SARAJEVO JOURNAL OF MATHEMATICS
Vol.7 (20) (2011), 265–275

THE CYCLIC QUADRANGLE IN THE ISOTROPIC PLANE

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ABSTRACT. In [15], [2] we focused on the geometry of the non-tangential quadrilateral and in [3], [14] we turned our attention to the non-cyclic quadrangle in the isotropic plane. This paper gives some of the results concerning the geometry of a cyclic quadrangle in the isotropic plane. A cyclic quadrangle is called standard if a circle with the equation $y = x^2$ is circumscribed to it. In order to prove geometric facts for each cyclic quadrangle, it is sufficient to give a proof for the standard quadrangle. Diagonal points and the diagonal triangle of the cyclic quadrangle are introduced. Some properties of the cyclic quadrangle are given where most of them are related to its diagonal triangle.

1. PRELIMINARIES

The isotropic plane is a real affine plane where the metric is introduced by an absolute figure (ω, Ω) where ω is a line at infinity and Ω is a point incident with it. If $T = (x_0, x_1, x_2)$ denotes any point in the plane presented in homogeneous coordinates then usually a projective coordinate system where $\Omega = (0, 1, 0)$ and the line ω with the equation $x_2 = 0$ is chosen. The line ω is said to be the *absolute line* and the point Ω the *absolute point*. *Isotropic points* are the points incident with the absolute line ω and the *isotropic lines* are the lines passing through the absolute point Ω. Two lines are *parallel* if they have the same isotropic point, and two points are *parallel* if they are incident with the same isotropic line. The classification of conics and the definitions of the notions concerning them can be reached in [4] and [12]. Onwards in the paper we will often consider the circle in the isotropic plane. The conic in the isotropic plane is a *circle* if the absolute line ω is tangent to it at the absolute point Ω.

2000 *Mathematics Subject Classification.* 51N25.
Key words and phrases. Isotropic plane, cyclic quadrangle, diagonal triangle, diagonal points.

By taking $x = \dfrac{x_0}{x_2}$ and $y = \dfrac{x_1}{x_2}$, all projective transformations that preserve the absolute figure form a 5-parametric group G_5

$$\begin{aligned} \overline{x} &= a + px & a,b,c,p,q \in \mathbb{R} \\ \overline{y} &= b + cx + qy, & pq \neq 0, \end{aligned}$$

known as the *group of similarities* of the isotropic plane. The isotropic plane is represented in the Euclidean model. Let us recall the definitions of the basic metric quantities in the isotropic plane.

For two non parallel points $T_1 = (x_1, y_1)$ and $T_2 = (x_2, y_2)$ a *distance* between these two points is defined as $d(T_1, T_2) := x_2 - x_1$. On the other hand, in the case of parallel points, e. g. $T_1 = (x, y_1)$ and $T_2 = (x, y_2)$, $s(T_1, T_2) := y_2 - y_1$ defines the *span* between the points T_1, T_2. It is obvious that the distance and the span are directed quantities. Two non isotropic lines p_1 and p_2 in the isotropic plane can be given with $y = k_i x + l_i, \quad k_i, l_i \in \mathbb{R}, i = 1, 2$; and represented by $p_i = (k_i, l_i), \quad i = 1, 2$ in line coordinates. Hence, an *angle* formed by these two lines is defined by $\varphi = \angle(p_1, p_2) := k_2 - k_1$. The angle of two non isotropic lines is directed as well. Any two points $T_1 = (x_1, y_1)$ and $T_2 = (x_2, y_2)$ have the midpoint $M = (\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2))$ and any two lines with the equations $y = k_i x + l_i \quad (i = 1, 2)$ have the bisector with the equation $y = \frac{1}{2}(k_1 + k_2)x + \frac{1}{2}(l_1 + l_2)$.

All above mentioned quantities are kept invariant under the subgroup G_3 of G_5 being of the form

$$\begin{aligned} \overline{x} &= a + x \\ \overline{y} &= b + cx + y, \end{aligned} \quad a, b, c \in \mathbb{R}.$$

G_3 is called the *motion group* of the isotropic plane. It consists of translations and rotations. In the Euclidean model of the isotropic plane, rotation is understood as shear along the y- axis.

All the notions related to the geometry of the isotropic plane can be found in [12] and [13].

As the principle of duality is valid in the projective plane, it is preserved in the isotropic plane as well. This very duality is significant for further research in the isotropic plane.

2. The cyclic quadrangle in the isotropic plane

In [8] and [1] one can find an elegant method of standardization of an allowable triangle in the isotropic plane. This standardization was used to present properties concerning every allowable triangle. A triangle is called *allowable* if none of its sides is isotropic. The same authors have been studying the geometry of a triangle in a few more papers, [9] and [16], as well as M. Spirova and H. Martini in [10] and [11]. The geometry of a quadrilateral and a quadrangle has naturally arisen in a very similar way to the geometry of a triangle in [8] and [1]. Therefore, in [15], [2] and [3], [14] we have been studying the properties of non tangential quadrilaterals and non cyclic quadrangles in the isotropic plane, respectively. In this paper we will discuss the geometry of the cyclic quadrangle.

Referring to [12] and [4], the circle in the isotropic plane can be written in the form

$$y = ux^2 + vx + w, \quad u \neq 0, u, v, w \in \mathbb{R}. \tag{1}$$

Without loss of generality, applying the substitution $x \to x, y \to w+vx+uy$, the equation (1) turns into

$$y = x^2. \tag{2}$$

In this sense, we will study the cyclic quadrangle $ABCD$ having the circle (2) as the circumscribed one. Therefore, we can choose A, B, C, D in the form

$$A = (a, a^2), B = (b, b^2), C = (c, c^2), D = (d, d^2), \tag{3}$$

a, b, c, d being mutually different real numbers.

For further investigation, the following abbreviations will be useful:

$$\begin{aligned} s &= a + b + c + d, \\ q &= ab + ac + ad + bc + bd + cd, \\ r &= abc + abd + acd + bcd, \\ p &= abcd. \end{aligned} \tag{4}$$

Choosing the y-axis to coincide with the diameter of the circle (2), passing through the centroid of the quadrangle $ABCD$, and the x-axis as a tangent of the circle (2) at its point parallel to the centroid,

$$s = a + b + c + d = 0 \tag{5}$$

follows.

We will call the direction of the x-axis the *main direction of our cyclic quadrangle.*

As (5) is valid, $a^2 + b^2 + c^2 + d^2 = -2q$ can be easily achieved. Hence, $q < 0$ and the centroid is of the form

$$G = (0, -\frac{q}{2}). \tag{6}$$

Therefore, we have:

Lemma 1. *For any cyclic quadrangle $ABCD$ there exist four distinct real numbers a, b, c, d such that, in the defined canonical affine coordinate system,* *the vertices have the form* (3)*, the circumscribed circle has the equation* (2) *and the sides are given by*

$$\begin{array}{ll} AB \ldots y = (a+b)x - ab, & AD \ldots y = (a+d)x - ad, \\ BC \ldots y = (b+c)x - bc, & AC \ldots y = (a+c)x - ac, \\ CD \ldots y = (c+d)x - cd, & BD \ldots y = (b+d)x - bd. \end{array} \tag{7}$$

Tangents $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ *of the circle* (2) *at the points* (3) *are of the form*

$$\begin{array}{ll} \mathcal{A} \ldots y = 2ax - a^2, & \mathcal{B} \ldots y = 2bx - b^2, \\ \mathcal{C} \ldots y = 2cx - c^2, & \mathcal{D} \ldots y = 2dx - d^2. \end{array} \tag{8}$$

This lemma is easily proved.

For the cyclic quadrangle $ABCD$ from Lemma 1 it is said to be in a *standard position* or it is a *standard quadrangle.* Due to Lemma 1 every cyclic quadrangle can be represented in the standard position. In order to prove geometric facts for each cyclic quadrangle, it is sufficient to give a proof for the standard quadrangle.

Let us now introduce the notion of a *diagonal triangle* of the cyclic quadrangle.

The vertices of the diagonal triangle are the points $U = AC \cap BD$, $V = AB \cap CD$ and $W = AD \cap BC$, the intersection points of the opposite sides of the quadrangle. These vertices will be called *diagonal points.* Mutually joint diagonal points form the sides of the diagonal triangle.

In the further investigation we will study only those cyclic quadrangles with allowable diagonal triangles. Any such cyclic quadrangle we will call an *allowable cyclic quadrangle.* Triangles that are not allowable, due to the isotropic direction of one of its sides, can not be studied in the isotropic plane (see [12]).

It is easy to prove that the forms of the diagonal points and the sides of the diagonal triangle are given in the lemma that follows.

Lemma 2. *The diagonal points* U, V, W *of the allowable cyclic quadrangle* $ABCD$ *are of the form*

$$\begin{aligned} U &= \left(\frac{ac-bd}{a+c-b-d}, \frac{ac(b+d)-bd(a+c)}{a+c-b-d}\right), \\ V &= \left(\frac{ab-cd}{a+b-c-d}, \frac{ab(c+d)-bd(a+c)}{a+b-c-d}\right), \\ W &= \left(\frac{ad-bc}{a+d-b-c}, \frac{ad(b+c)-bc(a+d)}{a+d-b-c}\right), \end{aligned} \tag{9}$$

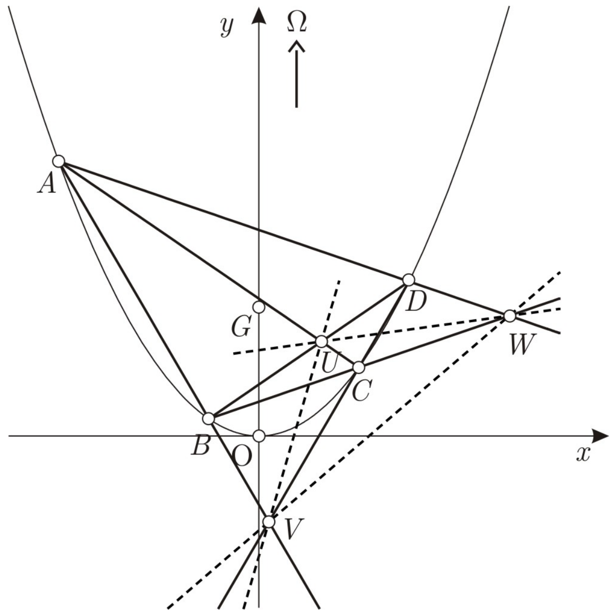

FIGURE 1. The allowable cyclic quadrangle

and the sides of the diagonal triangle are given by

$$UV \ldots y = \frac{2(ad-bc)}{a+d-b-c}x - \frac{ad(b+c)-bc(a+d)}{a+d-b-c},$$

$$UW \ldots y = \frac{2(ab-cd)}{a+b-c-d}x - \frac{ab(c+d)-cd(a+b)}{a+b-c-d}, \tag{10}$$

$$VW \ldots y = \frac{2(ac-bd)}{a+c-b-d}x - \frac{ac(b+d)-bd(a+c)}{a+c-b-d},$$

where $a+c-b-d \neq 0,\ a+b-c-d \neq 0,\ a+d-b-c \neq 0.$

Note 1. Conditions $a+c-b-d \neq 0,\ a+b-c-d \neq 0,\ a+d-b-c \neq 0$ are the conditions for the cyclic quadrangle $ABCD$ to be allowable.

The allowable cyclic quadrangle $ABCD$ with its diagonal triangle UVW is shown in Figure 1.

3. On the properties of the cyclic quadrangle

In this section we focus on the properties of the cyclic quadrangle, where most of them are related to its diagonal triangle.

First, let us recall the meaning of the isogonal lines [[4], p. 6]: lines through the vertex of an angle which are symmetric with respect to the bisector of that angle, are called *isogonal lines* with respect to the sides of the angle.

Theorem 1. *Let $ABCD$ be an allowable cyclic quadrangle. Choosing the pair of opposite sides AC, BD the point A' represents the point of intersection of the side BD and the isogonal line to AC with respect to the sides AB and AD. The points B', C' and D' are defined in an analogous way. The points A', B', C' and D' have the same centroid as the points A, B, C, D.*

For the Euclidean version of the theorem see [5].

Proof. The line $\mathcal{A}'$ with the equation

$$y = (a + b - c + d)x - a(b - c + d)$$

is the line isogonal to AC with respect to the sides AD and AB.

Namely, out of

$$a^2 = (a + b - c + d)a - a(b - c + d)$$

it follows that A is incident with the line $\mathcal{A}'$, while

$$(a + c) + (a + b - c + d) = (a + b) + (a + d)$$

proves that the lines $\mathcal{A}'$ and AC are isogonal with respect to the sides AB and AD. It is easy to prove that the point $A' = (a', a'')$ with the coordinates

$$a' = \frac{a(b + d) - ac - bd}{a - c},$$
$$a'' = \frac{a(b + d)^2 - ac(b + d) - bd(b + d) - bd(a - c)}{a - c}$$

is incident with the line $\mathcal{A}'$ and with the line BD in (7). Analogously,

$$c' = \frac{c(b + d) - ac - bd}{c - a},$$
$$c'' = \frac{c(b + d)^2 - ac(b + d) - bd(b + d) - bd(c - a)}{c - a}$$

stands for the coordinates of the point $C' = (c', c'')$.

As $a' + c' = b + d$ and $a'' + c'' = b^2 + d^2$ are valid, then the pairs of points A', C' and B, D have a common midpoint. The same statement is valid for the pairs of points B', D' and A, C as well, and the claim of the theorem is proved. □

Analogous claims are valid for the pairs of opposite sides AD, BC and AB, CD.

Corollary 1. *Under the assumptions of Theorem 1, the pairs of points A, C and B', D', and the pairs of points B, D and A', C', are each incident with the same lines and they have common midpoints.*

The following theorem gives a simple but nice result concerning the diagonal points of the cyclic quadrangle. The Euclidean case of the theorem is very well known (see e.g. [6]).

Theorem 2. *The angle bisectors of three pairs of opposite sides of the allowable cyclic quadrangle $ABCD$ are parallel to the main direction of the quadrangle.*

Proof. The opposite sides of the standard quadrangle $ABCD$ are the pairs of lines AB, CD; AD, BC, and AC, BD. By taking for instance AB, CD and summing up their equations given in (7) the equation of its bisector is achieved:

$$y = -\frac{ab + cd}{2}.$$

□

In the sequel we discuss the properties of the diagonal points of the allowable cyclic quadrangle $ABCD$ with respect to the bisectors of its sides.

Theorem 3. *S_{AD}, the point of intersection of the bisector of the sides BA, AD, and the bisector of AD, DC, and S_{BC}, the point of intersection of the bisector of the sides AB, BC and the bisector of BC, CD, are points parallel to the diagonal point V (the point of intersection of the opposite sides AB, CD). Analogously, the points S_{AB} and S_{CD} are parallel to the diagonal point W, and the points S_{AC} and S_{BD} are parallel to the diagonal point U.*

Proof. From the proof of the previous theorem, the bisectors of the sides AB, BC and BC, CD are

$$y = \frac{a + 2b + c}{2}x - \frac{ab + bc}{2},$$

$$y = \frac{b + 2c + d}{2}x - \frac{bc + cd}{2}.$$

They intersect at the point S_{BC} with the abscissa $x = \dfrac{ab - cd}{a + b - c - d}$ as it is proved by obtaining the equality

$$(a + 2b + c)(ab - cd) - (ab + bc)(a + b - c - d) = (b + 2c + d)(ab - cd)$$
$$-(bc + cd)(a + b - c - d).$$

Because of the symmetry on the pairs of numbers, a, b and c, d, the abscissa of S_{AD} is

$$x = \frac{ab - cd}{a + b - c - d}.$$

Comparing the obtained abscissae with the abscissa of the diagonal point V, from (9), we have proved the claim of the theorem. □

Theorem 4. *Let $ABCD$ be an allowable cyclic quadrangle. The lines AD and BC meet at W, the lines AB and CD meet at V, the tangents $\mathcal{A}$ and $\mathcal{C}$ at the points A and C of the circumscribed circle of the quadrangle intersect in the point G, and the tangents $\mathcal{B}$ and $\mathcal{D}$ at the points B and D intersect in the point H. Then the circles circumscribed to the triangles ABW, CDW, BCV, ADV, ACG, and BDH have a common point on the line $WVGH$.*

For the Euclidean version of the theorem see [7].

Proof. Let $\mathcal{U}$ be the line with the equation

$$y = \frac{2(ac - bd)}{a + c - b - d}x - \frac{ac(b + d) - bd(a + c)}{a + c - b - d}.$$

A short calculation yields that the points $G = (\frac{a+c}{2}, ac)$, V, $H = (\frac{b+d}{2}, bd)$ and W are incident with the line $\mathcal{U}$.

The circle circumscribed to the triangle ACG is given by

$$y = 2x^2 - (a + c)x + ac. \tag{11}$$

The upper equation can be easily verified when inserting the coordinates of A, C and G.

Because of the symmetry on a, c and on b, d we conclude that the same circle for the triangle BDH is of the form

$$y = 2x^2 - (b + d)x + bd. \tag{12}$$

It can be easily proved that the circles (11) and (12) meet at the point

$$K = (\frac{ac - bd}{a + c - b - d}, bd - \frac{(ac - bd)(b + d)}{a - b + c - d} + \frac{2(ac - bd)^2}{(a - b + c - d)^2}).$$

Moreover, in exactly the same manner as before, we obtain

$$(a - b)y = (a + d - b - c)x^2 + (a + b)(c - d)x - ab(c - d), \tag{13}$$

the equation of the circumscribed circle of the triangle ABW.

Indeed, inserting the coordinates of A and W in (13), one gets two valid equalities.

Onwards, because of

$$bd - \frac{(ac-bd)(b+d)}{a+c-b-d} + \frac{2(ac-bd)^2}{(a-b+c-d)^2} =$$

$$= \frac{(a+b)(c-d)(ac-bd)}{(a-b)(a-b+c-d)} - \frac{(-a+b+c-d)(ac-bd)^2}{(a-b)(a-b+c-d)^2} - \frac{ab(c-d)}{a-b},$$

and

$$bd - \frac{(ac-bd)(b+d)}{a+c-b-d} + \frac{2(ac-bd)^2}{(a+c-b-d)^2}$$

$$= \frac{2(ac-bd)^2}{(a+c-b-d)^2} - \frac{ac(b+d)-bd(a+c)}{a-b+c-d},$$

the point K is incident with (13) and the line $\mathcal{U}$, respectively. □

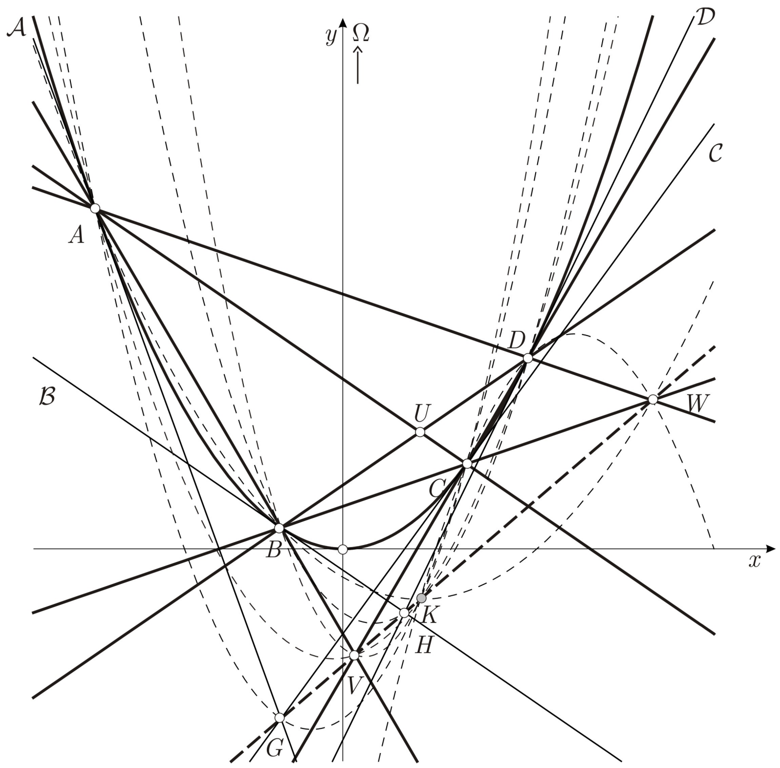

FIGURE 2. The visualization of Theorem 4

There is another interesting observation concerning the point K from the proof of the previous theorem: K is parallel to the third diagonal point $U = AC \cap BD$ from (9).

The visualization of Theorem 4 is given in Figure 2.

Acknowledgement. The authors are grateful to the referees for their valuable suggestions.

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(Received: September 16, 2010)
(Revised: February 1, 2011)

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