SARAJEVO JOURNAL OF MATHEMATICS
Vol.7 (20) (2011), 277–287

ON REAL COHOMOLOGY GENERATORS OF COMPACT HOMOGENEOUS SPACES

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To Professor Veselin Perić with deep esteem

ABSTRACT. In this paper we discuss the degrees of real cohomology generators of compact homogeneous spaces. We relate these degrees to rational homotopy groups and, furthermore, we discuss the formality and geometric formality of compact homogeneous spaces in the light of their cohomology generators. For generalized symmetric spaces the explicit formulas are obtained.

1. INTRODUCTION

A compact homogeneous space is the quotient G/H of a compact connected Lie group G by its closed connected subgroup H. In this paper we study the real cohomology generators of such spaces.

Compact homogeneous spaces represent an important class of manifolds and plays significant role in many branches of mathematics and physics such as theory of characteristic classes, representation theory, string topology and quantum physics. The problem of computation of real cohomology algebras of these spaces was theoretically solved a long time ago [1] by the famous Cartan theorem. This theorem led to the formulas for the cohomology algebras for a wide class of homogeneous spaces [1], [14], [16]. It is of interest from the point of view of both geometry and topology to further investigate the cohomology algebras of these spaces in the sense of determining the degrees of their generators.

In this paper we obtain explicit formulas for the degrees of real cohomology generators for all generalized symmetric spaces. These are the spaces that generalize the notion of symmetric spaces by omitting the involutivity condition. We appeal to rational homotopy theory and results on rational homotopy groups of these spaces obtained in [17]. We also discuss the formality and geometric formality question of compact homogeneous spaces in terms of their cohomology generators.

The paper is organized as follows. In Section 2 we provide some general facts on real cohomology of compact connected Lie groups and classifying spaces, as

2000 *Mathematics Subject Classification.* 53C25, 57R57, 58A14, 57R17.

well as on Cartan algebras on real cohomology of compact homogeneous spaces. We directly deduce from this the statement on degrees of some real cohomology generators of Cartan pair of homogeneous spaces. In Section 3 we relate the degrees of real cohomology generators of compact homogeneous spaces to the ranks of their homotopy groups. Then in Section 4 we give explicit formulas for the degrees of real cohomology generators of generalized symmetric spaces. In Section 5 we deduce the formality criterion for compact homogeneous spaces in terms of their cohomology generators and in that light we discuss the geometric formality issue.

2. On real cohomology of compact homogeneous spaces

2.1. **Some general facts.** Let G be a compact connected Lie group, $\mathfrak{g}$ its Lie algebra and BG its classifying space. It is classical Hopf theorem [1] that the real cohomology algebra of G is an exterior algebra:

$$H^*(G,\mathbb{R}) \cong \wedge(z_1,\dots,z_n)\ .$$

Here $z_1,\dots,z_n$ are universal transgressive generators whose degrees are given with $\deg z_i = 2k_i - 1$, where $k_1,\dots,k_n$ are the exponents of the group G (see [2]) and $n = \operatorname{rk} G$ denotes the dimension of the maximal torus in G.

It is also a classical result that the real cohomology algebra of the classifying space BG is the algebra of polynomials on the maximal abelian subalgebra $\mathfrak{t}$ for $\mathfrak{g}$ that are invariant under the action of the Weyl group W_G for G:

$$H^*(BG,\mathbb{R}) \cong \mathbb{R}[\mathfrak{t}]^{W_G}\ .$$

Recall that the Weyl group W_G is defined with $W_G = N_G(T)/T$, where $N_G(T)$ denotes the normalizer of the maximal torus T in G.

For the algebra $H^*(BG,\mathbb{R})$ it is well known to have generating polynomials $P_1,\dots,P_n$, called Weyl invariant generators, i. e.:

$$H^*(BG,\mathbb{R}) \cong \mathbb{R}[P_1,\dots,P_n]\ .$$

The polynomials $P_1,\dots,P_n$ correspond to $z_1,\dots,z_n$ by transgression in the universal bundle for G and their degrees are given by the exponents of the group G meaning that $\deg P_i = 2k_i$, $1 \le i \le n$ (see [1]).

2.2. **Compact homogeneous spaces.** We consider the homogeneous space G/H, where G is a compact connected Lie group and H its connected closed subgroup. Denote by $\mathfrak{s} \subset \mathfrak{t}$ the maximal abelian subalgebra for the Lie algebra $\mathfrak{h}$ for H. Note that $W_H \subset W_G$, so if restrict polynomials from $H^*(BG,\mathbb{R})$ to $\mathfrak{s}$, we obtain polynomials from $H^*(BH,\mathbb{R})$.

To the homogeneous space G/H it can be assigned a differential graded algebra (C,d), called a *Cartan algebra* (see [1],[12],[7]) as follows:

$$C = H^*(BH,\mathbb{R}) \otimes H^*(G,\mathbb{R}),\ \ d(b\otimes 1) = 0,\ \ d(1\otimes z_i) = \rho^*(P_i)\otimes 1\ ,$$

where $\rho^* : \mathbb{R}[\mathfrak{t}]^{W_G} \to \mathbb{R}[\mathfrak{s}]^{W_H}$ denotes the restriction. The restriction ρ^* is determined by the values $\rho^*(P_i)$, $1 \leq i \leq n$ that are polynomials in Weyl invariant generators $Q_1, \ldots, Q_r$ for $H^*(BH, \mathbb{R})$, where $r = \operatorname{rk} H$.

The famous Cartan theorem [1] states that:

Theorem 1.
$$H^*(G/H, \mathbb{R}) \cong H^*(C, d) \ .$$

There is a wide class of homogeneous spaces, called Cartan pair homogeneous spaces [7], which behave nicely from the point of view of real cohomology and therefore of rational homotopy theory. We say that the homogeneous space G/H is a Cartan pair homogeneous space if one can choose n algebraic independent generators $P_1, \ldots, P_n \in \mathbb{R}[\mathfrak{t}]^{W_G}$ such that $\rho^*(P_{r+1}), \ldots, \rho^*(P_n)$ belong to the ideal in $\mathbb{R}[\mathfrak{s}]^{W_H}$ which is generated by $\rho^*(P_1), \ldots, \rho^*(P_r)$. Furthermore, in this case one can choose $P_{r+1}, \ldots, P_n$ such that $\rho^*(P_{r+1}) = \ldots = \rho^*(P_n) = 0$. Then the Cartan theorem directly implies the known formula for cohomology of these spaces:

$$H^*(G/H, \mathbb{R}) \cong \mathbb{R}[\mathfrak{s}]^{W_H} / \left\langle \rho^*(P_1), \ldots, \rho^*(P_r) \right\rangle \otimes \wedge(z_{r+1}, \ldots z_n). \tag{1}$$

This formula has the following direct consequence:

Lemma 1. *Let $k_1, \ldots, k_n$ and $l_1, \ldots, l_r$ be the exponents for G and H respectively, where G is a compact connected Lie group and H its closed connected subgroup such that G/H is a Cartan pair homogeneous space. Then:*

- *if $p \notin \{l_1, \ldots,\ l_r\}$ the cohomology algebra $H^*(G/H, \mathbb{R})$ has no generator of degree $2p$.*
- *If $p \notin \{k_1, \ldots, k_n\}$ the cohomology algebra $H^*(G/H, \mathbb{R})$ has no generator of degree $2p-1$.*
- *If $l_i \notin \{k_1, \ldots, k_n\}$ then the number of generators in the cohomology algebra $H^*(G/H, \mathbb{R})$ of degree $2l_i$ is equal to the multiplicity of the exponent l_i.*

Proof. The first and the second statement immediately follows from (1). To prove the third statement note that some generator from $\mathbb{R}[\mathfrak{s}]^{W_H}$ is eliminated as the generator in $H^*(G/H, \mathbb{R})$ if and only if it appears as a linear part in $\rho^*(P_i)$ for some $1 \leq i \leq n$. Therefore the degree of the eliminated element has to be equal $2k_i$ for some exponent k_i of the group G. $\square$

Remark 1. The class of Cartan pair homogeneous spaces comprises, among the others, homogeneous spaces with a positive Euler characteristic, compact symmetric spaces, generalized symmetric spaces as well as homogeneous spaces having free cohomology algebra.

Remark 2. For homogeneous spaces G/H with a positive Euler characteristic, meaning that $\operatorname{rk} H = \operatorname{rk} G$, it is well known [1] that $H^*(G/H, \mathbb{R})$ has no generators of odd degree.

Example 1. Let us take $G = Sp(n)$ and $H = S(U(1) \times U(1) \times U(n-1))$. Then H is the subgroup of G, the Lie algebra for H is $\eta = \mathfrak{t}^2 \oplus A_{n-2}$ and the Lie algebra for G is $\mathfrak{g} = C_n$. The exponents for η are $1, 1, 2, 3, \ldots, n-1$ and the exponents for $\mathfrak{g}$ are $2, 4, \ldots, 2n$. It follows by Corollary 1 that the cohomology algebra $H^*(Sp(n)/U(1)\times U(1)\times SU(n-1), \mathbb{R})$ may have even degree generators only in degrees $2, 4, \ldots, 2(n-1)$. Furthermore, it has two generators of degree 2, while it has one generator of degrees $2p$ for odd p, $3 \leq p \leq n-1$.

Example 2. Lemma 1 is generally not true for non-Cartan pair homogeneous spaces. Namely, we can consider the space $M = SU(6)/(SU(3) \times SU(3))$, which according to [7] is not a Cartan pair as its Cartan algebra can be reduced to $C = \mathbb{R}[x_4, x_6] \otimes \wedge(z_7, z_9, z_{11})$ with the differentials $d(x_4) = d(x_6) = 0$ $d(z_7) = x_4^2$, $d(z_9) = x_4 x_6$ and $d(z_{11}) = x_6^2$, where $\deg x_i = i$ and $\deg z_j = j$. Since $d(x_6 z_7 - x_4 z_9) = 0$ we have that $x_6 z_7 - x_4 z_9$ is a cohomology generator for this space and its degree is 13, while the exponents for $SU(6)$ are $2, 3, 4, 5, 6$.

3. Cohomology generators and rational homotopy groups

3.1. **On a minimal model.** We refer to [6] as the comprehensive reference for minimal model theory.

Let (A, d_A) be a commutative graded differential algebra over real numbers. A differential graded algebra (μ_A, d) is called a *minimal model* for (A, d_A) if

- there exists a differential graded algebra morphism $h_A : (\mu_A, d) \to (A, d_A)$ inducing an isomorphism in their cohomology algebras (such h_A is called a quasi-isomorphism);
- (μ_A, d) is a free algebra in the sense that $\mu_A = \wedge V$ is an exterior algebra over a graded vector space V;
- differential d is indecomposable meaning that for a fixed set $V = \{P_\alpha, \alpha \in I\}$ of free generators of μ_A for any $P_\alpha \in V$, $d(P_\alpha)$ is a polynomial in generators P_β with no linear part.

Two algebras are said to be *weakly equivalent* or quasi-isomorphic if there exists quasi-isomorphism between them. Note that, by the definition of the minimal model, weakly equivalent algebras have isomorphic minimal models.

An algebra (A, d_A) is said to be *formal* if it is weakly equivalent to the algebra $(H^*(A), 0)$. Thus, an algebra (A, d_A) is formal if and only if its minimal model coincides with the minimal model of its cohomology algebra.

For a smooth connected manifold M the minimal model is by definition the minimal model of its de Rham algebra of differential forms $\Omega_{DR}(M)$. In the case when M is a simply connected manifold its minimal model completely classifies its rational homotopy type what means that the two simply connected manifolds have the same rational homotopy type if and only if they have isomorphic minimal models. For a simply connected manifold M, the minimal model also contains complete information on its rational homotopy groups. More precisely:

Theorem 2. *Let M be a simply connected manifold of finite type and let $\mu^{++}(M)$ denotes the set of all indecomposable elements in the minimal model $\mu(M)$. Then $\mu^{++}(M)_r \cong Hom(\pi_r(M), \mathbb{R})$, $r \geq 1$.*

The above theorem implies that

$$\operatorname{rk} \pi_r(M) = \dim \mu^{++}(M)_r \ . \tag{2}$$

Remark 3. The same is true for a more general class of topological spaces which are nilpotent and of finite type. These spaces satisfy that they have finitely generated homology groups and that their fundamental group is nilpotent and acts nilpotently on the higher homotopy groups.

In the case when $M = G/U$ is a compact connected homogeneous space it turned out that, in this context, de Rham algebra can be replaced with its Cartan algebra.

Theorem 3. *The Cartan algebra of a compact homogeneous spaces is weakly equivalent to its de Rham algebra of differential forms.*

Corollary 1. *The minimal model of the de Rham algebra of differential forms of a compact connected homogeneous space is isomorphic to the minimal model of its Cartan algebra.*

Remark 4. As the Cartan algebra of the homogeneous space G/H is a simply connected free algebra, one can apply algorithm given by Sullivan for the computation of its minimal model. Namely, let $\tilde{d}$ denotes the linear part in the differential d of the Cartan algebra (C, d). It was proved (see [13], [6]) that the minimal model for $(C = \wedge V, d)$ is given with $(\wedge \bar{V}, \bar{d})$, where $\bar{V}$ is the complement of $\Im \tilde{d}$ in $\operatorname{Ker} \tilde{d}$ and $\bar{d}$ is the restriction of the differential d to $\wedge \bar{V}$. It implies that $\operatorname{rk} \pi_r(G/H)$ is given by the number of generators of degree r in $\bar{V}$ which is further equal to the number of generators in V of degree r which are from the complement of $\Im \tilde{d}$ in $\operatorname{Ker} \tilde{d}$.

We prove the result which gives the relation between degrees of cohomology generators and rational homotopy groups of Cartan pair homogeneous spaces.

Proposition 1. *Let G be a semisimple compact connected Lie group, and H its connected closed subgroup such that G/H is a Cartan pair homogeneous space. Then for each even p, the number of generators in $H^*(G/H)$ of degree p is equal to $\operatorname{rk} \pi_p(G/H)$.*

Proof. For Cartan pair homogeneous spaces Lemma 1 implies that the candidates for even degree generators in $H^*(G/H, \mathbb{R})$ are the classes corresponding to the

Weyl invariant generators in $\mathbb{R}[\mathfrak{s}]^{W_H} \cong H^*(BH,\mathbb{R})$ since d is identically equal to zero on $\mathbb{R}[\mathfrak{s}]^{W_H}$. The elimination of some of them is done by the differential d in Cartan algebra. Namely, those generators Q_i, $1 \leq i \leq r$, in $\mathbb{R}[\mathfrak{s}]^{W_H}$ appearing as a linear part in the differential d meaning that for some generator P_j, $1 \leq j \leq n$, in $\mathbb{R}[\mathfrak{t}]^{W_G}$ it is satisfied that $d(P_j)$ contains the summand of the form aQ_i for $a \in \mathbb{R}$, $a \neq 0$ are eliminated. Thus, for p even we obtain that the number of generators in degree p is equal to the number of generators in $H^*(BH,\mathbb{R})$ of degree p minus the number of generators in $\Im \tilde{d}$ of degree p. The last one is, by Remark 4, equal to $\operatorname{rk}\pi_p(G/H)$. $\square$

Remark 5. Note that in the case $\operatorname{rk} G = \operatorname{rk} H$, by [1], we may have only even degree cohomology generators and using the above proposition their number for each even degree p is equal to $\operatorname{rk}\pi_p(G/H)$.

Remark 6. The analogous statement to Proposition 1 is not valid for odd degree cohomology generators. For example, if we consider the sphere S^{2n} then it has no odd degree cohomology generators, while it is easy to see that $\operatorname{rk}_{4n-1}\pi(S^{2n}) = 1$.

4. Degrees of real cohomology generators of generalized symmetric spaces

Symmetric spaces might be generalized in several ways [11], [18], [20],...
We consider here the spaces that generalize the notion of symmetric spaces by omitting the condition of involutivity.

Definition 1. *A generalized symmetric space of order m is a triple (G,H,Θ), where G is a compact Lie group, $H \subset G$ is a closed subgroup, and Θ is an automorphism of finite order m of the group G satisfying*

$$G_0^{\Theta} \subseteq H \subseteq G^{\Theta} ,$$

where Θ is the fixed point set of Θ and G_0^{Θ} its identity component.

If we assume G to be semisimple and simply connected, G^{Θ} is connected and, thus, generalized symmetric spaces are represented by the triples (G, G^{Θ}, Θ).

The explicit formulas on real cohomology algebras of generalized symmetric spaces are given [16], proving at the same time that these spaces are all Cartan pair homogeneous spaces. But, one very often needs more detailed or complete information on their cohomology algebra structure, for example, for the purpose of getting the obstructions for the existence of a different geometric structures. In this sense, it is very often useful to know the degrees of real cohomology generators.

Recalling [16] and [17], we give explicit formulas for the degrees of real cohomology generators of generalized symmetric spaces.

The following Theorem is obtained using results on rational homotopy groups of generalized symmetric spaces [17] together with Proposition 1.

Theorem 4. *Let G/H be a generalized symmetric space of a simple compact Lie group G, let $\{k_1, \ldots, k_n\}$ be the exponents of the group G and $\{l_1, \ldots, l_r\}$ the exponents of the subgroup U. Then for any integer p, the number of cohomology generators in degree $2p$ is:*

(1) *equal to zero for $p \notin \{l_1, \ldots, l_r\}$*
(2) *equal to $\nu(l_i)$ for $p = l_i \notin \{k_1, \ldots, k_n\}$*
(3) *if $p = l_i \in \{k_1, \ldots, k_n\}$*
  - *for $\mathrm{rk}\, H = \mathrm{rk}\, G$ equal to $\nu(l_i) - 1$,*
  - *for $\mathrm{rk}\, H < \mathrm{rk}\, G$ equal to $\nu(l_i)$ for p odd, and $\nu(l_i) - 1$ for p even.*

For Cartan pair homogeneous spaces, odd degree generators in $H^*(G/H)$ are given by those generators in $H^*(G)$ on which d vanishes, i.e. the number of generators of degree $2p-1$ is equal to $\dim \mathrm{Ker}\, d$ on $H^{2p-1}(G)$. In the case of generalized symmetric spaces using formulas from [16] on their cohomology for the case $\mathrm{rk}\, H < \mathrm{rk}\, G$ we immediately get the number and degrees of odd degree generators from the exterior algebra part in their cohomology.

Theorem 5. *The number of cohomology generators of degree $2p-1$ of the generalized symmetric space G/H is:*

(1) *equal to zero for $p \notin \{k_1, \ldots, k_n\}$*
(2) *if $p \in \{k_1, \ldots k_n\}$*
  - *for $\mathrm{rk}\, H = \mathrm{rk}\, G$ equal to 0 for $(\mathfrak{g}, p) \neq (D_{2n}, 2n)$ or $\mathfrak{g} = D_{2n}$ and $\mathfrak{g}' = A_{2n-1}$, while it is equal to 1 for $\mathfrak{g} = D_{2n}$ and $\mathfrak{g}' = A_{2n-1}$.*
  - *for $\mathrm{rk}\, H < \mathrm{rk}\, G$ equal to 1 if p is odd or $(\mathfrak{g}, p) = (D_{2k}, 2k)$, while it is equal to zero if p is even and $(\mathfrak{g}, p) \neq (D_{2k}, 2k)$.*

Example 3. We complete the description of the degrees of cohomolgy generators for the space $Sp(n)/S(U(1) \times U(1) \times U(n-2))$ from Example 1. It follows from [16] that this space is a 3-symmetric space. Theorem 4 implies that the number of generators of degree $2p$ for $p = 2, 4, \ldots, n-1$ and p even, is by one less then the reciprocity of these exponents related to the subgroup, which is this case equal to zero. Together with Example 1, it implies that this space has two generators in degree 2 and one generator in degrees $2p$ where $p = 3, 5, \ldots, n-1$ and p is odd.

5. On formality question

A smooth manifold M is said to be formal (in the sense of Sullivan) if its de Rham algebra of differential forms is weakly equivalent to its cohomology algebra. If M is a compact homogeneous space G/U then by Theorem 3 it will be formal if and only if its Cartan algebra is weakly equivalent to its cohomology algebra. The following theorem is well known:

Theorem 6. *Let G/H be a compact homogeneous space such that $\operatorname{rk} H = r$ and $\operatorname{rk} G = n$. Then the space G/H is formal if and only if it is a Cartan pair of homogeneous spaces.*

Using above theorem which is proved in [9] we have:

Theorem 7. *All generalized symmetric spaces are formal in the sense of Sullivan.*

As a consequence of Theorem 6 we deduce now that, in terms of cohomology generators, the criterion for formality of homogeneous spaces can be formulated in the following way.

Theorem 8. *A compact homogeneous space is formal if and only if the set of its even degree cohomology generators is free and its number is equal to the number of functionally independent cohomology relations between them.*

Proof. If G/H is formal Theorem 6 implies that one can choose the generators $P_1, \ldots, P_n$ in $H^*(BG, \mathbb{R})$ such that $\rho^*(P_{r+1}), \ldots, \rho^*(P_n)$ belong to the ideal generated by $\rho^*(P_1), \ldots, \rho^*(P_r)$. Therefore, the generators of this ideal produce the relations among the even degree generators in $H^*(BH, \mathbb{R})$. Because G/H has finite dimensional cohomology, $\rho^*(P_1), \ldots, \rho^*(P_r)$ are functionally independent. Now any of these relation that contains some of the generators $Q_1, \ldots, Q_r$ of $H^*(BH, \mathbb{R})$ as a linear summand are eliminated in the expression for cohomology of G/H, but at the same time it also eliminates some of the generators Q_i which it contains linearly. It follows that the number of remaining even degree cohomology generators Q_i for G/H, which are obviously free, is equal to the number of remaining relations $\rho^*(P_i)$. For the opposite direction, note that the set of functionally independent relations between even degree generators in $H^*(G/H, \mathbb{R})$ contains those given by functionally independent elements among $\rho^*(P_1), \ldots, \rho^*(P_n)$. Since the set of even degree generators is free, and cohomology is finite dimensional, we deduce that the number of functionally independent elements among $\rho^*(P_1), \ldots, \rho^*(P_n)$ is equal to the number of the cohomology generators in $H^*(G/H, \mathbb{R})$ which come from $H^*(BH, \mathbb{R})$. This implies that G/H is a Cartan pair homogeneous space, and, thus formal by Theorem 6. □

The notion of geometric formality is introduced in [8]. A smooth manifold is said to be *geometrically formal* if it admits the Riemannian metric for which the wedge product of any two harmonic forms is a harmonic form.

Remark 7. The notion of geometric formality is much more restrictive then the notion of formality. Namely, any geometrically formal manifold is formal since, using Hodge theory [4], it is straightforward to see that the minimal model for the algebra of harmonic forms of geometrically formal manifolds will be the minimal model for its de Rham algebra of differential forms as well as for its cohomology algebra. On the other hand the symmetric spaces are geometrically formal [5], while it is proved in [9] that the most of generalized symmetric spaces are not geometrically formal. The examples of non-symmetric geometrically formal homogeneous spaces that are not homology sphere are provided in [10].

Geometrically formal homogeneous spaces satisfy the following property.

Proposition 2. *If a homogeneous space G/H is geometrically formal than the number of its even degree linearly independent harmonic forms is less or equal than the number of non-trivial relations between them.*

Proof. Using Hodge theory we have that any cohomology class for G/H contains a unique harmonic representative. It implies that if G/H is geometrically formal than any relation between cohomology generators for G/H is valid for their harmonic representatives. Then Theorem 8 gives that the number of linearly independent harmonic forms is equal to the number of relations between them. Among these relations might be some whose degrees are greater than the dimension of the manifold what means that they are trivial on the level of differential forms. □

The following observations are related to the existence of symplectic structure on compact homogeneous spaces and geometrically formal manifolds. Recall that the symplectic structure on a smooth manifold M of dimension $2n$ is given by the closed non-degenerate 2-form ω, meaning that for any $p \in M$ there is no non-zero vector $X \in T_pM$ such that $\omega_p(X,Y) = 0$ for all $Y \in T_pM$.

Remark 8. Note that in the case of a compact symplectic manifold M there must exist $x \in H^2(M,\mathbb{R})$ such that $x^n \neq 0 \in H^{2n}(M^{2n},\mathbb{R})$, see [3].

It turns out that for the compact homogeneous spaces this condition is also sufficient for the existence of symplectic structure.

Lemma 2. *A homogeneous space G/U of dimension $2n$ is symplectic if and only if there exists $x \in H^2(G/U,\mathbb{R})$ such that $x^n \neq 0$.*

Proof. It is well known that the cohomology algebra of the homogeneous space G/U coincides with the cohomology algebra of the algebra of differential forms on G/H which are invariant under the canonical action of the group G on G/U. Let α be an invariant form representing class x. Then α^n is the non-zero top degree invariant form that implies that α^n is the volume form up to a non-zero constant c. This implies that the form α must be non-degenerate and therefore symplectic. □

Lemma 3. *A geometrically formal manifold M of dimension $2n$ is symplectic if and only if there exists $x \in H^2(M,\mathbb{R})$ such that $x^n \neq 0$.*

Proof. Let α be the unique harmonic form which represents the class x. Since M is geometrically formal α^n will again be harmonic form. If $x^n \neq 0$ we obtain that α^n is a non-zero top degree harmonic form which is equal by [8] to the volume form up to a non-zero constant. It follows that the form α is non-degenerate and thus a symplectic form on M. □

Remark 9. Note that this property is not true in general in the sense that there are examples of manifolds not admitting symplectic structure, but for which the cohomology condition from Remark 8 is satisfied. Consider the four-dimensional manifold given as the connected sum of two copies of $\mathbb{C}P^2$, i. e. $M = \mathbb{C}P^2\#\mathbb{C}P^2$. It is easy to see that the real, as well as integral, cohomology ring for M has two generators x_1 and x_2 from $H^2(M,\mathbb{R})$ which are subject to the relations $x_1x_2 = 0$ and $x_1^2 = x_2^2 = y$, where y generates $H^4(M,\mathbb{R})$. Thus, the cohomology condition given in Remark 8 is satisfied. However, arguing with characteristic Chern classes and the Hirzebruch signature theorem, it is proved in [3] that M does not admit any symplectic structure.

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(Received: November 11, 2010)
(Revised: February 25, 2011)

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