ is a subsheaf of $\mathcal{G}$.

Using Stirling's formula it is seen that condition (ii) in Definition 2 is equivalent to

(iii)
$$(\exists \eta > 0)(\exists a > 0)(\exists \varepsilon_0 \in (0,1))$$
$$\sup_{x \in B} |\partial^\alpha g_\varepsilon(x_0)| \leq \eta(\eta|\alpha|)^{|\alpha|} \varepsilon^{-a},\ 0 < \varepsilon < \varepsilon_0,\ \alpha \in \mathbb{N}_0^d.$$

The use of Taylor expansion and condition (*ii*) of Definition 2 imply that $(g_\varepsilon)_\varepsilon$ admits a holomorphic extension in a complex ball $B = \{z \in \mathbb{C}^d; |z - x_0| < r\}$ which is independent of ε. Consequently we get a holomorphic extension G of $[(g_\varepsilon)_\varepsilon]$ and then $f|B = G|B$. (It is clear from the context whether B is a complex or real ball.)

In the sequel we will use a multidimensional notation:
$\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d,\ \alpha! = \alpha_1!...\alpha_d!, \eta^\alpha = \eta^{\alpha_1} \ldots \eta^{\alpha_d} (\eta > 0);$
$(iy)^j = (iy_1)^{j_1} \ldots (iy_d)^{j_d}, i = \sqrt{-1}, y \in \mathbb{R}^d, j = (j_1, \ldots, j_d) \in \mathbb{N}^d;$

Let $f \in \mathcal{G}_A(\Omega)$. For every compactly supported generalized point $x_0 = [(x_{0,\varepsilon})_\varepsilon]$ ($x_0 \in \tilde{\omega}_c$) and every generalized point $x = [(x_\varepsilon)_\varepsilon]$ in a sharp neighborhood V of x_0 the series $F^N = [(F^N_\varepsilon)_\varepsilon], N \in \mathbb{N}$, converges in the sense of sharp topology, where

$$F^N_\varepsilon(x_\varepsilon) = \sum_{|j|=0}^{N} f^{(j)}_\varepsilon(x_{0,\varepsilon})(x_\varepsilon - x_{0,\varepsilon})^j/j! \ \ (\varepsilon < \varepsilon_0).$$

The existence of a global holomorphic representative of a holomorphic generalized function in the one dimensional-case given in the previous section, implies the following theorem.

Theorem 6. *Let $f \in \mathcal{G}_{\mathcal{A}}(\omega), \omega \subset \mathbb{R}$.*

(i) *f admits a global real analytic representative, that is a representative $(f_\varepsilon)_\varepsilon$ such that (i) and (ii) in Definition 1 is satisfied by f_ε instead of g_ε for all $x_0 \in \omega$ and B depending on x_0.*

(ii) *For every open set $\omega_1 \subset\subset \omega$ there exist an open set Ω_1 such that $\Omega_1 \cap \mathbb{R}^d = \omega_1$ and a net of holomorphic functions $(F_\varepsilon)\varepsilon \in \mathcal{E}_M(\Omega_1)$ such that $F_\varepsilon|\omega_1 = f_\varepsilon$, $\varepsilon \in (0,1)$.*

(iii) *If $d = 1$ then the assertion in (ii) holds globally, with Ω instead of Ω_1.*

The existence of a global holomorphic representative of $f \in \mathcal{G}_H(\Omega)$ depends on $\Omega \subset \mathbb{C}^d$ is an open problem. We have such representation for appropriate domains $\Omega \subset \mathbb{C}^n$. Also if $f \in \mathcal{G}_A(\omega)$, $\omega \subset \mathbb{R}^d$, the existence of a global representative is not established yet.

It is proved in [19] that a generalized holomorphic function in an open set $\Omega \subset \mathbb{C}$ is equal to zero if and only if it is equal to zero at every classical point of Ω. The proof of the first part of the next theorem is a consequence of this result and the result of [8] and [31], where it is proved that a ($p-$dimensional) holomorphic generalized function equals zero in $\Omega \subset \mathbb{C}^p$, if it equals zero in an open subset of this set. Actually, the basic assertion which will be used, is that a generalized function equals zero if and only if it equals zero at every compactly supported generalized point [29].

Theorem 7. a) *Let Ω be an open set in $\mathbb{C}^p, p > 1$ and $f = [(f_\varepsilon)_\varepsilon] \in \mathcal{G}_H(\Omega)$ such that $f(x) = 0$ for every $x \in \Omega$ ($(f_\varepsilon(x))_\varepsilon \in \mathcal{N}(\Omega)$). Then $f \equiv 0$.*

b) *Let ω be an open set in $\mathbb{R}^d, d \in \mathbb{N}$ and $f = [(f_\varepsilon)_\varepsilon] \in \mathcal{G}_A(\omega)$ such that $f(x) = 0$ for every $x \in \omega$. Then $f \equiv 0$.*

The next theorem is a consequence of theorems of the next section.

Theorem 8. *Let $f \in \mathcal{E}'(\omega)$ and $f_\varepsilon = f * \phi_\varepsilon, \varepsilon \in (0,1)$ be its regularization by a net $\phi_\varepsilon = \frac{1}{\varepsilon}\phi(\cdot/\varepsilon), \varepsilon < 1$, where $(\phi_\varepsilon)_\varepsilon$ is a net of mollifiers. Then*

$$\textit{singsupp}_{\ a} f = \ \textit{singsupp}_{\ ga}[(f_\varepsilon)_\varepsilon],$$

where the analytic singular support of the distribution f is on the left hand side.

4.1. **Analytic wave front set.** We will use the notation $\hat{\phi}$ for the Fourier transformation

$$\mathcal{F}(\phi)(\xi) = \hat{\phi}(\xi) = \int_{\mathbb{R}^d} e^{-ix\xi}\phi(x)dx,\ \xi \in \mathbb{R}^d,\ \phi \in L^1(\mathbb{R}^d).$$

Definition 3. *Let ω denote an open set in $\mathbb{R}^d$. A sequence $(u_n)_n$ in $\mathcal{G}_c(\omega)$ will be called bounded if there exists a sequence of representatives $(u_{n,\varepsilon})_\varepsilon$, $n \in \mathbb{N}$, such that*

$$(\exists K \subset\subset \omega)(\exists m \in \mathbb{R})(\exists b > 0)$$
$$\operatorname{supp}(u_{n,\varepsilon}) \subset\subset K, \varepsilon \in (0,1] \quad \textit{and} \quad \sup_{\xi\in\mathbb{R}^d}(1+|\xi|)^{-m}|\widehat{u_{n,\varepsilon}}(\xi)| = O(\varepsilon^{-b}).$$

The following lemma in [15] will be needed in the sequel.

Lemma 1. ([15], Chapter 9, Lemma 22) *Let K be a compact set of $\mathbb{R}^d$, $r > 0$ and $K_r = \{x : d(x,K) \leq r\}$. There exist $C > 0$ and a sequence of smooth functions $(\varphi_n)_n$, such that:*

(1) *$\varphi_n = 1$ on K and* $\operatorname{supp}(\varphi_n) \subset\subset K_r$ *for all $n \in \mathbb{N}_0$;*
(2) $\sup_{x\in\mathbb{R}^d,|\alpha|\leq n} |\partial^\alpha \varphi_n(x)| \leq C(Cn/r)^{|\alpha|}$, *$n \in \mathbb{N}_0$.*

We have the following characterization of analytic generalized functions:

Theorem 9. *Let ω denote an open set in $\mathbb{R}^d$ and $x_0 \in \omega$. A function $f \in \mathcal{G}(\omega)$ is analytic at x_0 if and only if there exist a bounded sequence $(u_n)_n$ in $\mathcal{G}_c(\omega)$ and $B(x_0,r) \subset \omega$ such that the following two conditions hold.*

(i) *$u_n = f$ in $B(x_0,r)$ for all $n \in \mathbb{N}_0$;*
(ii) *A sequence of representatives $(u_{n,\varepsilon})_\varepsilon \in \mathcal{E}_{M,c}(\omega)$, $n \in \mathbb{N}$, satisfies*

$$(\exists C > 0)(\exists a > 0)(\exists \varepsilon_0 \in (0,1])$$
$$|\widehat{u_{n,\varepsilon}}(\xi)| \leq (Cn)^n(1+|\xi|)^{-n}\varepsilon^{-a},\ \xi \in \mathbb{R}^d,\ 0 < \varepsilon < \varepsilon_0,\ n \in \mathbb{N}_0.$$

Theorem 9 leads to the following definition.

Definition 4. *Let $f \in \mathcal{G}(\omega)$. Then, the analytic wave front set of f,* $\mathrm{WF}_{ga}(f)$*, is the closed subset of $\omega \times \mathbb{R}^n \setminus \{0\}$ whose complement consists of points (x_0, ξ_0) satisfying the following conditions:*

There exist $B(x_0,r)$, an open conic neighborhood Γ of ξ_0 and a bounded sequence $(u_n)_n$ in $\mathcal{G}_c(\omega)$ such that $u_n = f$, in $B(x_0,r), n \in \mathbb{N}$ and for each n, u_n has a representative $(u_{n,\varepsilon})_\varepsilon \in \mathcal{E}_{M,c}(\omega)$ such that there exist constants $a > 0, \varepsilon_0 \in (0,1]$ and $C > 0$ such that

$$|\widehat{u_{n,\varepsilon}}(\xi)| \leq (Cn)^n(1+|\xi|)^{-n}\varepsilon^{-a},\ \xi \in \Gamma,\ 0 < \varepsilon < \varepsilon_0,\ n \in \mathbb{N}.$$

We denote by $(\kappa_n)_n$ a sequence of functions in $\mathcal{D}(\mathbb{R})$ with the property that for every $\alpha \in \mathbb{N}$ there exists $C_\alpha > 0$, such that

$$|\kappa_n^{(\alpha+\beta)}(x)| \leq C_\alpha (C_\alpha n)^\beta, \ \beta \leq n, \ n \in \mathbb{N}.$$

The existence of such a sequence is proved in [15], Theorem 1.4.2. For this sequence, in the same way as in [15], Lemma 8.4.4, one can show that the next definition is equivalent to the previous one.

Definition 5. *Let $f \in \mathcal{G}(\Omega)$ and $(x_0, \xi_0) \in \Omega \times \mathbb{R} \setminus \{0\}$. Then we say that f is $g-$microanalytic at this point if there is a cone Γ_{ξ_0} around ξ_0 and if there exist $a > 0$, $C > 0$ and ε_0 such that*

$$|\widehat{\kappa_n f_\varepsilon}(\xi)| \leq C\varepsilon^{-a}(\frac{Cn}{1+|\xi|})^n, \ n \in \mathbb{N}, \ \xi \in \Gamma, \ \varepsilon \in (0, \varepsilon_0).$$

Then, the analytic wave front set, $WF_{ga} f$ is the complement of the set of points where f is g-microanalytic.

The next theorem corresponds to Hörmander's Theorem 8.4.5 in [15], and it is already proved in [22]. In fact in [22] the wave front which corresponds to a general sequence (L_k) is considered; in our case $L_k = k$ or in the notation for ultradifferentiable functions $M_k = k!$, $k \in \mathbb{N}_0$.

Theorem 10. *Let $f \in \mathcal{G}(\Omega)$. Then $pr_1 \, WF_{ga} f = singsupp_{\ ga} f$.*

The next theorem also can be proved in a more general case but since we are dealing with the analytic class, we will present it in our context.

Theorem 11. *Let $f \in \mathcal{E}'(\Omega)$ and $f_\varepsilon = f * \phi_\varepsilon|_\omega$, $\varepsilon \in (0, 1)$ be the representative of embedded distribution ($(\phi_\varepsilon)_\varepsilon$ is a net of mollifiers described in the introduction). Then, with $F = [(f_\varepsilon)_\varepsilon]$,*

$$WF_{\ a} f = WF_{ga} F,$$

where the analytic wave front set of the distribution f is on the left side.

5. Harmonic generalized function [34]

The term "real valued generalized function" refers to elements having a representative net whose elements are smooth real valued functions. We denote by $Har(\Omega)$ the space of harmonic functions in Ω.

Definition 6. *We call a generalized function $G \in \mathcal{G}(\Omega)$ a harmonic generalized function, if $\Delta G = 0$ holds in $\mathcal{G}(\Omega)$. The linear space of harmonic generalized functions in Ω is denoted by $\mathcal{G}_{Har}(\Omega)$.*

As Δ is a continuous linear mapping, it induces a $\widetilde{\mathbb{C}}$-linear continuous mapping on Colombeau extensions; we can immediately conclude that $\mathcal{G}_{Har}$ is a closed subsheaf of the sheaf of $\widetilde{\mathbb{C}}$-modules $\mathcal{G}$.

By the use of the last theorem of Section 3 we have, with $P(D) = \Delta$: that every harmonic generalized function $G \in \mathcal{G}(\Omega)$ admits a harmonic representative $(G_\varepsilon)_\varepsilon$, that is, for each $\varepsilon \leq 1$, G_ε is harmonic.

We call $(G_\varepsilon)_\varepsilon$ *a global harmonic representative* of G. Also, by the functoriality we could have defined harmonic generalized functions. We derive the same objects since there is a canonic embedding

$$\kappa : \mathcal{G}[Har(\Omega)] = \frac{\mathcal{E}[Har(\Omega)]}{\mathcal{N}[Har(\Omega)]} \to G(\Omega) \text{ and } \kappa(G[Har(\Omega)]) = G_{Har}(\Omega).$$

We establish below the relation of harmonic generalized functions and analytic generalized functions.

Clearly, $\Omega \to \mathcal{G}_A(\Omega)$ is a subsheaf of the sheaf of algebras $\mathcal{G}^\infty$.

The additional prescribed growth-property in the smoothing parameter in the definition of analyticity in $\mathcal{G}(\Omega)$ is natural. Indeed, every generalized function locally admits a real analytic representative.

Recall, the Poisson kernel for the ball $B(0,R)$ and its boundary $S_R = \partial B(0,R)$ is given by

$$P_R(x,t) = \frac{R^2 - |x|^2}{d\omega_d R|x-t|^d}, \ x \in B(0,R) \ t \in S_R,$$

where $\omega_d = 2\pi^{d/2}/(d\Gamma(d/2))$ is the volume of the unit ball; the volume of the ball $B(0,R)$ is $R^d\omega_d$ and the surface area of S_R equals $d\omega_d R^{d-1}$. We will use the normalized surface measure denoted by σ_R so that $\sigma(S_R) = 1$; if $R = 1$, then we write σ for σ_1 as well as P for P_R. Thus, $d\omega_d R^{n-1} d\sigma_R = ds_R$, where ds_R is the $d-1$ dimensional area element of B_R. In our notation the mean value theorem and the Poisson formula for the ball $B(0,R)$ are given by

$$u(x) = \int_{S_R} u(t) d\sigma_R(t), \ P_R[u](x)$$
$$= R^{n-2} \int_{S_R} \frac{R^2 - |x|^2}{|x-t|^d} u(t) d\sigma_R(t), \ x \in B(0,R).$$

As one can expect:

Proposition 1. *Every harmonic generalized function is a real analytic generalized function.*

Having shown that $\mathcal{G}_{Har}(\Omega)$ is a submodule of $\mathcal{G}_A(\Omega)$ the following consequence is immediate (cf. [33]).

Theorem 12. *Let Ω be a connected open subset of $\mathbb{R}^d$ and $f \in \mathcal{G}_{Har}(\Omega)$. If there exists $A \subset \Omega$ of positive Lebesgue measure ($\mu(A) > 0$) such that $f(x) = 0$ for every $x \in A$, then $f \equiv 0$.*

Recall the definition of a heavy set ([19]). *Let* $\widetilde{A} \subset \widetilde{\Omega}_c$. *A bounded net* $(V_\varepsilon)_\varepsilon$ *of open sets in* Ω *is said to be absorbing for* $\widetilde{A}$ *if for every generalized point* $\widetilde{a} \in \widetilde{A}$ *and its every representative* $(a_\varepsilon)_\varepsilon$ *there exists* ε_0 *such that* $a_\varepsilon \in V_\varepsilon, \varepsilon < \varepsilon_0$. *A sharply bounded set* $\widetilde{A}$ *of generalized points is said to be heavy set in* $\widetilde{\Omega}_c$ *if there exists some positive constant* c *such that for all bounded nets* $(V_\varepsilon)_\varepsilon$ *absorbing for* $\widetilde{A}$ *the following holds:* $\limsup \mu(V_\varepsilon) > c$. It is proved in [19] that if a holomorphic generalized function in $\Omega \subset \mathbb{C}^n \equiv \mathbb{R}^{2d}$, takes zero values in all generalized points of a heavy set, then it is equal to zero in Ω. This result can be transferred to real analytic generalized function and thus to harmonic generalized functions.

Theorem 13. *Let* Ω *be a connected open subset of* $\mathbb{R}^d$, *A be a bounded heavy set of* $\widetilde{\Omega}_c$ *and* $f \in \mathcal{G}_{Har}(\Omega)$. *If* $g(\widetilde{x}) = 0$ *for every* $\widetilde{x} \in A$, *then* $f = 0$ *in* Ω.

5.1. **The main theorems for harmonic generalized functions.** Let K be a compact set of Ω, $\widetilde{x}_0 \in \widetilde{\Omega}_c$ be supported by K and let $r > 0$ be such that $K + B(0, r) \subset\subset \Omega$. With such an $\widetilde{x}_0$, we denote by $B(\widetilde{x}_0, r)$ the following subset of $\widetilde{\Omega}_c$

$$B(\widetilde{x}_0, r) = \{\widetilde{t} = [(t_\varepsilon)_\varepsilon] \in \widetilde{\Omega}_c;\ |x_{0,\varepsilon} - t_\varepsilon| \le r,\ \varepsilon \le 1\}. \tag{5.2}$$

We call the set $B(\widetilde{x}_0, r) \subseteq \widetilde{\Omega}_c$ a semi-ball in Ω. Note that by now we distinguish between balls $B(x_0, r) \subset \mathbb{K}$, balls $\widetilde{B}(\widetilde{x}_0, r)$ in $\widetilde{\mathbb{K}}$ and semi-balls $B(\widetilde{x}_0, r)$.

Theorem 14. (Maximum principle) *Let* G *be a real-valued harmonic generalized function in an open set* Ω. *Then the following holds:*

(1) *Let* $r > 0$ *and* $\widetilde{x}_0 \in \widetilde{\Omega}_c$ *with representative* $(x_{0,\varepsilon})_\varepsilon$ *be given so that* $B(x_{0,\varepsilon}, 2r) \subset\subset \Omega$, $\varepsilon \le 1$ *(that is the semi-ball* $B(\widetilde{x}_0, 2r) \subset \widetilde{\Omega}_c$*). Suppose* $G(\widetilde{x}_0) \ge G(\widetilde{t})$ *for each* $\widetilde{t} \in B(\widetilde{x}_0, r)$. *Then* G *is a constant generalized function in* $B(\widetilde{x}_0, r)$.
(2) *Let* Ω *be connected. If there exists a compactly supported point* $\widetilde{x}_0 \in \widetilde{\Omega}_c$ *such that* $G(\widetilde{x}_0) \ge G(\widetilde{t})$, $\widetilde{t} \in \widetilde{\Omega}_c$, *then* G *is a constant generalized function in* Ω.

Proposition 2. *Let* $G = [(G_\varepsilon)_\varepsilon] \in \mathcal{G}(\Omega)$ *such that for every* $\widetilde{x} = [(x_\varepsilon)_\varepsilon] \in \widetilde{\Omega}_c$ *and every* $R > 0$ *such that the semi-ball* $\tilde{B}(\widetilde{x}, R) \subset \widetilde{\Omega}_c$,

$$[(G_\varepsilon(\widetilde{x}))_\varepsilon] = \Big[\Big(\frac{1}{V_R}\int_{B(x_\varepsilon, R)} G_\varepsilon(t)dt\Big)_\varepsilon\Big]. \tag{5.3}$$

Then $G \in \mathcal{G}_{Har}(\Omega)$.

Theorem 15. *Let* Ω *be connected.*

(i) *With the assumptions of Theorem* 14 (2), G *is a constant generalized function in* Ω.

(ii) *If there exists a near standard point* $\widetilde{x}_0 \in \widetilde{\Omega}_{ns}$ *such that* $G(\widetilde{x}_0) \geq G(\widetilde{t})$ *for each near standard point* $\widetilde{t} \in \widetilde{\Omega}_{ns}$, *then* G *is a constant generalized function in* Ω.

Corollary 1.

(i) *Let* u *be a complex harmonic generalized function in a connected open set* Ω. *If* $|u|$ *has a maximum* $\widetilde{M} \in \widetilde{\mathbb{R}}$ *at* $\widetilde{x}_0 \in \widetilde{\Omega}_c$, *then* $u \equiv \widetilde{A} = u(\widetilde{x}_0) \in \widetilde{\mathbb{C}}$, $|\widetilde{A}| = \widetilde{M}$.

(ii) *Let* $G \in \mathcal{G}_{Har}(\Omega)$ *be a non-constant real valued generalized function. Then*

  (1) G *does not attain its maximum inside* Ω, *that is at a generalized point* $\widetilde{t}_0 \in \widetilde{\Omega}_c$.

  (2) *Let* $\Omega' \subset\subset \Omega$ *be open. Then the maximum of* G *in* $\overline{\Omega'}$ *is attained at a generalized point supported by the boundary of* Ω'.

Now we will present generalizations of Liouville's theorem for harmonic functions. First, we present a list of different positivity notions for generalized functions (cf. [30], [23]).

Let $u \in \mathcal{G}(\mathbb{R}^d)$. We call u:

(i) non-negative, if for each compact set K there exists a representative $(u_\varepsilon)_\varepsilon$ of u such that for each $\varepsilon > 0$, $\inf_{x\in K} u_\varepsilon(x) \geq 0$.

(ii) strictly positive, if for each representative $(u_\varepsilon)_\varepsilon$ of u and for each compact set K there exists constants m and ε_0 such that for each $\varepsilon < \varepsilon_0$, $\inf_{x\in K} u_\varepsilon(x) \geq \varepsilon^m$.

(iii) A harmonic generalized function u is said to be globally non-negative, if it admits a global harmonic representative $(G_\varepsilon)_\varepsilon$ so that G_ε is non-negative for each $\varepsilon \leq 1$.

(iv) We call a harmonic generalized function u H-non-negative (and write $u \geq_H 0$), if it admits a global harmonic representative $(G_\varepsilon)_\varepsilon$ with the following property:

$$(\forall m > 0)(\forall a > 0)(\exists \varepsilon_{a,m} \in (0,1])(\forall \varepsilon < \varepsilon_{a,m})(\forall t : |t| < \frac{1}{\varepsilon^m})(u_\varepsilon(t) + \varepsilon^a \geq 0) \tag{5.4}$$

Furthermore, a harmonic generalized function u is said to be H-bounded from above (resp. below) by $\widetilde{c} \in \widetilde{\mathbb{R}}$, if for a representative $(c_\varepsilon)_\varepsilon$ of $\widetilde{c}$, the global harmonic representative $(G_\varepsilon)_\varepsilon - (c_\varepsilon)_\varepsilon$ satisfies condition (5.4). A harmonic generalized function u is said to be H-bounded if it is H-bounded from above and from below.

Definition 7. *We mention the following implications: u globally non-negative harmonic $\Rightarrow$ u is H-non-negative; u is H-non-negative harmonic $\not\Rightarrow$ u globally non-negative harmonic.*

Remark 1.

(i) Note, that a harmonic generalized function u in $\mathbb{R}^d$, with a global harmonic representative $(G_\varepsilon)_\varepsilon$ that is globally bounded from below by a net $(C_\varepsilon)_\varepsilon$ of real numbers, is simply constant. This follows by applying Liouville's theorem on G_ε for each ε.

(ii) However, if u is just bounded from below (we use here part (i) of the above definition), the above conclusion does not hold. Consider, for instance, $u_\varepsilon(x,y) = (x+1/\varepsilon)+(y+1/\varepsilon), (x,y) \in \mathbb{R}^2, \varepsilon \leq 1$. Then the corresponding u is a harmonic generalized function and bounded from below. But u is not constant. Note, however, that u is not globally bounded.

Theorem 16. *A harmonic generalized function u in $\mathbb{R}^d$ which is H-bounded from below is a constant.*

A direct consequence of this theorem is that every H-bounded harmonic generalized function $u \in \mathcal{G}(\mathbf{R}^d)$ is a constant.

Now we give the definition of an isolated singularity of a harmonic generalized function, and show that such singularities can be removed. To begin with, we start with basic observations.

Example 1. (1) Let $G \in \mathcal{G}(\Omega)$. If there exists a representative $(G_\varepsilon)_\varepsilon$ such that for all $\varepsilon \leq 1$, G_ε is harmonic in $\Omega\backslash\{x_0\}$, then G is a harmonic generalized function. In fact, we know that for all $\varepsilon \leq 1$, $\Delta G_\varepsilon = m_\varepsilon$, where m_ε is smooth and $m_\varepsilon = 0$ whenever $x \neq x_0$. By continuity, $m_\varepsilon = 0, \varepsilon \leq 1$, also at x_0, and therefore, $\Delta G = 0$ in $\mathcal{G}(\Omega)$.

(2) Let $G \in \mathcal{G}(\Omega)$ and $G|_{\Omega\backslash\{x_0\}} \in \mathcal{G}_H(\Omega \setminus \{x_0\})$. Then G need not be a harmonic generalized function in Ω. Assume for the sake of simplicity that $\Omega = \mathbb{R}^d$. Take $\rho \in \mathcal{D}(\mathbb{R}^d)$ so that $\int \rho(t)dt = 1$ and set $u = [(\rho_\varepsilon)_\varepsilon]$ $(\rho_\varepsilon = \varepsilon^{-d}\rho(\cdot/\varepsilon))$. Then for each $a > 0$, $\Delta\rho_\varepsilon = m_\varepsilon, \varepsilon \leq 1$, with $(m_\varepsilon)_\varepsilon$ negligible in $\{x : |x| > a\}$, but $\Delta u \neq 0$ in $\mathcal{G}(\mathbb{R}^d)$.

Thus one has to find an appropriate way to find conditions which allow removing "singularities".

Definition 8. *Let Ω be an open set in $\mathbb{R}^d$ and $x_0 \in \Omega$. A generalized function $G \in \mathcal{G}(\Omega\setminus\{x_0\})$ (resp. $G \in \mathcal{G}_{Har}(\Omega\setminus\{x_0\})$) is said to have an isolated (resp. isolated harmonic) singularity at x_0. Moreover, if there exists $F \in \mathcal{G}(\Omega)$ (resp. $F \in \mathcal{G}_{Har}(\Omega)$) such that $F|_{\Omega\backslash\{x_0\}} = G$, then we say that G has a removable (resp. harmonic removable) singularity.*

Example 2. We give an example of a generalized function in $\mathcal{G}_H(\Omega \setminus \{x_0\})$ which can be extended in Ω as a generalized function but which does not have a harmonic extension in Ω. Let $\psi \in C^\infty((0,\infty))$ be such that $\psi \equiv 0$ in the interval $(0, 1/4)$ and $\psi \equiv 1$ in the interval $(1/3, \infty)$. Consider in $\mathbb{R}^d$, $d > 2$, a net

$$g_\varepsilon(x) = |x|^{2-d}\psi(|x|/\varepsilon), x \in B(0,1), \varepsilon \leq 1.$$

Let $K \subset\subset B(0,1)$ and $0 \in K$. Then, there exists ε_0 such that for $\varepsilon \leq \varepsilon_0$,

$$\sup_{x\in K} |g_\varepsilon(x)| = \sup_{\substack{x\in K\\ |x|>\varepsilon/4}} |g_\varepsilon(x)| \leq (4/\varepsilon)^{d-2} \sup_{\substack{x\in K\\ |x|>\varepsilon/4}} \psi(|x|/\varepsilon) \leq C\varepsilon^{2-d}.$$

If $K \subset\subset B(0,1)$ and $0 \notin K$, then for $\varepsilon \leq \varepsilon_0$,

$$\sup_{x\in K} |g_\varepsilon(x)| = \sup_{x\in K} |x|^{2-d}.$$

The derivatives of g_ε can be estimated in the same way. Thus $(g_\varepsilon)_\varepsilon$ determines a harmonic generalized function in $B(0,1) \setminus \{0\}$. Furthermore it can be extended to a generalized function in $B(0,1)$ by merely choosing the same representative $(g_\varepsilon)_\varepsilon$. But as a harmonic generalized function, it has a harmonic singularity which cannot be removed. To see this, one has to consider the ball $B(0,1/2) \subset \mathbb{R}^d$.

Theorem 17 below states assertions on harmonic generalized functions in pierced domains. First we need a definition of H-boundedness in a neighborhood of x_0 which corresponds to H-boundedness at infinity.

Definition 9. *Let $G \in \mathcal{G}_{Har}(\Omega \setminus \{x_0\})$ and let $B(x_0, R) \subset \Omega$. We say that it is $H-$bounded in a neighborhood of x_0 if there exists $M = [(M_\varepsilon)_\varepsilon] > 0$ and a global harmonic representative $(G_\varepsilon)_\varepsilon$ in $\Omega \setminus \{x_0\}$ such that for every $m \in \mathbb{N}$ there exists $\varepsilon_m \in (0,1]$ such that*

$$|G_\varepsilon(x)| < M_\varepsilon, x \in \{\varepsilon^m < |x - x_0| < R, \varepsilon < \varepsilon_m\}.$$

Theorem 17. *Let $G \in \mathcal{G}_{Har}(\Omega \setminus \{x_0\})$. The following holds:*

(1) *Assume additionally that $G \in \mathcal{G}(\Omega)$, and that for every sharp neighborhood V of x_0 G has a representative $(G_\varepsilon)_\varepsilon$ so that for every $\varepsilon \leq 1$, G_ε is harmonic outside V_ε, where $V = [(V_\varepsilon)_\varepsilon]$. Then $G \in \mathcal{G}_{Har}(\Omega)$.*
(2) *If G is $H-$bounded at x_0, then G extends uniquely to an element of $\mathcal{G}_{Har}(\Omega)$.*

As an application of our theory in a new setting we provide a solution to the Dirichlet problem for the Laplace equation in the unit ball.

Given $E = C(S_1)$, the space of continuous functions in the unit sphere in $\mathbb{R}^d$, with sup-norm $||\cdot||_{S_1}$, $\mathcal{G}[E] = \mathcal{G}[C(S_1)]$ denotes the Colombeau extension of E.

Theorem 18. *Let $f \in \mathcal{G}[C(S_1)]$ and $(f_\varepsilon)_\varepsilon$ be its representative. Then there exists a generalized function G with a global harmonic representative $(G_\varepsilon)_\varepsilon$ in $B(0,1)$, such that:*

(1) *for each $\varepsilon \leq 1$, G_ε can be continuously extended to f in S_1;*
(2) *the construction of G is independent of the choice of a representative $(f_\varepsilon)_\varepsilon$ of f.*

Suppose that $u \in \mathcal{G}[C_b(\mathbb{R}^{d-1} \times \{0\})]$, where C_b is the space of continuous and bounded functions(with the sup-norm) and that $(u_\varepsilon)_\varepsilon$ is its representative. Then there exists a generalized function $G \in \mathcal{G}(H)$ with a global harmonic representative $(G_\varepsilon)_\varepsilon$ in H, such that:

(1) for each $\varepsilon \leq 1$, G_ε can be continuously extended to u_ε in $\mathbb{R}^{d-1} \times \{0\}$;
(2) the construction of G is independent of the choice of representative $(u_\varepsilon)_\varepsilon$ of u.

Let Ω be a bounded open set in $\mathbb{R}^d$ with a C^2- boundary.

Theorem 19. *Let $F_n, F \in \mathcal{G}[C(\partial\Omega)], n \in \mathbb{N}$. There exist harmonic generalized functions $G_n, n \in \mathbb{N}$ and G in Ω such that $G_{n|\partial\Omega} = F_n$ and $G_{|\partial\Omega} = F$. If $F_n \to F$, as $n \to \infty$, in the sense of the sharp topology of $\mathcal{G}[C(\partial\Omega)]$, then $G_n \to G$ in the sense of the sharp topology of $\mathcal{G}_{Har}(\Omega)$ as well as of $\mathcal{G}[C(\overline{\Omega})]$.*

Remark 2. There exists a solution u to a Dirichlet problem (in $B(0,1)$) $\Delta u = 0$ $u_{|S_1} = f \in \mathcal{G}[C(\partial\Omega)]$, obtained by the Poisson formula (cf. Theorem 18) so that for a given generalized point $\widetilde{x} \in \widetilde{S}_1$ and a sequence of generalized points $(\widetilde{y}_n)_n$ in $\widetilde{\overline{\Omega}}$ which converges sharply to $\widetilde{x}$ in $\widetilde{\overline{\Omega}}$, $(u(\widetilde{y}_n))_n$ does not converge sharply to $f(\widetilde{x})$.

In order to obtain the continuity in the sense of sharp topology we have to assume a stronger condition on f.

Let $\mathcal{G}[C^{0,\alpha}(S_R)]$ be the Colombeau extension of the Hölder space $C^{0,\alpha}$ (S_R), $\alpha \in (0,1)$. So $(F_\varepsilon)_\varepsilon$ is a representative of some element in $\mathcal{G}[C^{0,\alpha}(S_R)]$ if there exist $C > 0, M > 0$ and $\varepsilon_0 \in (0,1]$ such that

$$\sup_{t,s \in S_R} \left|\frac{F_\varepsilon(t) - F_\varepsilon(s)}{(t-s)^\alpha}\right| \leq C\varepsilon^{-M}, \varepsilon \leq \varepsilon_0. \tag{5.5}$$

We have the following theorem.

Theorem 20. *Let $F \in \mathcal{G}[C^{0,\alpha}(S_R)]$. Then there exists a unique $G \in \mathcal{G}_{Har}(B(0,R)) \cap \mathcal{G}[C(\overline{B(0,R)})]$ and so that $\Delta G = 0$ in $B(0,R)$, $G_{|S_R} = F$ so that if $(\widetilde{y}_n)_n \to \widetilde{x} \in \widetilde{S}_R$ in the sharp topology of $\widetilde{\overline{B}(0,R)}$, then $G(\widetilde{y}_n)$ converges to $G(\widetilde{x})$ in the sharp topology of $\widetilde{\mathbb{C}}$.*

Note that if $F \in \mathcal{G}(\partial\Omega)$, where $\partial\Omega$ is smooth, then the Dirichlet problem is solvable and the sharp continuity of the solution, up to the boundary, holds.

Remark 3. As we explained, the elements of algebras of generalized functions are represented by nets $(f_\varepsilon)_\varepsilon$ of smooth functions, with appropriate growth as $\varepsilon \to 0$, and for the space of smooth functions the corresponding algebra of smooth generalized functions is $\mathcal{G}^\infty$ (see [28]). In our further investigations we find out the conditions with respect to the growth order in ε which characterize spaces of absolutely continuous functions AC^k, Sobolev spaces $W^{k,p}$, Zigmund spaces C_* and Hölder spaces $H^{k,p}$.

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References

[1] S. Axler, P. Bourdon and W. Ramey, *Harmonic Function Theory*, Second edition. Graduate Texts in Mathematics, 137. Springer-Verlag, New York, 2001.

[2] J. Aragona, *On existence theorems for the $\bar{\partial}$ operator on generalized differential forms*, Proc. London Math. Soc., 53 (1986), 474–488.

[3] J. Aragona, R. Fernandez and S. O. Juriaans, *Discontinuous Colombeau differential calculus*, Monatsh. Math., 144 (2005), 13–29.

[4] H. A. Biagioni, *A Nonlinear Theory of Generalized Functions*, Springer-Verlag, Berlin-Hedelberg-New York, 1990.

[5] J. F. Colombeau, *New Generalized Functions and Multiplications of Distributions*, North Holland, 1982.

[6] J. F. Colombeau, *Elementary Introduction in New Generalized Functions*, North Holland, 1985.

[7] J. F. Colombeau and J. E. Galé, *Holomorphic generalized functions*, J. Math. Anal. Appl., 103 (1984), 117–133.

[8] J. F. Colombeau and J. E. Galé, *Analytic continuation of generalized functions*, Acta Math. Hung., 52 (1988), 57-60.

[9] C. Garetto, *Topological structures in Colombeau algebras: investigation of the duals of $\mathcal{G}_c(\Omega)$, $\mathcal{G}(\Omega)$ and $\mathcal{GS}(\mathbb{R}^n)$*, Monatsh. Math., 146 (2005), 203-226.

[10] C. Garetto, *Topological structures in Colombeau algebras: topological $\widetilde{\mathbb{C}}$-modules and duality theory*, Acta Appl. Math., 88 (2005), 81-123.

[11] C. Garetto and G. Hörmann, *Microlocal analysis of generalized functions: pseudodifferential techniques and propagation of singularities*, Proc. Edinburgh Math. Soc., 48, (2005), 603-629.

[12] M. Grosser, M. Kunzinger, M. Oberguggenberger and R. Steinbauer, *Geometric Generalized Functions with Applications to General Relativity*, Kluwer, 2001.

[13] R.C. Gunning and H. Rossi, *Analytic Functions of Several Complex Variables*, Englewood Cliffs: Prentice-Hall, 1965.

[14] C. Hanel, E. Mayerhofer, S. Pilipović and H. Vernaeve, *Homogeneity in algebras of generalized functions*, J. Math. Anal. Appl., 339 (2008), 889–904.

[15] L. Hörmander, *The Analysis of Linear Partial Differential Operators. I*, Grundlehren der Mathematischen Wissenschaften, 256, Springer-Verlag, Berlin, 1983.

[16] G. Hörmann, *Hölder-Zygmund regularity in algebras of generalized functions*, Z. Anal. Anwendungen, 23 (2004), 139-165.

[17] L. Hormander, *Uniqueness theorems and wave front sets for solutions of linear differential equations with analytic coefficients*, Commun. Pure Appl. Math., 24 (1971) 671-704.

[18] A. Kaneko, Introduction to Hyperfunctions, Dordercht, Kluwer, 1982.
[19] A. Khelif and D. Scarpalezos, *Zeros of generalized holomorphic functions*, Monatsh. Math., 149 (2006), 323–335.
[20] I. Kmit, M. Kunzinger and R. Steinbauer, *Generalized solutions of the Vlasov-Poisson system with singular data*, J. Math. Anal. Appl., 340 (2008), 575–587.
[21] S. Konjik and M. Kunzinger, *Generalized group actions in a global setting*, J. Math. Anal. Appl., 322 (2006), 420–436.
[22] J. A. Marti, *$\mathcal{G}^L$-Microlocal analysis of generalized functions*, Int. Tansf. Spec. Funct., 17, (2006), 119 - 125.
[23] E, Mayerhofer, *The wave equation on static singular space-times*, PhD Thesis, arXiv:0802.1616.
[24] E. Michael, *Continuous selections I*, Annal. Math, 63 (1956), 361–382.
[25] E. Michael, *A short proof of the Arens-Eells embedding theorem*, Proc. Amer. Math. Soc., 15 (1964), 415–416
[26] M. Morimoto, *An Introduction to Sato's Hyperfunctions*, Providence, Rhode Island, Amer. Math. Soc., 1993.
[27] M. Oberguggenberger, *Rotationally invariant Colombeau generalized functions*, In: A. Delcroix, M. Hasler, J-A. Marti and V. Valmorin, (eds)*Nonlinear Algebraic Analysis and Applications*, Proceedings of the International Conference on Generalized Functions (ICGF 2000), 227–236, 2004. Cottenham, Cambridge Scientific.
[28] M. Oberguggenberger, *Multiplication of distributions and application to partial differential equations*, Pitman Res. Notes Math. Ser., 259, Longman, Harlow, 1992.
[29] M. Oberguggenberger, M. Kunzinger, *Characterization of Colombeau generalized functions by their pointvalues*, Math. Nachr., 203 (1999), 147–157.
[30] M. Oberguggenberger, S. Pilipovic and D. Scarpalezos, *Positivity and positive definiteness in generalized function algebras*, J.Math. Anal. Appl., 328 (2007), 1321–1335.
[31] M. Oberguggenberger, S. Pilipovic and V. Valmorin, *Global holomorphic representatives of holomorphic generalized functions*, Monatsh. Math., 151 (2007), 67–74.
[32] S. Pilipovic, *Generalized hyperfunctions and algebra of megafunctions*, Tokyo J. Math., 28 (2005), 1–12.
[33] S. Pilipovic, D. Scarpalezos and V. Valmorin, *Real analytic generalized functions,* Monatsh. Math., 156 (2009), 85–102.
[34] S. Pilipovic and D. Scarpalezos, *Harmonic generalized function algebras*, Monatsh. Math., 2010.
[35] D. Scarpalezos, *Topologies dans les espaces de nouvelles fonctions généralisées de Colombeau. $\widetilde{\mathbb{C}}$–modules topologiques*, Preprint ser. Université Paris 7, Université Paris 7, 1992.
[36] L. Schwartz, *Theorie des distributiones*, Hermann, Paris, 1950.
[37] V. Valmorin, *Vanishing theorems in $\mathcal{G}^\infty$ classes of Colombeau generalized functions*, Preprint AOC, 2005.

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