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# ON THE NORM OF A MULTIDIMENSIONAL HILBERT-TYPE OPERATOR

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ABSTRACT. In this paper we consider multidimensional Hilbert-type integral inequalities with non-conjugate exponents and homogeneous kernels of negative degree. As an application, we define related Hilbert-type integral operators and consider their norms. The problem of determining the norm of such operators is equivalent to the problem of the best possible constant factors involved in the right-hand sides of related inequalities. In such a way, we obtain the norms and the best possible constant factors in some general settings with conjugate exponents. In particular, we obtain generalizations of some recent results from the literature.

## 1. INTRODUCTION AND PRELIMINARIES

Let $|\cdot|_\alpha$ denote a $\alpha-$norm of the vector $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n$, that is

$$|\mathbf{x}|_\alpha = \left( \sum_{i=1}^{n} {x_i}^\alpha \right)^{\frac{1}{\alpha}}, \quad \alpha > 0. \tag{1.1}$$

Let $\phi, \psi : \mathbb{R}_+^n \to \mathbb{R}$ be non-negative measurable power functions defined by $\phi(\mathbf{x}) = {|\mathbf{x}|_\alpha}^{p\left(\frac{n}{q} - \frac{s}{r_1}\right)}$ and $\psi(\mathbf{y}) = {|\mathbf{y}|_\alpha}^{q\left(\frac{n}{p} - \frac{s}{r_2}\right)}$, where $s > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{r_1} + \frac{1}{r_2} = 1$, with $p, r_1 > 1$. Further, let $||\cdot||_{L^r_\Omega(\mathbb{R}_+^n)}$ denote the norm of a non-negative measurable function $f : \mathbb{R}_+^n \to \mathbb{R}$, with respect to a non-negative measurable weight function $\Omega : \mathbb{R}_+^n \to \mathbb{R}$, that is

$$||f||_{L^r_\Omega(\mathbb{R}_+^n)} = \left[ \int_{\mathbb{R}_+^n} \Omega(\mathbf{x}) |f(\mathbf{x})|^r d\mathbf{x} \right]^{\frac{1}{r}}. \tag{1.2}$$

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In relation (1.2), $L^r_\Omega(\mathbb{R}^n_+)$ denotes the measure space

$$L^r_\Omega(\mathbb{R}^n_+) = \left\{ f : \mathbb{R}^n_+ \to \mathbb{R}; ||f||_{L^r_\Omega(\mathbb{R}^n_+)} < \infty \right\}, \tag{1.3}$$

and $d\mathbf{x}$ is a Lebesgue measure.

The Hilbert-type integral operator $\mathcal{T}_s : L^p_\phi(\mathbb{R}^n_+) \to L^p_{\psi^{1-p}}(\mathbb{R}^n_+)$ is defined by

$$(\mathcal{T}_s f)(\mathbf{y}) = \int_{\mathbb{R}^n_+} k_s\left(|\mathbf{x}|_\alpha, |\mathbf{y}|_\alpha\right) f(\mathbf{x}) d\mathbf{x}, \quad \mathbf{y} \in \mathbb{R}^n_+, \tag{1.4}$$

where $k_s : \mathbb{R}^2_+ \to \mathbb{R}$ is non-negative measurable homogeneous function of degree $-s$, $s > 0$, that is $k_s(tu, tv) = t^{-s} k_s(u, v)$ for all $t, u, v \geq 0$.

In [7], Y. Bicheng obtained that the norm of the operator $\mathcal{T}_s$ is

$$||\mathcal{T}_s|| = \frac{\Gamma^n\left(\frac{1}{\alpha}\right)}{\alpha^{n-1}\Gamma\left(\frac{n}{\alpha}\right)} \int_0^\infty k_s(t, 1) t^{\frac{s}{r_1}-1} dt, \tag{1.5}$$

where $\Gamma$ denotes the usual Gamma function under the assumption that the integral in formula (1.5) converges. Here, we assume that the norm of the operator $\mathcal{T}_s$ is defined as

$$||\mathcal{T}_s|| = \sup_{f \in L^p_\phi(\mathbb{R}^n_+), f \neq 0} \frac{||\mathcal{T}_s f||_{L^p_{\psi^{1-p}}(\mathbb{R}^n_+)}}{||f||_{L^p_\phi(\mathbb{R}^n_+)}}. \tag{1.6}$$

The obtained norm is established by means of two equivalent inequalities

$$\int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} k_s\left(|\mathbf{x}|_\alpha, |\mathbf{y}|_\alpha\right) f(\mathbf{x}) g(\mathbf{y}) d\mathbf{x} d\mathbf{y} \leq ||\mathcal{T}_s|| \cdot ||f||_{L^p_\phi(\mathbb{R}^n_+)} \cdot ||g||_{L^p_\psi(\mathbb{R}^n_+)} \tag{1.7}$$

and

$$\left\{ \int_{\mathbb{R}^n_+} |\mathbf{y}|_\alpha^{\frac{ps}{r_2}-n} \left[ \int_{\mathbb{R}^n_+} k_s\left(|\mathbf{x}|_\alpha, |\mathbf{y}|_\alpha\right) f(\mathbf{x}) d\mathbf{x} \right]^p d\mathbf{y} \right\}^{\frac{1}{p}} \leq ||\mathcal{T}_s|| \cdot ||f||_{L^p_\phi(\mathbb{R}^n_+)}, \tag{1.8}$$

which hold for conjugate exponents $p$ and $q$, that is $\frac{1}{p} + \frac{1}{q} = 1$, with $p > 1$, and for non-negative measurable functions $f, g : \mathbb{R}^n_+ \to \mathbb{R}$. In [7], it is verified that the constant factor $||\mathcal{T}_s||$ is the best possible in both inequalities (1.7) and (1.8), which is equivalent that the norm of operator $\mathcal{T}_s$ is given by (1.5).

Inequality (1.7) is a generalization of the classical Hilbert inequality, and will be referred to as a Hilbert-type inequality. On the other hand, inequality (1.8) will be referred to as a Hardy-Hilbert type inequality.

If $0 < p < 1$ then the signs of inequality in (1.7) and (1.8) are reversed. Also, equalities in (1.7) and (1.8) hold if and only if $f = 0$ or $g = 0$ a.e. on $\mathbb{R}^n_+$.

Previous results are some extensions of papers [5], [11] and [18]. Further, if $n=1$ and $k_s(x,y)=(x+y)^{-s}$, inequality (1.7) reduces to

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^s}dxdy \\ \leq B\left(\frac{s}{r_1},\frac{s}{r_2}\right)\left[\int_0^\infty x^{p\left(\frac{1}{q}-\frac{s}{r_1}\right)}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q\left(\frac{1}{p}-\frac{s}{r_2}\right)}g^q(y)dy\right]^{\frac{1}{q}}, \quad (1.9)$$

where $B$ is the usual Beta function (see [2], [12], and [13]). Clearly, for $p=q=r_1=r_2=2$ and $s=1$, inequality (1.9) reduces to a classical Hilbert integral inequality, which is one of the most important inequalities in analysis and its applications (see [6], [10] and [16]). In recent years, some authors also studied a few kinds of Hilbert-type operators and their relating inequalities (see [1], [2], [3] and [4]).

The main objective of this paper is a generalization of a Hilbert-type operator (1.4) and related inequalities (1.7) and (1.8). More precisely, such operators and inequalities can be treated in a more general setting with non-conjugate exponents and with more general weight functions.

The paper is organized in the following way: After this Introduction, in Section 2 we introduce a unified treatment of the Hilbert and Hardy-Hilbert type inequalities with non-conjugate exponents, developed in a recent paper [9]. In Section 3 we obtain a pair of equivalent inequalities which extend the inequalities (1.7) and (1.8) to the case of non-conjugate exponents and with more general power weights. Further, we define a related Hilbert-type operator and, as a consequence, we obtain the upper bound for its norm. In Section 4 we restrict obtained results to a conjugate setting and analyze the conditions which yield the best possible constant factors in obtained inequalities and the norm of the related operator. Section 5 is dedicated to Hardy-type operators, deduced from the defined Hilbert-type operator. Finally, in the last section we apply derived results to some special homogeneous kernels and compare our results with some previously known results from the literature.

**Conventions.** Throughout this paper, let $r'$ be the conjugate exponent to a positive real number $r\neq 1$, that is, $\frac{1}{r}+\frac{1}{r'}=1$, or $r'=\frac{r}{r-1}$. A weight function is assumed to be any non-negative measurable function. Further, definitions (1.1), (1.2), and (1.3) of the norms $|\cdot|_\alpha$, $||\cdot||_{L^r_\Omega(\mathbb{R}^n_+)}$, and the measure space $L^r_\Omega(\mathbb{R}^n_+)$ will be valid throughout the whole paper. Similarly, the norm of the operator between two measure spaces is defined according to definition (1.6). In addition, all functions are assumed to be non-negative and measurable. Expressions of the form $0\cdot\infty$, $\frac{0}{\infty}$, $\frac{\infty}{\infty}$, and $\frac{0}{0}$ are taken to be equal to zero.

Techniques that will be used in the proofs are mainly based on classical real analysis. So, before we state and prove our main results, we have to introduce an extension of the Hilbert-type inequalities to the case of non-conjugate exponents.

## 2. An unified treatment of Hilbert-type inequalities with non-conjugate exponents

Suppose $p$ and $q$ are real parameters, such that

$$p > 1, \ q > 1, \ \frac{1}{p} + \frac{1}{q} \geq 1, \tag{2.1}$$

and let $p' = \frac{p}{p-1}$ and $q' = \frac{q}{q-1}$ respectively be their conjugate exponents, that is, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Further, define

$$\lambda = \frac{1}{p'} + \frac{1}{q'} \tag{2.2}$$

and observe that $0 < \lambda \leq 1$ holds for all $p$ and $q$ as in (2.1). In particular, equality $\lambda = 1$ holds in (2.2) if and only if $q = p'$, that is, only if $p$ and $q$ are mutually conjugate. Otherwise, we have $0 < \lambda < 1$, and such parameters $p$ and $q$ will be referred to as non-conjugate exponents.

Considering $p$, $q$, and $\lambda$ as in (2.1) and (2.2), Hardy, Littlewood, and Pólya, [10], obtained an extension of the classical Hilbert inequality to the case of non-conjugate exponents. A more simple proof of that extension, based on a single application of the Hölder inequality, was given by F. F. Bonsall, [8] later.

Very recently, A. Čižmešija et. al. developed in paper [9] a unified treatment of the general Hilbert-type inequalities extended to the case of non-conjugate exponents. Their main result, based on one Bonsall's idea from paper [8], is the following.

**Theorem 2.1.** *Let real parameters $p$, $q$, and $\lambda$ be as in* (2.1) *and* (2.2)*, and let $X$ and $Y$ be measure spaces with positive $\sigma$-finite measures $\mu_1$ and $\mu_2$ respectively. Let $K$ be a non-negative measurable function on $X \times Y$, $\varphi$ a measurable, a.e. positive function on $X$, and $\psi$ a measurable, a.e. positive function on $Y$. If the functions $F$ on $X$ and $G$ on $Y$ are defined by*

$$F(x) = \left[ \int_Y K(x,y) \psi^{-q'}(y) \, d\mu_2(y) \right]^{\frac{1}{q'}}, \quad x \in X, \tag{2.3}$$

*and*

$$G(y) = \left[ \int_X K(x,y) \varphi^{-p'}(x) \, d\mu_1(x) \right]^{\frac{1}{p'}}, \quad y \in Y, \tag{2.4}$$

*then for all non-negative measurable functions $f$ on $X$ and $g$ on $Y$ the inequalities*

$$\int_X \int_Y K^\lambda(x,y) f(x) g(y)\, d\mu_1(x) d\mu_2(y) \le \|\varphi F f\|_{L^p(\mu_1)} \|\psi G g\|_{L^q(\mu_2)} \tag{2.5}$$

*and*

$$\left\{ \int_Y \left[ (\psi G)^{-1}(y) \int_X K^\lambda(x,y) f(x)\, d\mu_1(x) \right]^{q'} d\mu_2(y) \right\}^{\frac{1}{q'}} \le \|\varphi F f\|_{L^p(\mu_1)} \tag{2.6}$$

*hold and are equivalent. Here, $L^p(\mu)$ denotes the measure space $L^p(X;\mu)$ and $||f||_{L^p(\mu)} = \left[\int_X f^p(x) d\mu(x)\right]^{\frac{1}{p}}$ is the corresponding norm.*

Clearly, inequalities (2.5) and (2.6) are generalizations of the classical Hilbert and Hardy inequalities. So, inequalities deduced from (2.5) will be referred to as the Hilbert-type inequalities and inequalities deduced from (2.6), as the Hardy-Hilbert type inequalities.

Further, in paper [9] it was shown that necessary and sufficient conditions for equalities in (2.5) and (2.6) are described with the relations

$$f = \alpha \varphi^{-p'} F^{\frac{q'}{p}-1} \quad \text{a.e. on } X \quad \text{and} \quad g = \beta \psi^{-q'} G^{\frac{p'}{q}-1} \quad \text{a.e. on } Y, \tag{2.7}$$

and

$$K = \gamma F^{q'} G^{p'} \quad \text{a.e. on } X \times Y, \tag{2.8}$$

where $\alpha, \beta$ and $\gamma$ are some positive real constants.

The same authors also considered reversed inequalities in (2.5) and (2.6), these are the inequalities with the reversed sign of inequality. They obtained that reverse inequalities in (2.5) and (2.6) hold if

$$p < 0,\ q \in \langle 0,1 \rangle,\ \frac{1}{p} + \frac{1}{q} \le 1, \tag{2.9}$$

$$p \in \langle 0,1 \rangle,\ q < 0,\ \frac{1}{p} + \frac{1}{q} \le 1, \tag{2.10}$$

or

$$p \in \langle 0,1 \rangle,\ q \in \langle 0,1 \rangle, \tag{2.11}$$

since the crucial step in proving Theorem 2.1 was applying the Hölder inequality, and parameters given by (2.9), (2.10) or (2.11) provide the so-called the reversed Hölder inequality (for details, see e.g. [16, Chapter V]). Note that in conjugate case, reverse inequalities in (2.5) and (2.6) are achieved if $0 < p < 1$ and $q < 0$.

## 3. Basic result and definition of Hilbert-type operator

We proceed with an application of Theorem 2.1 to homogeneous kernels of negative degree and certain power weights, with respect to Lebesgue measure in $\mathbb{R}_+^n$, $n \in \mathbb{N}$. The results will be given in two equivalent forms in non-conjugate setting, analogous to (2.5) and (2.6). However, the best possible constants will be obtained only in the conjugate case, since the described problem in the non-conjugate case seems very difficult and remains open.

At the beginning, we need some auxiliary results. In [7], one can find the following lemma.

**Lemma 3.1.** *Suppose $m \in \mathbb{N}$, $\alpha > 0$, and $R > 0$. If $\Psi : [0,1] \to \mathbb{R}$ is a non-negative measurable function, then*

$$\int_{D_R^m} \Psi\left[\sum_{i=1}^{m}\left(\frac{x_i}{R}\right)^{\alpha}\right] d\mathbf{x} = \frac{R^m \Gamma^m\left(\frac{1}{\alpha}\right)}{\alpha^m \Gamma\left(\frac{m}{\alpha}\right)} \int_0^1 \Psi(u) u^{\frac{m}{\alpha}-1} du, \tag{3.1}$$

*where $D_R^m = \left\{\mathbf{x} = (x_1, x_2, \ldots, x_m) \in \mathbb{R}_+^m : \sum_{i=1}^m {x_i}^{\alpha} \leq R^{\alpha}\right\}$ and $\Gamma$ is the usual Gamma function.*

As a direct consequence of Lemma 3.1 we easily get the following result.

**Lemma 3.2.** *If $m \in \mathbb{N}$, $\alpha > 0$, and $\varepsilon \geq 0$ then*

$$\int_{\{\mathbf{x}\in\mathbb{R}_+^m : |\mathbf{x}|_\alpha \geq 1\}} {|\mathbf{x}|_\alpha}^{-n-\varepsilon} d\mathbf{x} = \begin{cases} \frac{\Gamma^m\left(\frac{1}{\alpha}\right)}{\varepsilon\alpha^{m-1}\Gamma\left(\frac{m}{\alpha}\right)}, & \varepsilon > 0, \\ \infty, & \varepsilon = 0, \end{cases} \tag{3.2}$$

*where $\Gamma$ is the usual Gamma function.*

*Proof.* We use Lemma 3.1. Let $R > 1$. We define $\Theta : [0,1] \to \mathbb{R}$ as

$$\Theta(u) = \begin{cases} 0 & ,\ 0 \leq u < R^{-\alpha}, \\ \left(Ru^{\frac{1}{\alpha}}\right)^{-m-\varepsilon} & ,\ R^{-\alpha} \leq u \leq 1. \end{cases}$$

Now, by using (3.1) and the notation from Lemma 3.1 we have

$$\begin{aligned} \int_{\{\mathbf{x}\in\mathbb{R}_+^m : |\mathbf{x}|_\alpha \geq 1\}} {|\mathbf{x}|_\alpha}^{-m-\varepsilon} d\mathbf{x} &= \lim_{R\to\infty} \int_{D_R^m} \Theta\left[\sum_{i=1}^{m}\left(\frac{x_i}{R}\right)^{\alpha}\right] d\mathbf{x} \\ &= \lim_{R\to\infty} \frac{R^m \Gamma^m\left(\frac{1}{\alpha}\right)}{\alpha^m \Gamma\left(\frac{m}{\alpha}\right)} \int_0^1 \Theta(u) u^{\frac{m}{\alpha}-1} du \\ &= \frac{\Gamma^m\left(\frac{1}{\alpha}\right)}{\alpha^m \Gamma\left(\frac{m}{\alpha}\right)} \lim_{R\to\infty} \frac{1}{R^{\varepsilon}} \int_{R^{-\alpha}}^1 u^{-\frac{\varepsilon}{\alpha}-1} du \\ &= \frac{\Gamma^m\left(\frac{1}{\alpha}\right)}{\varepsilon\alpha^{m-1} \Gamma\left(\frac{m}{\alpha}\right)} \lim_{R\to\infty} \left(1 - \frac{1}{R^{\varepsilon}}\right), \end{aligned}$$

and the result follows. □

In this paper we assume that $k_s : \mathbb{R}_+^2 \to \mathbb{R}$ is a non-negative, measurable homogeneous function of degree $-s$, $s > 0$, and we consider the integral

$$c_s(\eta) = \int_0^\infty k_s(1,t)t^{-\eta}dt, \tag{3.3}$$

defined in terms of the function $k_s$. Clearly, the convergence of integral (3.3) depends on the function $k_s$, so the parameters $\eta$ are assumed to be chosen so that integral (3.3) converges.

Further, we consider inequalities related to (1.7) and (1.8) with power weight functions $\Phi_{\mathbf{A}} : \mathbb{R}_+^m \to \mathbb{R}$, $\Psi_{\mathbf{A}} : \mathbb{R}_+^n \to \mathbb{R}$ defined by

$$\Phi_{\mathbf{A}}(\mathbf{x}) = |\mathbf{x}|_\alpha{}^{p(A_1-A_2)+(n-s)\frac{p}{q'}} \quad \text{and} \quad \Psi_{\mathbf{A}}(\mathbf{y}) = |\mathbf{y}|_\beta{}^{q(A_2-A_1)+(m-s)\frac{q}{p'}}, \tag{3.4}$$

where $m, n \in \mathbb{N}$ and $A_1, A_2 \in \mathbb{R}$.

Now we are ready to state and prove the result that will be the base in our investigation of Hilbert-type operators and their norms.

**Theorem 3.3.** *Let real parameters $p$, $q$, $\lambda$ be as in* (2.1) *and* (2.2)*, let $m, n \in \mathbb{N}$, and let $\alpha, \beta > 0$. Suppose $k_s : \mathbb{R}_+^2 \to \mathbb{R}$ is a homogeneous function of degree $-s$, $s > 0$, and $A_1$, $A_2$ are real parameters such that $c_s(q'A_2 + 1 - n) < \infty$, $c_s(m + 1 - p'A_1 - s) < \infty$. If the functions $\Phi_{\mathbf{A}}$ on $\mathbb{R}_+^m$ and $\Psi_{\mathbf{A}}$ on $\mathbb{R}_+^n$ are defined by* (3.4)*, then the inequalities*

$$\int_{\mathbb{R}_+^m}\int_{\mathbb{R}_+^n} k_s{}^\lambda\left(|\mathbf{x}|_\alpha, |\mathbf{y}|_\beta\right) f(\mathbf{x})g(\mathbf{y})d\mathbf{x}d\mathbf{y} \le C_{\mathbf{A}}||f||_{L^p_{\Phi_{\mathbf{A}}}(\mathbb{R}_+^m)}||g||_{L^q_{\Psi_{\mathbf{A}}}(\mathbb{R}_+^n)} \tag{3.5}$$

*and*

$$\left\{\int_{\mathbb{R}_+^n} |\mathbf{y}|_\beta{}^{q'(A_1-A_2)+(s-m)\frac{q'}{p'}}\left[\int_{\mathbb{R}_+^m} k_s{}^\lambda\left(|\mathbf{x}|_\alpha, |\mathbf{y}|_\beta\right) f(\mathbf{x})d\mathbf{x}\right]^{q'} d\mathbf{y}\right\}^{\frac{1}{q'}} \le C_{\mathbf{A}}||f||_{L^p_{\Phi_{\mathbf{A}}}(\mathbb{R}_+^m)}, \tag{3.6}$$

*where*

$$C_{\mathbf{A}} = \left[\frac{\Gamma^m\left(\frac{1}{\alpha}\right)}{\alpha^{m-1}\Gamma\left(\frac{m}{\alpha}\right)}\right]^{\frac{1}{p'}}\left[\frac{\Gamma^n\left(\frac{1}{\beta}\right)}{\beta^{n-1}\Gamma\left(\frac{n}{\beta}\right)}\right]^{\frac{1}{q'}} c_s{}^{\frac{1}{q'}}(q'A_2+1-n)c_s{}^{\frac{1}{p'}}(m+1-p'A_1-s), \tag{3.7}$$

*hold for all non-negative measurable functions $f : \mathbb{R}_+^m \to \mathbb{R}$, $g : \mathbb{R}_+^n \to \mathbb{R}$ and are equivalent. Equalities in* (3.5) *and* (3.6) *hold if and only if $f = 0$ a.e. on $\mathbb{R}_+^m$ or $g = 0$ a.e. on $\mathbb{R}_+^n$. Moreover, if $p$ and $q$ are as in* (2.9), (2.10) *or* (2.11)*, the signs of inequality in* (3.5) *and* (3.6) *are reversed.*

*Proof.* We apply Theorem 2.1 to the homogeneous kernel $k_s : \mathbb{R}_+^2 \to \mathbb{R}$ and certain power weights. More precisely, we substitute the power weights

$\varphi(\mathbf{x}) = |\mathbf{x}|^{A_1}$, $\mathbf{x} \in \mathbb{R}_+^m$ and $\psi(\mathbf{y}) = |\mathbf{y}|^{A_2}$, $\mathbf{y} \in \mathbb{R}_+^n$, in inequality (2.5). Now, since

$$|\mathbf{y}|_\beta = R\left[\sum_{i=1}^n \left(\frac{y_i}{R}\right)^\beta\right]^{\frac{1}{\beta}}, R > 0,$$

by using the notation from Lemma 3.1, we have

$$F^{q'}(\mathbf{x}) = \lim_{R\to\infty}\int_{D_R^n} R^{-q'A_2} k_s\left(|\mathbf{x}|_\alpha, R\left[\sum_{i=1}^n \left(\frac{y_i}{R}\right)^\beta\right]^{\frac{1}{\beta}}\right)\left[\sum_{i=1}^n \left(\frac{y_i}{R}\right)^\beta\right]^{-\frac{q'A_2}{\beta}} d\mathbf{y}, \tag{3.8}$$

where the function $F : \mathbb{R}_+^m \to \mathbb{R}$ is defined by (2.3).

Further, by using relation (3.1), with $\Psi(u) = R^{-q'A_2} k_s\left(|\mathbf{x}|_\alpha, Ru^{\frac{1}{\beta}}\right) u^{-\frac{q'A_2}{\beta}}$, the above formula (3.8) reads

$$F^{q'}(\mathbf{x}) = \frac{\Gamma^n\left(\frac{1}{\beta}\right)}{\beta^n \Gamma\left(\frac{n}{\beta}\right)} \lim_{R\to\infty} R^{n-q'A_2}\int_0^1 k_s\left(|\mathbf{x}|_\alpha, Ru^{\frac{1}{\beta}}\right) u^{\frac{n-\beta-q'A_2}{\beta}} du. \tag{3.9}$$

Finally, by using substitution $Ru^{\frac{1}{\beta}} = t|\mathbf{x}|_\alpha$ and homogeneity of function $k_s$, we have

$$\begin{aligned} F^{q'}(\mathbf{x}) &= \frac{\Gamma^n\left(\frac{1}{\beta}\right)}{\beta^{n-1}\Gamma\left(\frac{n}{\beta}\right)} |\mathbf{x}|_\alpha{}^{n-s-q'A_2} \lim_{R\to\infty}\int_0^{R|\mathbf{x}|_\alpha^{-1}} k_s(1,t) t^{n-1-q'A_2} dt \\ &= \frac{\Gamma^n\left(\frac{1}{\beta}\right)}{\beta^{n-1}\Gamma\left(\frac{n}{\beta}\right)} |\mathbf{x}|_\alpha{}^{n-s-q'A_2} \int_0^{\infty} k_s(1,t) t^{n-1-q'A_2} dt. \end{aligned}$$

Hence,

$$F(\mathbf{x}) = \left[\frac{\Gamma^n\left(\frac{1}{\beta}\right)}{\beta^{n-1}\Gamma\left(\frac{n}{\beta}\right)}\right]^{\frac{1}{q'}} c_s{}^{\frac{1}{q'}}(q'A_2 + 1 - n) |\mathbf{x}|_\alpha{}^{\frac{n-s}{q'} - A_2}. \tag{3.10}$$

In a similar way we obtain

$$G(\mathbf{y}) = \left[\frac{\Gamma^m\left(\frac{1}{\alpha}\right)}{\alpha^{m-1}\Gamma\left(\frac{m}{\alpha}\right)}\right]^{\frac{1}{p'}} c_s{}^{\frac{1}{p'}}(m + 1 - p'A_1 - s) |\mathbf{y}|_\beta{}^{\frac{m-s}{p'} - A_1}. \tag{3.11}$$

Finally, by substituting (3.10) and (3.11) in (2.5) and (2.6) we obtain relations (3.5) and (3.6).

It remains to investigate the case of equality in the obtained inequalities. According to conditions (2.7) and (2.8), the necessary condition for equality in (3.5) and (3.6) is that the function $f$ has the form

$$f(\mathbf{x}) = \nu |\mathbf{x}|_\alpha{}^{-p'A_1+(1-\lambda)(n-s-q'A_2)}, \quad \text{a.e. on} \quad \mathbb{R}_+^m, \tag{3.12}$$

for some positive constant $\nu$. For such a choice of function $f$, the first integral on the right-hand side of inequality (3.5) becomes

$$\nu \int_{\mathbb{R}_+^m} |\mathbf{x}|_\alpha{}^{n-s-p'A_1-q'A_2} d\mathbf{x},$$

which clearly diverges. Hence, we came to a contradiction so the equality in (3.5) and (3.6) holds only if $f = 0$ a.e. on $\mathbb{R}_+^m$ or $g = 0$ a.e. on $\mathbb{R}_+^n$. The proof is now complete. □

**Remark 3.4.** If we choose the parameters $A_1 = \frac{1}{p'}(m - s_1)$ and $A_2 = \frac{1}{q'}(n - s_2)$, where $s_1 + s_2 = s$, inequalities (3.5) and (3.6) respectively read

$$\int_{\mathbb{R}_+^m}\int_{\mathbb{R}_+^n} k_s{}^\lambda\left(|\mathbf{x}|_\alpha, |\mathbf{y}|_\beta\right) f(\mathbf{x}) g(\mathbf{y}) d\mathbf{x} d\mathbf{y} \le C_{\mathbf{s}} ||f||_{L^p_{\Phi_{\mathbf{s}}}(\mathbb{R}_+^m)} ||g||_{L^q_{\Psi_{\mathbf{s}}}(\mathbb{R}_+^n)} \tag{3.13}$$

and

$$\left\{\int_{\mathbb{R}_+^n} |\mathbf{y}|_\beta{}^{\lambda p s_2 - n}\left[\int_{\mathbb{R}_+^m} k_s{}^\lambda\left(|\mathbf{x}|_\alpha, |\mathbf{y}|_\beta\right) f(\mathbf{x}) d\mathbf{x}\right]^{q'} d\mathbf{y}\right\}^{\frac{1}{q'}} \le C_{\mathbf{s}} ||f||_{L^p_{\Phi_{\mathbf{s}}}(\mathbb{R}_+^m)}, \tag{3.14}$$

where

$$C_{\mathbf{s}} = \left[\frac{\Gamma^m\left(\frac{1}{\alpha}\right)}{\alpha^{m-1}\Gamma\left(\frac{m}{\alpha}\right)}\right]^{\frac{1}{p'}} \left[\frac{\Gamma^n\left(\frac{1}{\beta}\right)}{\beta^{n-1}\Gamma\left(\frac{n}{\beta}\right)}\right]^{\frac{1}{q'}} c_s{}^\lambda(1 - s_2) \tag{3.15}$$

and

$$\Phi_{\mathbf{s}}(\mathbf{x}) = |\mathbf{x}|_\alpha{}^{p\left(\frac{m}{p'} - \lambda s_1\right)} \quad \text{and} \quad \Psi_{\mathbf{s}}(\mathbf{y}) = |\mathbf{y}|_\beta{}^{q\left(\frac{n}{q'} - \lambda s_2\right)}. \tag{3.16}$$

Of course, inequalities (3.13) and (3.14) are valid if $c_s(1 - s_2) < \infty$. Now, if we substitute $s_1 = \frac{s}{r_1}$ and $s_2 = \frac{s}{r_2}$ in (3.13) and (3.14), where $r_1, r_2$ is a pair of conjugate parameters, these inequalities become (1.7) and (1.8) (in the conjugate case and for $\alpha = \beta$, $m = n$). Thus, we may regard relations (3.13) and (3.14) as extensions of relations (1.7) and (1.8), from the Introduction, to the case of non-conjugate exponents.

On the other hand, if we substitute $m = n = \alpha = \beta = 1$ in (3.13), we get the result from paper [6], also in the conjugate case.

**Remark 3.5.** In the paper [17], the author obtained the related Hilbert and Hardy-Hilbert type inequalities for a homogeneous kernel with $n$ variables, but with the same norms of appropriate vectors. So, we can regard inequalities (3.5) and (3.6) as a slight generalizations of results from [17], for $n = 2$. One can obtain the extensions of (3.5) and (3.6) for a homogeneous kernel in $n$ variables. Such results are here omitted since we shall investigate the operators which include the homogeneous kernel in two variables.

The Hardy-Hilbert type inequality (3.6) allows us a precise definition of the Hilbert-type operator in a non-conjugate setting, and some conclusions about its norm, as well.

Hence, under the assumptions of Theorem 3.3, we define the Hilbert-type operator $\mathcal{H}_s : L^p_{\Phi_{\mathbf{A}}}(\mathbb{R}^m_+) \to L^{q'}_{\Psi_{\mathbf{A}}^{1-q'}}(\mathbb{R}^n_+)$ as

$$(\mathcal{H}_s f)(\mathbf{y}) = \int_{\mathbb{R}^m_+} {k_s}^{\lambda}(|\mathbf{x}|_\alpha, |\mathbf{y}|_\beta) f(\mathbf{x}) d\mathbf{x}, \quad \mathbf{y} \in \mathbb{R}^n_+. \tag{3.17}$$

Clearly, the operator $\mathcal{H}_s$ is well defined since from inequality (3.6) we conclude that $\mathcal{H}_s f \in L^{q'}_{\Psi_{\mathbf{A}}^{1-q'}}(\mathbb{R}^n_+)$. Moreover, inequality (3.6) reads in the operator form

$$||\mathcal{H}_s f||_{L^{q'}_{\Psi_{\mathbf{A}}^{1-q'}}(\mathbb{R}^n_+)} \leq C_{\mathbf{A}} ||f||_{L^p_{\Phi_{\mathbf{A}}}(\mathbb{R}^m_+)}. \tag{3.18}$$

On the other hand, since the norm of the operator $\mathcal{H}_s$ is defined by

$$||\mathcal{H}_s|| = \sup_{f \in L^p_{\Phi_{\mathbf{A}}}(\mathbb{R}^m_+), f \neq 0} \frac{||\mathcal{H}_s f||_{L^{q'}_{\Psi_{\mathbf{A}}^{1-q'}}(\mathbb{R}^n_+)}}{||f||_{L^p_{\Phi_{\mathbf{A}}}(\mathbb{R}^m_+)}}, \tag{3.19}$$

inequality (3.18) yields the upper bound for the norm of operator $\mathcal{H}_s$:

$$||\mathcal{H}_s|| \leq C_{\mathbf{A}}. \tag{3.20}$$

In the next section we shall investigate some general cases in which we can find the norm of operator $\mathcal{H}_s$. Obviously, that problem is equivalent to the problem of the best possible constant factors involved in the right-hand sides of inequalities (3.5) and (3.6).

## 4. On the best possible constant and norm

As already mentioned, we are not able to find an explicit value for the norm of the operator (3.17) in a non-conjugate setting.

In spite of that, we can solve the mentioned problem in some general settings with conjugate exponents. Hence, in this section parameters $p$ and $q$ are assumed to be conjugate, that is, when $p = q'$, $q = p'$ and $\lambda = 1$. We consider constant factor $C_{\mathbf{A}}$ involved in the right-hand sides of inequalities (3.5) and (3.6). Of course, it is enough to investigate the problem for the Hilbert-type inequality (3.5) since we have pair of equivalent inequalities.

The main idea in obtaining the best possible constant factor is a reduction of the constant $C_{\mathbf{A}}$, defined by (3.7), in the form without some exponents, by the appropriate choice of parameters $A_1$ and $A_2$. Thus, it is natural to set the condition

$$qA_1 + pA_2 = m + n - s, \tag{4.1}$$

where $p$ and $q$ are conjugate exponents, since $c_s(pA_2 + 1 - n) = c_s(m + 1 - qA_1 - s)$ in that case.

Under condition (4.1) in the conjugate case, the constant factor (3.7) becomes

$$\overline{C}_{\mathbf{A}} = \left[\frac{\Gamma^m\left(\frac{1}{\alpha}\right)}{\alpha^{m-1}\Gamma\left(\frac{m}{\alpha}\right)}\right]^{\frac{1}{q}} \left[\frac{\Gamma^n\left(\frac{1}{\beta}\right)}{\beta^{n-1}\Gamma\left(\frac{n}{\beta}\right)}\right]^{\frac{1}{p}} c_s(pA_2 + 1 - n). \tag{4.2}$$

Further, if the constraint (4.1) is satisfied, inequalities (3.5) and (3.6) from Theorem 3.3 become respectively

$$\int_{\mathbb{R}^m_+}\int_{\mathbb{R}^n_+} k_s\left(|\mathbf{x}|_\alpha, |\mathbf{y}|_\beta\right) f(\mathbf{x})g(\mathbf{y})d\mathbf{x}d\mathbf{y} \le \overline{C}_{\mathbf{A}}||f||_{L^p_{\overline{\Phi}_{\mathbf{A}}}(\mathbb{R}^m_+)}||g||_{L^q_{\overline{\Psi}_{\mathbf{A}}}(\mathbb{R}^n_+)} \tag{4.3}$$

and

$$\left\{\int_{\mathbb{R}^n_+} |\mathbf{y}|_\beta{}^{pqA_1+p(s-m)-n}\left[\int_{\mathbb{R}^m_+} k_s\left(|\mathbf{x}|_\alpha, |\mathbf{y}|_\beta\right) f(\mathbf{x})d\mathbf{x}\right]^p d\mathbf{y}\right\}^{\frac{1}{p}} \le \overline{C}_{\mathbf{A}}||f||_{L^p_{\overline{\Phi}_{\mathbf{A}}}(\mathbb{R}^m_+)}, \tag{4.4}$$

where $\overline{C}_{\mathbf{A}}$ is defined by (4.2) and

$$\overline{\Phi}_{\mathbf{A}}(\mathbf{x}) = |\mathbf{x}|_\alpha{}^{pqA_1-m} \quad \text{and} \quad \overline{\Psi}_{\mathbf{A}}(\mathbf{y}) = |\mathbf{y}|_\beta{}^{pqA_2-n}. \tag{4.5}$$

Clearly, if $0 < p < 1$ and $q < 0$, then the reverse inequalities in (4.3) and (4.4) hold.

The following result provides the best possible constant factors in inequalities (4.3) and (4.4).

**Theorem 4.1.** *Suppose $p$ and $q$ are conjugate exponents, $A_1$ and $A_2$ are real parameters satisfying constraint* (4.1)*, and suppose there exist $\delta > 0$ such that $c_s(\eta) < \infty$ for $\eta \in [pA_2 + 1 - n - \delta, pA_2 + 1 - n]$. Then the constant factor $\overline{C}_{\mathbf{A}}$ is the best possible in both inequalities* (4.3) *and* (4.4)*.*

*Proof.* We consider two cases depending on whether $p > 1$ or $0 < p < 1$.

If $p > 1$ then $q > 1$. Suppose the constant factor $\overline{C}_{\mathbf{A}}$, defined by (4.2), is not the best possible in inequality (4.3). That implies that there exists a constant factor $K$, smaller than $\overline{C}_{\mathbf{A}}$, such that the inequality (4.3) is still valid if we replace $\overline{C}_{\mathbf{A}}$ with $K$.

Let $\varepsilon > 0$. We substitute functions $f_\varepsilon : \mathbb{R}_+^m \to \mathbb{R}$ and $g_\varepsilon : \mathbb{R}_+^n \to \mathbb{R}$ defined by

$$f_\varepsilon(\mathbf{x}) = \begin{cases} 0 & , 0 < |\mathbf{x}|_\alpha < 1, \\ |\mathbf{x}|_\alpha{}^{-qA_1 - \frac{\varepsilon}{p}} & , |\mathbf{x}|_\alpha \geq 1 \end{cases}$$

and

$$g_\varepsilon(\mathbf{x}) = \begin{cases} 0 & , 0 < |\mathbf{y}|_\beta < 1, \\ |\mathbf{y}|_\beta{}^{-pA_2 - \frac{\varepsilon}{q}} & , |\mathbf{y}|_\beta \geq 1 \end{cases}.$$

in inequality (4.3), with the constant factor $K$ instead of $\overline{C}_{\mathbf{A}}$. Then, by using relation (3.2) from Lemma 3.2, the right-hand side of (4.3) becomes

$$R_\varepsilon = \frac{K}{\varepsilon} \left[ \frac{\Gamma^m\left(\frac{1}{\alpha}\right)}{\alpha^{m-1}\Gamma\left(\frac{m}{\alpha}\right)} \right]^{\frac{1}{p}} \left[ \frac{\Gamma^n\left(\frac{1}{\beta}\right)}{\beta^{n-1}\Gamma\left(\frac{n}{\beta}\right)} \right]^{\frac{1}{q}}. \tag{4.6}$$

Now we compute the left-hand side $L_\varepsilon$ of inequality (4.3) for the above choice of functions $f_\varepsilon : \mathbb{R}_+^m \to \mathbb{R}$ and $g_\varepsilon : \mathbb{R}_+^n \to \mathbb{R}$. We have

$$L_\varepsilon = \int_{\{\mathbf{x} \in \mathbb{R}_+^m : |\mathbf{x}|_\alpha \geq 1\}} |\mathbf{x}|_\alpha{}^{-qA_1 - \frac{\varepsilon}{p}} \left[ \int_{\{\mathbf{y} \in \mathbb{R}_+^n : |\mathbf{y}|_\beta \geq 1\}} k_s\left(|\mathbf{x}|_\alpha, |\mathbf{y}|_\beta\right) |\mathbf{y}|_\beta{}^{-pA_2 - \frac{\varepsilon}{q}} \right] d\mathbf{x}. \tag{4.7}$$

Clearly,

$$\begin{aligned} &\int_{\{\mathbf{y} \in \mathbb{R}_+^n : |\mathbf{y}|_\beta \geq 1\}} k_s\left(|\mathbf{x}|_\alpha, |\mathbf{y}|_\beta\right) |\mathbf{y}|_\beta{}^{-pA_2 - \frac{\varepsilon}{q}} d\mathbf{y} \\ &= \int_{\mathbb{R}_+^n} k_s\left(|\mathbf{x}|_\alpha, |\mathbf{y}|_\beta\right) |\mathbf{y}|_\beta{}^{-pA_2 - \frac{\varepsilon}{q}} d\mathbf{y} - \int_{\{\mathbf{y} \in \mathbb{R}_+^n : |\mathbf{y}|_\beta \leq 1\}} k_s\left(|\mathbf{x}|_\alpha, |\mathbf{y}|_\beta\right) |\mathbf{y}|_\beta{}^{-pA_2 - \frac{\varepsilon}{q}} d\mathbf{y}, \end{aligned} \tag{4.8}$$

since the set $\{\mathbf{y} \in \mathbb{R}_+^n : |\mathbf{y}|_\beta = 1\}$ has measure 0. On the other hand, by using relation (3.1), we have

$$\begin{aligned} &\int_{D_R^n} k_s\left(|\mathbf{x}|_\alpha, |\mathbf{y}|_\beta\right) |\mathbf{y}|_\beta{}^{-pA_2 - \frac{\varepsilon}{q}} d\mathbf{y} \\ &\qquad = |\mathbf{x}|_\alpha{}^{n - pA_2 - s - \frac{\varepsilon}{q}} \frac{\Gamma^n\left(\frac{1}{\beta}\right)}{\beta^{n-1}\Gamma\left(\frac{n}{\beta}\right)} \int_0^{R|\mathbf{x}|_\alpha^{-1}} k_s(1,t) t^{n - pA_2 - 1 - \frac{\varepsilon}{q}} dt, \end{aligned} \tag{4.9}$$

where the set $D_R^n$ is defined in a statement of Lemma 3.1. Hence, if we substitute $R = 1$ and $R = \infty$ in (4.9), then from (4.8) we get

$$\int_{\{\mathbf{y}\in\mathbb{R}_+^n:|\mathbf{y}|_\beta\geq 1\}} k_s\left(|\mathbf{x}|_\alpha, |\mathbf{y}|_\beta\right) |\mathbf{y}|_\beta{}^{-pA_2-\frac{\varepsilon}{q}} d\mathbf{y}$$

$$= |\mathbf{x}|_\alpha{}^{n-pA_2-s-\frac{\varepsilon}{q}} \frac{\Gamma^n\left(\frac{1}{\beta}\right)}{\beta^{n-1}\Gamma\left(\frac{n}{\beta}\right)} \int_{|\mathbf{x}|_\alpha^{-1}}^{\infty} k_s(1,t) t^{n-pA_2-1-\frac{\varepsilon}{q}} dt. \quad (4.10)$$

Further, if we substitute (4.10) in (4.7) we get

$$L_\varepsilon = \frac{\Gamma^n\left(\frac{1}{\beta}\right)}{\beta^{n-1}\Gamma\left(\frac{n}{\beta}\right)} \int_{\{\mathbf{x}\in\mathbb{R}_+^m:|\mathbf{x}|_\alpha\geq 1\}} |\mathbf{x}|_\alpha{}^{-m-\varepsilon} \left[\int_{|\mathbf{x}|_\alpha^{-1}}^{\infty} k_s(1,t) t^{n-pA_2-1-\frac{\varepsilon}{q}} dt\right] d\mathbf{x}. \quad (4.11)$$

Now, we separate the inner integral in (4.11) into two integrals over the intervals $\langle |\mathbf{x}|_\alpha^{-1}, 1\rangle$ and $\langle 1, \infty\rangle$. Clearly, by using the Fubini theorem and substitution variable theorem, with $\mathbf{x} = t^{-1}\mathbf{z}$, we get

$$\begin{aligned}
&\int_{\{\mathbf{x}\in\mathbb{R}_+^m:|\mathbf{x}|_\alpha\geq 1\}} |\mathbf{x}|_\alpha{}^{-m-\varepsilon} \left[\int_{|\mathbf{x}|_\alpha^{-1}}^{1} k_s(1,t) t^{n-pA_2-1-\frac{\varepsilon}{q}} dt\right] d\mathbf{x} \\
&\quad = \int_0^1 k_s(1,t) t^{n-pA_2-1-\frac{\varepsilon}{q}} \left[\int_{\{\mathbf{x}\in\mathbb{R}_+^m:|\mathbf{x}|_\alpha\geq t^{-1}\}} |\mathbf{x}|_\alpha{}^{-m-\varepsilon} d\mathbf{x}\right] dt \\
&\quad = \int_0^1 k_s(1,t) t^{n-pA_2-1+\frac{\varepsilon}{p}} \left[\int_{\{\mathbf{z}\in\mathbb{R}_+^m:|\mathbf{z}|_\alpha\geq 1\}} |\mathbf{z}|_\alpha{}^{-m-\varepsilon} d\mathbf{z}\right] dt. \quad (4.12)
\end{aligned}$$

Finally, (3.2), (4.11) and (4.12) yield

$$\begin{aligned}
L_\varepsilon &= \frac{1}{\varepsilon} \cdot \frac{\Gamma^m\left(\frac{1}{\alpha}\right)}{\alpha^{m-1}\Gamma\left(\frac{m}{\alpha}\right)} \cdot \frac{\Gamma^n\left(\frac{1}{\beta}\right)}{\beta^{n-1}\Gamma\left(\frac{n}{\beta}\right)} \\
&\quad \cdot \left[\int_0^1 k_s(1,t) t^{n-pA_2-1+\frac{\varepsilon}{p}} + \int_1^\infty k_s(1,t) t^{n-pA_2-1-\frac{\varepsilon}{q}}\right]. \quad (4.13)
\end{aligned}$$

If $p > 1$, then the estimates

$$t^{n-pA_2-1+\frac{\varepsilon}{p}} \leq t^{n-pA_2-1}, \quad t \in \langle 0, 1\rangle$$
$$t^{n-pA_2-1-\frac{\varepsilon}{q}} \leq t^{n-pA_2-1}, \quad t \in \langle 1, \infty\rangle$$

are obviously valid, so

$$\int_0^1 k_s(1,t) t^{n-pA_2-1+\frac{\varepsilon}{p}} + \int_1^\infty k_s(1,t) t^{n-pA_2-1-\frac{\varepsilon}{q}} \leq c_s(pA_2 + 1 - n) < \infty.$$

Thus, according to the Lebesgue control convergent theorem (see [15]) we have

$$\int_0^1 k_s(1,t)t^{n-pA_2-1+\frac{\varepsilon}{p}} + \int_1^\infty k_s(1,t)t^{n-pA_2-1-\frac{\varepsilon}{q}} = c_s(pA_2+1-n)+o(1). \tag{4.14}$$

Finally, since $L_\varepsilon \le R_\varepsilon$, by using (4.6), (4.13) and (4.14) we have

$$\frac{1}{\varepsilon}\cdot\frac{\Gamma^m\left(\frac{1}{\alpha}\right)}{\alpha^{m-1}\Gamma\left(\frac{m}{\alpha}\right)}\cdot\frac{\Gamma^n\left(\frac{1}{\beta}\right)}{\beta^{n-1}\Gamma\left(\frac{n}{\beta}\right)}\cdot c_s(pA_2+1-n)\left(1+o(1)\right)$$
$$\le \frac{K}{\varepsilon}\left[\frac{\Gamma^m\left(\frac{1}{\alpha}\right)}{\alpha^{m-1}\Gamma\left(\frac{m}{\alpha}\right)}\right]^{\frac{1}{p}}\left[\frac{\Gamma^n\left(\frac{1}{\beta}\right)}{\beta^{n-1}\Gamma\left(\frac{n}{\beta}\right)}\right]^{\frac{1}{q}},$$

that is $\overline{C}_{\mathbf{A}}(1+o(1)) \le K$. Now, by letting $\varepsilon \searrow 0$ we get $\overline{C}_{\mathbf{A}} \le K$, which contradicts the fact that $K$ is smaller than $\overline{C}_{\mathbf{A}}$. Thus, $\overline{C}_{\mathbf{A}}$ is the best possible constant factor in (4.3), for $p>1$.

It remains to cover the case when $0<p<1$ and $q<0$. In that setting we have the reverse inequality in (4.3). We consider the same functions $f_\varepsilon : \mathbb{R}_+^m \to \mathbb{R}$ and $g_\varepsilon : \mathbb{R}_+^n \to \mathbb{R}$ as in the case when $p>1$, and substitute them in the reverse of (4.3). Now, for $\varepsilon < -\delta q$, the obvious estimate

$$t^{n-pA_2-1-\frac{\varepsilon}{q}} \le t^{n-pA_2-1+\delta}, \quad t \in \langle 1,\infty\rangle$$

implies that

$$\int_0^1 k_s(1,t)t^{n-pA_2-1+\frac{\varepsilon}{p}} + \int_1^\infty k_s(1,t)t^{n-pA_2-1-\frac{\varepsilon}{q}}$$
$$\le c_s(pA_2+1-n)+c_s(pA_2+1-n-\delta) < \infty,$$

so the relation (4.14) holds in the reverse case, as well. That means that we obtain the same expressions for the left-hand side and right-hand side of the reverse in (4.14), as in the previous case.

Now, suppose $\overline{C}_{\mathbf{A}}$ is not the best possible constant in the reverse case. That means that there exist a constant $M > \overline{C}_{\mathbf{A}}$ such that the reverse inequality in (4.3) holds if we replace $\overline{C}_{\mathbf{A}}$ with $M$. Similarly as in the first case, that is when $p>1$, we get $\overline{C}_{\mathbf{A}}(1+o(1)) \ge M$, which implies that $\overline{C}_{\mathbf{A}} \ge M$. Hence, we get a contradiction and the proof is now complete. □

**Remark 4.2.** Note that the constraint $c_s(\eta) < \infty$, $\eta \in [pA_2+1-n-\delta, pA_2+1-n]$ in the statement of Theorem 4.1 is necessary only in the reverse case, that is when $0<p<1$. If $p>1$, it is enough to set condition $c_s(pA_2+1-n)<\infty$.

**Remark 4.3.** Let $m = n$ and $\alpha = \beta$. We consider the parameters $A_1 = \frac{1}{q}\left(n - \frac{s}{r_1}\right)$ and $A_2 = \frac{1}{p}\left(n - \frac{s}{r_2}\right)$, where $p, q$ and $r_1, r_2$ are two pairs of conjugate exponents. Clearly, $A_1$ and $A_2$ satisfy relation (4.1). Now, if we substitute these parameters in (4.3) and (4.4), we get inequalities (1.7) and (1.8) from Introduction. Thus, relations (4.3) and (4.4) are generalizations of (1.7) and (1.8), with the best possible constant factors.

**Remark 4.4.** It is easy to see that the parameters $A_1$ and $A_2$, defined in Remark 3.4, satisfy relation (4.1). Thus, the constants involved in the right-hand sides of inequalities (3.13) and (3.14) are the best possible in the conjugate case.

According to the discussion in Section 3, Theorem 4.1 also provides the norm of the operator $\mathcal{H}_s$, defined by (3.17), in the case of conjugate exponents.

**Corollary 4.5.** *Let $p$ and $q$ be conjugate exponents, let the parameters $A_1$ and $A_2$ satisfy relation* (4.1)*, and let there exist $\delta > 0$ such that $c_s(\eta) < \infty$ for $\eta \in [pA_2 + 1 - n - \delta, pA_2 + 1 - n]$. Then the norm of the operator $\overline{\mathcal{H}}_s : L^p_{\overline{\Phi}_{\mathbf{A}}}(\mathbb{R}^m_+) \to L^p_{\overline{\Psi}_{\mathbf{A}}^{1-p}}(\mathbb{R}^n_+)$, defined by*

$$\left(\overline{\mathcal{H}}_s f\right)(\mathbf{y}) = \int_{\mathbb{R}^m_+} k_s\left(|\mathbf{x}|_\alpha, |\mathbf{y}|_\beta\right) f(\mathbf{x}) d\mathbf{x} \quad , \quad \mathbf{y} \in \mathbb{R}^n_+,$$

*is $||\overline{\mathcal{H}}_s|| = \overline{C}_{\mathbf{A}}$. Here, the weight functions $\overline{\Phi}_{\mathbf{A}} : \mathbb{R}^m_+ \to \mathbb{R}$, $\overline{\Psi}_{\mathbf{A}} : \mathbb{R}^n_+ \to \mathbb{R}$ are defined by* (4.5) *and the constant $\overline{C}_{\mathbf{A}}$ by* (4.2)*.*

## 5. The related Hardy-type operators

We can also generate some interesting Hardy-type operators, deduced from homogeneous kernels of negative degree. More precisely, if $k_s : \mathbb{R}^2_+ \to \mathbb{R}$ is a non-negative measurable homogeneous function of degree $-s$, $s > 0$, then the function $\hat{k}_s : \mathbb{R}^2_+ \to \mathbb{R}$, defined by

$$\hat{k}_s(u, v) = k_s(u, v)\chi_{u \le v}(u, v) = \begin{cases} k_s(u, v), & u \le v, \\ 0, & u > v, \end{cases} \tag{5.1}$$

is also homogeneous of degree $-s$. Further, since

$$c_s(\eta) = \int_0^\infty \hat{k}_s(1, t) t^{-\eta} dt = \int_1^\infty k_s(1, t) t^{-\eta} dt,$$

we define

$$\hat{c}_s(\eta) = \int_1^\infty k_s(1, t) t^{-\eta} dt. \tag{5.2}$$

Now, by inserting the homogeneous kernel (5.1) in inequalities (3.5) and (3.6) we have

**Corollary 5.1.** *Suppose $p$, $q$, $\lambda$ are as in* (2.1) *and* (2.2), *$m,n \in \mathbb{N}$, and $\alpha,\beta > 0$. Let $k_s : \mathbb{R}_+^2 \to \mathbb{R}$ be homogeneous function of degree $-s$, $s > 0$, and let $A_1$, $A_2$ be real parameters such that $\hat{c}_s(q'A_2 + 1 - n) < \infty$ and $\hat{c}_s(m + 1 - p'A_1 - s) < \infty$. If the functions $\Phi_{\mathbf{A}}$ on $\mathbb{R}_+^m$ and $\Psi_{\mathbf{A}}$ on $\mathbb{R}_+^n$ are defined by* (3.4), *then the inequalities*

$$\int_{\mathbb{R}_+^m} \int_{\{\mathbf{y}\in\mathbb{R}_+^n : |\mathbf{y}|_\beta \geq |\mathbf{x}|_\alpha\}} {k_s}^\lambda \left(|\mathbf{x}|_\alpha, |\mathbf{y}|_\beta\right) f(\mathbf{x}) g(\mathbf{y}) d\mathbf{x} d\mathbf{y} \leq \hat{C}_{\mathbf{A}} ||f||_{L^p_{\Phi_{\mathbf{A}}}(\mathbb{R}_+^m)} ||g||_{L^q_{\Psi_{\mathbf{A}}}(\mathbb{R}_+^n)} \tag{5.3}$$

*and*

$$\left\{ \int_{\mathbb{R}_+^n} {|\mathbf{y}|_\beta}^{q'(A_1-A_2)+(s-m)\frac{q'}{p'}} \left[ \int_{\{\mathbf{x}\in\mathbb{R}_+^m : |\mathbf{x}|_\alpha \leq |\mathbf{y}|_\beta\}} {k_s}^\lambda \left(|\mathbf{x}|_\alpha, |\mathbf{y}|_\beta\right) f(\mathbf{x}) d\mathbf{x} \right]^{q'} d\mathbf{y} \right\}^{\frac{1}{q'}} \leq \hat{C}_{\mathbf{A}} ||f||_{L^p_{\Phi_{\mathbf{A}}}(\mathbb{R}_+^m)}, \tag{5.4}$$

*where*

$$\hat{C}_{\mathbf{A}} = \left[ \frac{\Gamma^m\left(\frac{1}{\alpha}\right)}{\alpha^{m-1}\Gamma\left(\frac{m}{\alpha}\right)} \right]^{\frac{1}{p'}} \left[ \frac{\Gamma^n\left(\frac{1}{\beta}\right)}{\beta^{n-1}\Gamma\left(\frac{n}{\beta}\right)} \right]^{\frac{1}{q'}} \hat{c}_s^{\frac{1}{q'}}(q'A_2 + 1 - n) \hat{c}_s^{\frac{1}{p'}}(m + 1 - p'A_1 - s), \tag{5.5}$$

*hold for all non-negative measurable functions $f : \mathbb{R}_+^m \to \mathbb{R}$, $g : \mathbb{R}_+^n \to \mathbb{R}$ and are equivalent. Equalities in* (5.3) *and* (5.4) *hold if and only if $f = 0$ a.e. on $\mathbb{R}_+^m$ or $g = 0$ a.e. on $\mathbb{R}_+^n$. Further, reverse inequalities in* (5.3) *and* (5.4) *hold if $p$ and $q$ are as in* (2.9), (2.10) *or* (2.11).

On the other hand, we define the dual Hardy-type kernel deduced from the homogeneous kernel $k_s : \mathbb{R}_+^2 \to \mathbb{R}$. More precisely, we define $\check{k}_s : \mathbb{R}_+^2 \to \mathbb{R}$, by

$$\check{k}_s(u,v) = k_s(u,v)\chi_{u\geq v}(u,v) = \begin{cases} 0, & u < v, \\ k_s(u,v), & u \geq v. \end{cases} \tag{5.6}$$

Clearly, $\check{k}_s$ is a homogeneous function of degree $-s$, so if we denote

$$\check{c}_s(\eta) = \int_0^1 k_s(1,t) t^{-\eta} dt, \tag{5.7}$$

Theorem 3.3 takes the following form.

**Corollary 5.2.** *Let $p$, $q$, $\lambda$ be as in* (2.1) *and* (2.2), *let $m,n \in \mathbb{N}$, and let $\alpha,\beta > 0$. Suppose $k_s : \mathbb{R}_+^2 \to \mathbb{R}$ is the homogeneous function of degree $-s$, $s > 0$, and $A_1$, $A_2$ are real parameters such that $\check{c}_s(q'A_2 + 1 - n) < \infty$ and*

*$\check{c}_s(m+1-p'A_1-s)<\infty$. If the functions $\Phi_{\mathbf{A}}$ on $\mathbb{R}_+^m$ and $\Psi_{\mathbf{A}}$ on $\mathbb{R}_+^n$ are defined by* (3.4)*. Then the inequalities*

$$\int_{\mathbb{R}_+^m}\int_{\{\mathbf{y}\in\mathbb{R}_+^n:|\mathbf{y}|_\beta\le|\mathbf{x}|_\alpha\}} {k_s}^\lambda\left(|\mathbf{x}|_\alpha,|\mathbf{y}|_\beta\right)f(\mathbf{x})g(\mathbf{y})d\mathbf{x}d\mathbf{y} \le \check{C}_{\mathbf{A}}||f||_{L^p_{\Phi_{\mathbf{A}}}(\mathbb{R}_+^m)}||g||_{L^q_{\Psi_{\mathbf{A}}}(\mathbb{R}_+^n)} \tag{5.8}$$

*and*

$$\left\{\int_{\mathbb{R}_+^n}|\mathbf{y}|_\beta{}^{q'(A_1-A_2)+(s-m)\frac{q'}{p'}}\left[\int_{\{\mathbf{x}\in\mathbb{R}_+^m:|\mathbf{x}|_\alpha\ge|\mathbf{y}|_\beta\}}{k_s}^\lambda\left(|\mathbf{x}|_\alpha,|\mathbf{y}|_\beta\right)f(\mathbf{x})d\mathbf{x}\right]^{q'}d\mathbf{y}\right\}^{\frac{1}{q'}} \le \check{C}_{\mathbf{A}}||f||_{L^p_{\Phi_{\mathbf{A}}}(\mathbb{R}_+^m)}, \tag{5.9}$$

*where*

$$\check{C}_{\mathbf{A}}=\left[\frac{\Gamma^m\left(\frac{1}{\alpha}\right)}{\alpha^{m-1}\Gamma\left(\frac{m}{\alpha}\right)}\right]^{\frac{1}{p'}}\left[\frac{\Gamma^n\left(\frac{1}{\beta}\right)}{\beta^{n-1}\Gamma\left(\frac{n}{\beta}\right)}\right]^{\frac{1}{q'}}\check{c}_s^{\frac{1}{q'}}(q'A_2+1-n)\check{c}_s^{\frac{1}{p'}}(m+1-p'A_1-s), \tag{5.10}$$

*hold for all non-negative measurable functions $f:\mathbb{R}_+^m\to\mathbb{R}$, $g:\mathbb{R}_+^n\to\mathbb{R}$ and are equivalent. Equalities in* (5.8) *and* (5.9) *hold if and only if $f=0$ a.e. on $\mathbb{R}_+^m$ or $g=0$ a.e. on $\mathbb{R}_+^n$. Moreover, if $p$ and $q$ are as in* (2.9)*,* (2.10) *or* (2.11)*, the signs of inequality in* (5.8) *and* (5.9) *are reversed.*

Corollaries 5.1 and 5.2 determine the Hardy-type operators $\hat{\mathcal{H}}_s:L^p_{\Phi_{\mathbf{A}}}(\mathbb{R}_+^m)\to L^p_{\Psi_{\mathbf{A}}^{1-p}}(\mathbb{R}_+^n)$ and $\check{\mathcal{H}}_s:L^p_{\Phi_{\mathbf{A}}}(\mathbb{R}_+^m)\to L^p_{\Psi_{\mathbf{A}}^{1-p}}(\mathbb{R}_+^n)$ defined by

$$\left(\hat{\mathcal{H}}_s f\right)(\mathbf{y})=\int_{\{\mathbf{x}\in\mathbb{R}_+^m:|\mathbf{x}|_\alpha\le|\mathbf{y}|_\beta\}}{k_s}^\lambda\left(|\mathbf{x}|_\alpha,|\mathbf{y}|_\beta\right)f(\mathbf{x})d\mathbf{x},\quad \mathbf{y}\in\mathbb{R}_+^n \tag{5.11}$$

and

$$\left(\check{\mathcal{H}}_s f\right)(\mathbf{y})=\int_{\{\mathbf{x}\in\mathbb{R}_+^m:|\mathbf{x}|_\alpha\ge|\mathbf{y}|_\beta\}}{k_s}^\lambda\left(|\mathbf{x}|_\alpha,|\mathbf{y}|_\beta\right)f(\mathbf{x})d\mathbf{x},\quad \mathbf{y}\in\mathbb{R}_+^n. \tag{5.12}$$

**Remark 5.3.** Similarly as in Section 3, inequalities (5.4) and (5.9) yield the upper bounds for the norms of operators (5.11) and (5.12):

$$||\hat{\mathcal{H}}_s||\le\hat{C}_{\mathbf{A}}\quad\text{and}\quad||\check{\mathcal{H}}_s||\le\check{C}_{\mathbf{A}}. \tag{5.13}$$

Moreover, Corollary 4.5 implies that, in the case of conjugate exponents $p$ and $q$ and under condition (4.1), the norms of Hardy-type operators are

$$||\hat{\mathcal{H}}_s||=\hat{C}_{\mathbf{A}}\quad\text{and}\quad||\check{\mathcal{H}}_s||=\check{C}_{\mathbf{A}}. \tag{5.14}$$

Clearly, expressions for the constants $\hat{C}_{\mathbf{A}}$ and $\check{C}_{\mathbf{A}}$ simplify in the conjugate case according to condition (4.1), similarly as the constant $C_{\mathbf{A}}$ becomes $\overline{C}_{\mathbf{A}}$ (see Section 4).

## 6. Applications

The last section is dedicated to some special homogeneous functions and some special choices of parameters $A_1$ and $A_2$ from Theorem 3.3. Namely, we consider related Hilbert-type operators in such settings. According to Corollary 4.5, we shall find explicit formulas for the norms of operators in the conjugate case.

**Example 6.1.** If we consider the homogeneous function $k_s : \mathbb{R}_+^2 \to \mathbb{R}$, defined by

$$k_s(u,v) = \frac{1}{(u+v)^s}, \quad s > 0, \tag{6.1}$$

then the constant $C_{\mathbf{A}}$, from Theorem 3.3, is expressed in terms of the usual Beta and Gamma functions since

$$c_s(\eta) = \int_0^\infty k(1,t)t^{-\eta}dt = \int_0^\infty \frac{t^{-\eta}}{(1+t)^s}dt = B(1-\eta, \eta+s-1), \ 1-s < \eta < 1.$$

In papers [2], [9], [12], [13] one can find the Hilbert and Hardy-Hilbert type inequalities involving the homogeneous function (6.1). Moreover, the same homogeneous function is considered in papers [11] and [17], in multidimensional settings. Of course, our Theorem 3.3 generalizes all those results.

For example, let $m = 2$, $n = 1$, $\alpha = \beta = 1$ and let parameters $A_1, A_2$ be defined by $A_1 = \frac{1}{p'}\left(2 - \frac{s}{r_1}\right)$ and $A_2 = \frac{1}{q'}\left(1 - \frac{s}{r_2}\right)$, where $r_1, r_2 > 1$ is a pair of conjugate exponents. For these parameters and homogeneous kernel (6.1), inequality (3.6) reads

$$\left\{ \int_{\mathbb{R}_+} y^{\lambda \frac{ps}{r_2}-1} \left[ \int_{\mathbb{R}_+^2} \frac{1}{(x_1+x_2+y)^{\lambda s}} f(x_1,x_2)dx_1dx_2 \right]^{q'} dy \right\}^{\frac{1}{q'}}$$
$$\leq B^\lambda\left(\frac{s}{r_1}, \frac{s}{r_2}\right) ||f||_{L^p_{\widetilde{\phi}_{\mathbf{s}}}(\mathbb{R}_+^2)}, \tag{6.2}$$

where $\widetilde{\phi}_{\mathbf{s}}(x_1,x_2) = (x_1+x_2)^{p\left(\frac{2}{p'} - \frac{\lambda s}{r_1}\right)}$. Note that the inequality (6.2) is closely related to inequality (1.9) from the Introduction. Namely, (6.2) represents non-conjugate Hardy-Hilbert type extension of (1.9) on $\mathbb{R}_+^2 \times \mathbb{R}_+$. Now, we define the Hilbert-type operator $\mathcal{L}_s : L^p_{\widetilde{\phi}_{\mathbf{s}}}(\mathbb{R}_+^2) \to L^{q'}_{\widetilde{\psi}_{\mathbf{s}}^{1-q'}}(\mathbb{R}_+)$, as

$$(\mathcal{L}_s f)(y) = \int_{\mathbb{R}_+^2} \frac{1}{(x_1+x_2+y)^{\lambda s}} f(x_1,x_2)dx_1dx_2, \quad y \in \mathbb{R}_+, \tag{6.3}$$

where $\widetilde{\psi}_{\mathbf{s}}(y) = y^{q\left(\frac{1}{q'} - \frac{\lambda s}{r_2}\right)}$. Having in mind the discussion at the end of Section 3, we conclude that the constant $B^\lambda\left(\frac{s}{r_1}, \frac{s}{r_2}\right)$ represents the upper bound for the norm of operator $\mathcal{L}_s$. On the other hand, the defined parameters $A_1$ and $A_2$ satisfy relation (4.1) in the conjugate case. Thus, according to Corollary 4.5 we get that

$$||\mathcal{L}_s|| = B\left(\frac{s}{r_1}, \frac{s}{r_2}\right), \tag{6.4}$$

in the conjugate case, since $\lambda = 1$.

We conclude this paper with an example in which we shall be able to find the norm of a Hilbert-type operator and related Hardy-type operators, defined in Section 5, as well.

**Example 6.2.** Let $m = 2$, $n = 1$, $\alpha = \beta = 1$ and let $s = s_1 + s_2$, where $s > 0$. We consider homogeneous function $k_s : \mathbb{R}^2_+ \to \mathbb{R}$ defined by

$$k_s(u, v) = \frac{[\min\{u, v\}]^\gamma}{[\max\{u, v\}]^{s+\gamma}}, \tag{6.5}$$

where $\gamma > -\min\{s_1, s_2\}$. Hence, in this setting, with parameters $A_1 = \frac{1}{p'}(2 - s_1)$ and $A_2 = \frac{1}{q'}(1 - s_2)$, inequality (3.6) takes the form

$$\left\{ \int_{\mathbb{R}_+} y^{\lambda p s_2 - 1} \left[ \int_{\mathbb{R}^2_+} \frac{[\min\{x_1 + x_2, y\}]^{\lambda\gamma}}{[\max\{x_1 + x_2, y\}]^{\lambda(s+\gamma)}} f(x_1, x_2) dx_1 dx_2 \right]^{q'} dy \right\}^{\frac{1}{q'}}$$
$$\leq \left[ \frac{2\gamma + s}{(\gamma + s_1)(\gamma + s_2)} \right]^\lambda ||f||_{L^p_{\widetilde{\Phi}_{\mathbf{s}}}(\mathbb{R}^2_+)}, \tag{6.6}$$

where $\widetilde{\Phi}_{\mathbf{s}}(x_1, x_2) = (x_1 + x_2)^{p\left(\frac{2}{p'} - \lambda s_1\right)}$. Now, we define the Hilbert-type operator $\mathcal{M}_s : L^p_{\widetilde{\Phi}_{\mathbf{s}}}(\mathbb{R}^2_+) \to L^{q'}_{\widetilde{\Psi}_{\mathbf{s}}^{1-q'}}(\mathbb{R}_+)$, as

$$\left(\mathcal{M}_s f\right)(y) = \int_{\mathbb{R}^2_+} \frac{[\min\{x_1 + x_2, y\}]^{\lambda\gamma}}{[\max\{x_1 + x_2, y\}]^{\lambda(s+\gamma)}} f(x_1, x_2) dx_1 dx_2, \quad y \in \mathbb{R}_+, \tag{6.7}$$

where $\widetilde{\Psi}_{\mathbf{s}}(y) = y^{q\left(\frac{1}{q'} - \lambda s_2\right)}$. As we have already concluded in Section 3, inequality (6.6) provides the upper bound for the norm of operator $\mathcal{M}_s$. Further, since the defined parameters $A_1$ and $A_2$ satisfy relation (4.1) in the conjugate case, we get the norm of $\mathcal{M}_s$ in that setting ($\lambda = 1$):

$$||\mathcal{M}_s|| = \frac{2\gamma + s}{(\gamma + s_1)(\gamma + s_2)}. \tag{6.8}$$

On the other hand, Corollaries 5.1 and 5.2 provide definitions of Hardy-type operators $\hat{\mathcal{M}}_s, \check{\mathcal{M}}_s : L^p_{\widetilde{\Phi}_{\mathbf{s}}}(\mathbb{R}^2_+) \to L^{q'}_{\widetilde{\Psi}_{\mathbf{s}}^{1-q'}}(\mathbb{R}_+)$, deduced from the definition of operator $\mathcal{M}_s$. More precisely, operators $\hat{\mathcal{M}}_s$ and $\check{\mathcal{M}}_s$ are defined by the formula (6.7), where the set of integration $\mathbb{R}_+^2$ is respectively replaced with $[0, y] \times [0, y]$ and $[y, \infty] \times [y, \infty]$. Of course, the same sets of integration are also replaced in inequality (6.6). Finally, by using definitions (5.2), (5.7), and Remark 5.3, we easily get the norms of operators $\hat{\mathcal{M}}_s$ and $\check{\mathcal{M}}_s$ in the conjugate case:

$$||\hat{\mathcal{M}}_s|| = \frac{1}{\gamma + s_1} \quad \text{and} \quad ||\check{\mathcal{M}}_s|| = \frac{1}{\gamma + s_2}. \tag{6.9}$$

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