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NEW RANDOM FIXED POINT RESULTS FOR GENERALIZED ALTERING DISTANCE FUNCTIONS

HEMANT KUMAR NASHINE

Abstract. The aim of this work is to establish new random common fixed points for pair of mappings satisfying generalized weakly contractive conditions in the setting of complete metric spaces.

1. Introduction and preliminaries

During the last fifty years there have been so many exciting developments in the field of random operator theory. Probabilistic functional analysis is an important mathematical discipline because of its applications to probabilistic models in applied problems. Random operator theory is needed for the study of various classes of random equations. The theory of random fixed point theorems was initiated by the Prague school of probabilistic in the 1950s. The interest in this subject enhanced after publication of the survey paper by Bharucha Reid [6]. Random fixed point theory has received much attention in recent years.

The Banach contraction mapping principle is one of the pivotal results of analysis. It is widely considered as the source of metric fixed point theory. Also its significance lies in its vast applicability in a number of branches of mathematics.

Generalization of the aforesaid principle has been a heavily investigated branch of research. In particular, obtaining the existence and uniqueness of fixed points for self-maps on a metric space by altering distances between the points with the use of a control function is an interesting aspect. In this direction, Khan et al. [10] addressed a new category of fixed point problems for a single self-map with the help of a control function that alters distance between two points in a metric space which they called an altering distance function.

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Definition 1.1. (Altering distance function [10]). *A function* $\varphi : [0,\infty) \to [0,\infty)$ *is called an altering distance function if the following properties are satisfied:*

(a) $\varphi(0) = 0$,
(b) φ *is continuous and monotonically non-decreasing.*

Theorem 1.2. *Let* $(\mathcal{X}, d)$ *be a complete metric space, let* φ *be an altering distance function, and let* $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ *be a self-mapping which satisfies the following inequality:*

$$\varphi(d(\mathcal{T}x, \mathcal{T}y)) \leq c\varphi(d(x,y)) \tag{1.1}$$

for all $x, y \in \mathcal{X}$ *and for some* $0 < c < 1$. *Then* $\mathcal{T}$ *has a unique fixed point.*

In fact Khan et al. [10] proved a more general theorem [10, Theorem 2] of which the above result is a corollary. Another generalization of the contraction principle was suggested by Alber and Guerre-Delabriere [2] in Hilbert spaces by introducing the concept of weakly contractive mappings.

Definition 1.3. (weakly contractive mapping). *Let* $\mathcal{X}$ *be a metric space. A mapping* $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ *is called weakly contractive if for each* $x, y \in \mathcal{X}$,

$$d(\mathcal{T}x, \mathcal{T}y) \leq d(x,y) - \varphi(d(x,y)) \tag{1.2}$$

where $\varphi : [0,\infty) \to [0,\infty)$ *is positive on* $(0,\infty)$ *and* $\varphi(0) = 0$.

If one takes $\varphi(t) = kt$ where $0 < k < 1$, then (1.2) reduces to (1.1).

In fact, Alber and Guerre-Delabriere [2] assumed an additional condition on φ which is that $lim_{t\to\infty}\varphi(t) = \infty$. But Rhoades [11] obtained the result noted in following theorem without using this particular assumption.

Theorem 1.4. [11] . *If* $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ *is a weakly contractive mapping, where* $(\mathcal{X}, d)$ *is a complete metric space, then* $\mathcal{T}$ *has a unique fixed point.*

It may be observed that though the function φ has been defined in the same way as the altering distance function, the way it has been used in Theorem 1.4 is completely different from the use of altering distance function.

Definition 1.5. *A self mapping* $\mathcal{T}$ *of a metric space* $(\mathcal{X}, d)$ *is said to be weakly contractive with respect to a self mapping* $\mathcal{S} : \mathcal{X} \to \mathcal{X}$, *if for each* $x, y \in \mathcal{X}$,

$$d(\mathcal{T}x, \mathcal{T}y) \leq d(\mathcal{S}x, \mathcal{S}y) - \psi(d(\mathcal{S}x, \mathcal{S}y)),$$

where $\psi : [0,\infty) \to [0,\infty)$ *is a continuous and nondecreasing function such that* ψ *is positive on* $(0,\infty)$, $\psi(0) = 0$ *and* $lim_{t\to\infty}\psi(t) = \infty$.

Recently, Beg and Abbas [4] proved a generalization of the corresponding theorems of Rhoades [11] for a pair of mappings in which one is weakly contractive with respect to the other. This is further generalized by Azam and Shakeel [3] in convex metric spaces. Combining the generalization of the Banach contraction principle given by Khan et al. [10] and the generalization given by Rhoades [11], Dutta and Choudhury [8] obtained a result which is further extended by Abbas and Khan [1]. Choudhury [7] also proved similar types of results for generalized altering distance functions. Recently, Beg et al. [5] obtained random versions of these results in convex separable complete metric spaces.

In the Section 2 of this paper, a random common fixed point for a weakly contractive condition based on a generalized altering function is derived. A number of fixed point results may be obtained by assuming different forms for the functions φ_1 and φ_2. In particular, fixed point results under various contractive conditions are obtainable from the results of this section discussed under a relaxed condition of commuting of mappings. In the hypothesis, the condition of commuting of mappings is not necessary. These results generalize some recent well known comparable results in the literature. Our results improve and extend the results of Beg et al. [5] for pair of mappings. Thus, we shall prove new results for random mappings, which are extensions of the corresponding results for deterministic mappings of [1, 3, 4, 7, 8, 10, 11].

Throughout this paper, let $(\mathcal{X}, d)$ be a Polish space, i.e., a separable complete metric space and $(\Omega, \mathcal{A})$ be a measurable space. (i.e., Σ is a σ-algebra of subsets of Ω). A function $\xi : \Omega \to \mathcal{X}$ is said to be a Σ-measurable if for any open subset $\mathcal{B}$ of $\mathcal{X}$, $\xi^{-1}(\mathcal{B}) \in \Sigma$. A mapping $\mathcal{S} : \Omega \times \mathcal{X} \to \mathcal{X}$ is said to be a random map if and only if for each fixed $x \in \mathcal{X}$, the mapping $\mathcal{S}(., x) : \Omega \to \mathcal{X}$ is measurable. A random map $\mathcal{S} : \Omega \times \mathcal{X} \to \mathcal{X}$ is continuous if for each $\omega \in \Omega$, the mapping $\mathcal{S}(\omega, .) : \mathcal{X} \to \mathcal{X}$ is continuous. A measurable mapping $\xi : \Omega \to \mathcal{X}$ is a random fixed point of the random map $\mathcal{S} : \Omega \times \mathcal{X} \to \mathcal{X}$ if and only if $\mathcal{S}(\omega, \xi(\omega)) = \xi(\omega)$ for each $\omega \in \Omega$.

Definition 1.6. *A measurable mapping $\xi : \Omega \to \mathcal{K}$, is said to be a common fixed point of random operators $\mathcal{S} : \Omega \times \mathcal{K} \to \mathcal{K}$ and $\mathcal{T} : \Omega \times \mathcal{K} \to \mathcal{K}$ if for each $\omega \in \Omega$,*

$$\xi(\omega) = \mathcal{S}(\omega, \xi(\omega)) = \mathcal{T}(\omega, \xi(\omega)).$$

2. The random common fixed point theorem for a generalized altering distance function

Now we will give a random common fixed point theorem for a pair of maps. For this we need the following definition, which is given in [7].

Definition 2.1. [7]. *A function $\varphi : [0, \infty)^3 \to [0, \infty]$ is said to be a generalized altering distance function if*

(i) *$\varphi(x, y, z)$ is continuous,*
(ii) *φ is monotone increasing in all of the three variables and*

(iii) $\varphi(x, y, z) = 0$ *only if* $x = y = z = 0$.

Define $\psi(x) = \varphi(x, x, x)$ for $x \in [0, \infty)$. Clearly, $\psi(x) = 0$ if and only if $x = 0$. Examples of φ are
$\varphi(a, b, c) = k \max\{a, b, c\}$, for $k > 0$,
$\varphi(a, b, c) = a^p + b^q + c^r$, $p, q, r \geq 1$,
$\varphi(a, b, c) = (a + \alpha b^q)r + c^s$, where $p, q, r, s \geq 1$ and $\alpha > 0$.

Theorem 2.2. *Let $\mathcal{X}$ be a separable metric space and $\mathcal{K}$ be a nonempty Polish subspace of $\mathcal{X}$. Let $\mathcal{T}, \mathcal{S} : \Omega \times \mathcal{K} \to \mathcal{K}$ be continuous maps such that for each $x, y \in \mathcal{K}$*

$$\begin{aligned}\psi_1(d(\mathcal{T}(\omega, x), \mathcal{S}(\omega, y))) &\leq \varphi_1(d(x, y), d(x, \mathcal{T}(\omega, x)), d(y, \mathcal{S}(\omega, y))) \\ &\quad - \varphi_2(d(x, y), d(x, \mathcal{T}(\omega, x)), d(y, \mathcal{S}(\omega, y))) \qquad (2.1)\end{aligned}$$

where $\varphi_i (i = 1, 2)$ are generalized altering distance functions and $\psi_1(x) = \varphi_1(x, x, x)$. Then, there exists a measurable mapping $\xi : \Omega \to \mathcal{K}$ such that

$$\xi(\omega) = \mathcal{T}(\omega, \xi(\omega)) = \mathcal{S}(\omega, \xi(\omega)).$$

Proof. Let $\xi_0 : \Omega \to \mathcal{K}$ be a measurable but fixed mapping on $\mathcal{K}$, we set $\xi_1(\omega) = \mathcal{T}(\omega, \xi_0(\omega))$ and $\xi_2(\omega) = \mathcal{S}(\omega, \xi_1(\omega))$. Similarly we set $\xi_3(\omega) = \mathcal{T}(\omega, \xi_2(\omega))$ and $\xi_4(\omega) = \mathcal{S}(\omega, \xi_3(\omega))$. Inductively, we construct a sequence of measurable maps $\{\xi_n\}$ from Ω to $\mathcal{K}$ such that

$$\xi_{2n+1}(\omega) = \mathcal{T}(\omega, \xi_{2n}(\omega)) \text{ and } \xi_{2n+2}(\omega) = \mathcal{S}(\omega, \xi_{2n+1}(\omega)). \qquad (2.2)$$

Since $\mathcal{S}$ and $\mathcal{T}$ are continuous, by a result of Himmelberg [9], $\{\xi_n\}$ is a measurable sequence.

We complete the proof in four parts.

Part (I). First, we will prove that

$$d(\xi_n(\omega), \xi_{n+1}(\omega)) \leq d(\xi_{n-1}(\omega), \xi_n(\omega)).$$

Consider

$$\begin{aligned}&\psi_1(d(\xi_{2n+1}(\omega)), \xi_{2n+2}(\omega)))) = \psi_1(d(\mathcal{T}(\omega, \xi_{2n}(\omega)), \mathcal{S}(\omega, \xi_{2n+1}(\omega)))) \\ &\leq \varphi_1(d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)), d(\xi_{2n}(\omega), \mathcal{T}(\omega, \xi_{2n}(\omega))), d(\xi_{2n+1}(\omega), \\ &\quad \mathcal{S}(\omega, \xi_{2n+1}(\omega)))) - \varphi_2(d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)), d(\xi_{2n}(\omega), \mathcal{T}(\omega, \xi_{2n}(\omega))), \\ &\quad d(\xi_{2n+1}(\omega), \mathcal{S}(\omega, \xi_{2n+1}(\omega)))) \\ &= \varphi_1(d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)), d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)), d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))) \\ &\quad - \varphi_2(d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)), d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)), d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))) \\ &\leq \varphi_1(d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)), d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))) \\ &\quad - \varphi_2(d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)), d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))). \qquad (2.3)\end{aligned}$$

If $d(\xi_{2n+1}(\omega)),\xi_{2n+2}(\omega)) > d(\xi_{2n}(\omega),\xi_{2n+1}(\omega))$, then

$$\begin{aligned}&\psi_1(d(\xi_{2n+1}(\omega),\xi_{2n+2}(\omega)))\\&\le \varphi_1(d(\xi_{2n+1}(\omega),\xi_{2n+2}(\omega),d(\xi_{2n+1}(\omega)),\xi_{2n+2}(\omega),d(\xi_{2n+1}(\omega),\xi_{2n+2}(\omega)))\\&=\psi(d(\xi_{2n+1}(\omega),\xi_{2n+2}(\omega))),\end{aligned}\tag{2.4}$$

since φ_1 is monotone increasing in all variables and

$$\varphi_2(d(\xi_{2n+1}(\omega)),\xi_{2n+2}(\omega)),d(\xi_{2n+1}(\omega)))\neq 0$$

whenever $d(\xi_{2n+1}(\omega)),\xi_{2n+2}(\omega))\neq 0$, which is a contradiction. So, we have

$$d(\xi_{2n+1}(\omega)),\xi_{2n+2}(\omega))\le d(\xi_{2n}(\omega),\xi_{2n+1}(\omega)),\ \ n\ge 0.\tag{2.5}$$

Putting $x=\xi_{2n}(\omega)$ and $y=\xi_{2n-1}(\omega)$ in (2.1), we have

$$\begin{aligned}&\psi_1(d(\xi_{2n}(\omega)),\xi_{2n+1}(\omega))))\\&\le \varphi_1(d(\xi_{2n-1}(\omega),\xi_{2n}(\omega)),d(\xi_{2n-1}(\omega),\xi_{2n-1}(\omega)),d(\xi_{2n}(\omega),\xi_{2n}(\omega)))\\&\quad-\varphi_2(d(\xi_{2n-1}(\omega),\xi_{2n}(\omega)),d(\xi_{2n-1}(\omega),\xi_{2n-1}(\omega)),d(\xi_{2n}(\omega),\xi_{2n}(\omega)))\end{aligned}\tag{2.6}$$

By a similar argument, we have

$$d(\xi_{2n+2}(\omega),\xi_{2n+3}(\omega))\le d(\xi_{2n+1}(\omega),\xi_{2n+2}(\omega)).\tag{2.7}$$

From (2.5) and (2.7), we obtain

$$d(\xi_{n+1}(\omega),\xi_{n+2}(\omega))\le d(\xi_n(\omega),\xi_{n+1}(\omega)),\ \ \forall n\ge 0.\tag{2.8}$$

Therefore, from (2.3) and (2.8), we have for all $n\ge 0$

$$\begin{aligned}&\psi_1(d(\xi_{n+1}(\omega),\xi_{n+2}(\omega)))\\&\le\psi_1(d(\xi_n(\omega),\xi_{n+1}(\omega)))-\psi_2(d(\xi_n(\omega),\xi_{n+1}(\omega))),\ \text{ where }\ \psi_2=\varphi_2(x,x,x)\end{aligned}$$

or equivalently

$$\psi_2(d(\xi_{n+1}(\omega),\xi_{n+2}(\omega)))\le\psi_1(d(\xi_n(\omega),\xi_{n+1}(\omega)))-\psi_1(d(\xi_n(\omega),\xi_{n+1}(\omega))).$$

Summing up from (2.8), we obtain

$$\sum_{n=0}^{\infty}\psi_2(d(\xi_{n+1}(\omega),\xi_{n+2}(\omega)))\le\psi_1(d(\xi_0(\omega),\xi_1(\omega)))<\infty$$

which implies

$$\psi_2(d(\xi_n(\omega),\xi_{n+1}(\omega)))\to 0\ \ as\ \ n\to\infty.\tag{2.9}$$

Again, from (2.8), the sequence $\{d(\xi_n(\omega),\xi_{n+1}(\omega)))\}$ is monotone non-increasing and bounded. Hence, there exists a real number $r(\omega)\ge 0$ such that,

$$\lim_{n\to\infty}d(\xi_n(\omega),\xi_{n+1}(\omega))=r(\omega).$$

Then by the continuity of ψ, from (2.9), we obtain $\psi_2(r(\omega)) = 0$ which implies that by the property of function ψ, we have $r(\omega) = 0$. Thus

$$\lim_{n\to\infty} d(\xi_n(\omega)), \xi_{n+1}(\omega)) = 0. \tag{2.10}$$

Part (II.) Next, we claim that $\{\xi_n(\omega)\}$ is a Cauchy sequence in $\mathcal{K}$.

If possible, let $\{\xi_n(\omega)\}$ not be a Cauchy sequence. Then there exists $\epsilon > 0$ for which we can find subsequences $\{\xi_{n_i}(\omega)\}$ and $\{\xi_{m_i}(\omega)\}$ of $\{\xi_n(\omega)\}$ with $n_i > m_i > i$ such that

$$d(\xi_{m_i}(\omega), \xi_{n_i}(\omega)) \geq \epsilon. \tag{2.11}$$

Further, we can choose n_i corresponding to m_i in such a way that it is the smallest integer with $n_i > m_i$ satisfying (2.11). Then

$$d(\xi_{m_i}(\omega), \xi_{n_i-1}(\omega)) < \epsilon. \tag{2.12}$$

Using (2.11), (2.12) and the triangle inequality, we have

$$\begin{aligned}\epsilon \leq d(\xi_{m_i}(\omega), \xi_{n_i}(\omega)) &\leq d(\xi_{m_i}(\omega), \xi_{n_i-1}(\omega)) + d(\xi_{n_i-1}, \xi_{n_i}(\omega)) \\ &< \epsilon + d(\xi_{n_i-1}(\omega), \xi_{n_i}(\omega)).\end{aligned} \tag{2.13}$$

Letting $i \to \infty$ and using (2.10),

$$\lim_{i\to\infty} d(\xi_{m_i}(\omega), \xi_{n_i}(\omega)) = \epsilon. \tag{2.14}$$

Again, from the triangle inequality we get

$$\begin{aligned} d(\xi_{m_i}(\omega), \xi_{n_i}(\omega)) &\leq d(\xi_{m_i}(\omega), \xi_{m_i-1}(\omega))) + d(\xi_{m_i-1}(\omega), \xi_{n_i-1}(\omega)) \\ &\quad + d(\xi_{n_i-1}(\omega), \xi_{n_i}(\omega)) \\ d(\xi_{m_i-1}(\omega), \xi_{n_i-1}(\omega)) &\leq d(\xi_{m_i-1}(\omega), \xi_{m_i}(\omega)) + d(\xi_{m_i}(\omega), \xi_{n_i}(\omega)) \\ &\quad + d(\xi_{n_i}(\omega), \xi_{n_i-1}(\omega)). \end{aligned} \tag{2.15}$$

Letting $i \to \infty$ and using inequalities (2.10) and (2.13), we have

$$\lim_{i\to\infty} d(\xi_{m_i-1}(\omega), \xi_{n_i-1}(\omega)) = \epsilon. \tag{2.16}$$

Setting $x = \xi_{m_i}(\omega)$ and $y = \xi_{n_i}(\omega)$ in (2.1), we obtain

$$\begin{aligned} &\psi_1(d(\xi_{m_i-1}(\omega)), \xi_{n_i-1}(\omega))) = \psi_1(d(\mathcal{T}(\omega, \xi_{m_i}(\omega)), \mathcal{S}(\omega, \xi_{n_i}(\omega)))) \\ &\leq \varphi_1(d(\xi_{m_i}(\omega), \xi_{n_i}(\omega)), d(\xi_{m_i}(\omega), \mathcal{T}(\omega, \xi_{m_i}(\omega))), d(\xi_{n_i}(\omega), \mathcal{S}(\omega, \xi_{n_i}(\omega)))) \\ &- \varphi_2(d(\xi_{m_i}(\omega), \xi_{n_i}(\omega)), d(\xi_{m_i}(\omega), \mathcal{T}(\omega, \xi_{m_i}(\omega))), d(\xi_{n_i}(\omega), \mathcal{S}(\omega, \xi_{n_i}(\omega)))). \end{aligned} \tag{2.17}$$

Letting $i \to \infty$ in (2.17) and using inequalities (2.2), (2.11) and (2.12), we have

$$\begin{aligned}\psi_1(\epsilon) &\leq \lim_{n\to\infty}[\varphi_1(d(\mathcal{T}(\omega,\xi_{m_i}(\omega)),\mathcal{S}(\omega,\xi_{n_i}(\omega))))]\\ &\leq \lim_{n\to\infty}[\varphi_1(d(\xi_{m_i}(\omega),\xi_{n_i}(\omega)),d(\xi_{m_i}(\omega),\mathcal{T}(\omega,\xi_{m_i}(\omega))),d(\xi_{n_i}(\omega),\mathcal{S}(\omega,\xi_{n_i}(\omega))))\\ &\quad -\varphi(d(\xi_{m_i}(\omega),\xi_{n_i}(\omega)),d(\xi_{m_i}(\omega),\mathcal{T}(\omega,\xi_{m_i}(\omega))),d(\xi_{n_i}(\omega),\mathcal{S}(\omega,\xi_{n_i}(\omega)))]\\ &= \lim_{n\to\infty}[\varphi_1(d(\xi_{m_i}(\omega),\xi_{n_i}(\omega)),d(\xi_{m_i}(\omega),\xi_{m_i+1}(\omega))),d(\xi_{n_i}(\omega),\xi_{n_i+1}(\omega)))]\\ &\quad -\varphi_2(d(\xi_{m_i}(\omega),\xi_{n_i}(\omega)),d(\xi_{m_i}(\omega),\xi_{m_i+1}(\omega)),d(\xi_{n_i}(\omega),\xi_{n_i+1}(\omega))).\end{aligned} \tag{2.18}$$

Using inequalities (2.10), (2.11), (2.12) and (2.14), we have

$$\psi(\epsilon) \leq \varphi_1(\epsilon,0,0) - \varphi_2(\epsilon,0,0) < \varphi_1(\epsilon),$$

since φ_1 is monotone increasing in its variables and by the property of φ_2 that $\varphi(x,y,z) = 0$ if and only if $x = y = z = 0$. Thus, we arrive at a contradiction, as $\epsilon > 0$. Hence $\{\xi_n(\omega)\}$ is a Cauchy sequence in $\mathcal{K}$.

Part (III). Thirdly, we find a random common fixed point of $\mathcal{T}$ and $\mathcal{S}$.

Since $\{\xi_n(\omega)\}$ is a Cauchy sequence in the complete metric space $\mathcal{K}$, there exists $\xi : \Omega \to \mathcal{K}$ such that $\xi_n(\omega) \to \xi(\omega)$. Moreover, $\xi_{n+1}(\omega) \to \xi(\omega)$. We show that $\xi(\omega)$ is a random common point of $\mathcal{T}$ and $\mathcal{S}$. Consider,

$$\begin{aligned}&\psi_1(d(\mathcal{T}(\omega,\xi(\omega)),\xi_{n+1}(\omega))) = \psi_1(d(\mathcal{T}(\omega,\xi(\omega)),\mathcal{S}(\omega,\xi_n(\omega))))\\ &\leq \varphi_1(d(\xi(\omega),\xi_n(\omega)),d(\mathcal{T}(\omega,\xi_n(\omega)),\xi_n(\omega)),d(\mathcal{T}(\omega,\xi(\omega)),\xi(\omega)))\\ &\qquad - \varphi_2(d(\xi(\omega),\xi_n(\omega)),d(\mathcal{T}(\omega,\xi_n(\omega)),\xi_n(\omega)),d(\mathcal{T}(\omega,\xi(\omega)),\xi(\omega))).\end{aligned}$$

Letting limit $n \to \infty$ in the above inequality, we have

$$\begin{aligned}&\psi(d(\mathcal{T}(\omega,\xi(\omega)),\xi(\omega)))\\ &\leq \lim_{i\to\infty}[\varphi_1(d(\xi(\omega),\xi_n(\omega)),d(\mathcal{T}(\omega,\xi_n(\omega)),\xi_n(\omega)),d(\mathcal{T}(\omega,\xi(\omega)),\xi(\omega)))\\ &\qquad - \varphi_2(d(\xi(\omega),\xi_n(\omega)),d(\mathcal{T}(\omega,\xi_n(\omega)),\xi_n(\omega)),d(\mathcal{T}(\omega,\xi(\omega)),\xi(\omega)))]\\ &= \varphi_1(0,0,d(\mathcal{T}(\omega,\xi(\omega)),\xi(\omega))) - \varphi_2(0,0,d(\mathcal{T}(\omega,\xi(\omega)),\xi(\omega))).\end{aligned}$$

If $d(\mathcal{T}(\omega,\xi(\omega)),\xi(\omega)) \neq 0$, then using the property of monotone increasing of φ_1 and φ_2, and that $\varphi_2(x,y,z) = 0$ if and only if $x = y = z$, we get

$$\psi_1(d(\mathcal{T}(\omega,\xi(\omega)),\xi(\omega))) \leq \varphi_1(d(\mathcal{T}(\omega,\xi(\omega)),\xi(\omega))),$$

a contradiction. Hence, $d(\mathcal{T}(\omega,\xi(\omega)),\xi(\omega)) = 0$ or $\mathcal{T}(\omega,\xi(\omega)) = \xi(\omega)$. Now, using a similar argument as above, we conclude that $d(\xi(\omega),\mathcal{S}(\omega,\xi(\omega))) = 0$ or $\xi(\omega) = \mathcal{S}(\omega,\xi(\omega))$.
Hence $\xi(\omega)$ is a random common fixed point of $\mathcal{T}$ and $\mathcal{S}$.

Part (IV). Finally, we prove the uniqueness of the random common fixed point.

Let $\xi(\omega)$ and $\zeta(\omega)$ be two fixed points of $\mathcal{T}$ and $\mathcal{S}$, i.e, $\mathcal{S}(\omega, \xi(\omega)) = \xi(\omega) = \mathcal{T}(\omega, \xi(\omega))$ and $\mathcal{T}(\omega, \zeta(\omega)) = \zeta(\omega) = \mathcal{S}(\omega, \zeta(\omega))$. Using inequality (2.1), we have

$$\begin{aligned}
&\psi(d(\xi(\omega), \zeta(\omega))) = \psi(d(\mathcal{T}(\omega, \xi(\omega)), \mathcal{T}(\omega, \zeta(\omega))))\\
&\leq \varphi_1(d(\xi(\omega), \zeta(\omega)), d(\xi(\omega), \mathcal{T}(\omega, \xi(\omega))), d(\zeta(\omega), \mathcal{S}(\omega, \zeta(\omega))))\\
&\qquad - \varphi_2(d(\xi(\omega), \zeta(\omega)), d(\xi(\omega), \mathcal{T}(\omega, \xi(\omega))), d(\zeta(\omega), \mathcal{S}(\omega, \zeta(\omega))))\\
&= \varphi_1(d(\xi(\omega), \zeta(\omega)), 0, 0) - \varphi_2(d(\xi(\omega), \zeta(\omega)), 0, 0) < \varphi_1(d(\xi(\omega), \zeta(\omega)))
\end{aligned}$$

which is possible only when $\xi(\omega) = \zeta(\omega)$, since φ_1 is monotone increasing in all its variables and $\varphi(x, y, z) \leq 0$ if at least one of x, y, z is non-zero. Hence $\xi(\omega)$ is the unique random common fixed point of $\mathcal{T}$ and $\mathcal{S}$, i.e.,

$$\mathcal{S}(\omega, \xi(\omega)) = \xi(\omega) = \mathcal{T}(\omega, \xi(\omega)).$$

□

Remark 2.3. (i) Theorem 2.2 is a random version improvement of Theorem 1 [7].

(ii) Theorem 2.2 presents a random version improvement, extension and generalization of Abbas and Khan [1], Dutta and Choudhury [8, Theorem 2.1] and Rhoades [11] by considering generalized altering distance function.

(iii) Theorem 2.2 is generalization of Theorem 2.1 [5] for two mappings using considering a generalized altering distance function.

If in Theorem 2.2, $\mathcal{S} = \mathcal{T}$, then we have the following result as a particular case.

Theorem 2.4. *Let $\mathcal{X}$ be a separable metric space and $\mathcal{K}$ be a nonempty Polish subspace of $\mathcal{X}$, with $\mathcal{T} : \Omega \times \mathcal{K} \to \mathcal{K}$ and suppose that for each $x, y \in \mathcal{K}$*

$$\begin{aligned}
\psi_1(d(\mathcal{T}(\omega, x), \mathcal{T}(\omega, y))) \leq\ & \varphi_1(d(x, y), d(x, \mathcal{T}(\omega, x)), d(y, \mathcal{T}(\omega, y)))\\
&- \varphi_2(d(x, y), d(x, \mathcal{T}(\omega, x)), d(y, \mathcal{T}(\omega, y)))
\end{aligned}$$

where $\varphi_i (i = 1, 2)$ are generalized altering distance functions and $\psi_1(x) = \varphi_1(x, x, x)$. Then, there exists a measurable mapping $\xi : \Omega \to \mathcal{K}$ such that

$$\xi(\omega) = \mathcal{T}(\omega, \xi(\omega)).$$

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Department of Mathematics
Disha Institute of Management and Technology
Satya Vihar, Vidhansabha-Chandrakhuri Marg
Mandir Hasaud, Raipur-492101(Chhattisgarh)
India
E–mail: hemantnashine@rediffmail.com