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ON WEIGHTED STOLARSKY MEANS

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ABSTRACT. In this article we study a weighted variant of the well known Stolarsky means. Its advantage is the possibility of a simple and natural extension to the multi-variable case. We also investigate some common properties of those classes of means.

1. INTRODUCTION AND RESULTS

There are a plenty of papers (cf. [2], [3], [5], [6], [7], [8]) studying different properties of the so-called Stolarsky (or extended) two-parametric mean value, defined for positive values of $x, y, x \neq y$ by the following

$$E_{r,s}(x,y) = \begin{cases} \left(\frac{r(x^s-y^s)}{s(x^r-y^r)}\right)^{1/(s-r)}, & rs(r-s) \neq 0 \\ \exp\left(-\frac{1}{s} + \frac{x^s \log x - y^s \log y}{x^s-y^s}\right), & r = s \neq 0 \\ \left(\frac{x^s-y^s}{s(\log x - \log y)}\right)^{1/s}, & s \neq 0, r = 0 \\ \sqrt{xy}, & r = s = 0, \\ x, & y = x > 0. \end{cases}$$

In this form it was introduced by K. Stolarsky in [1].

Most of the classical two variable means are special cases of the class E. For example, $E_{1,2} = \frac{x+y}{2}$ is the arithmetic mean, $E_{0,0} = E_{-1,1} = \sqrt{xy}$ is the geometric mean, $E_{0,1} = \frac{x-y}{\log x - \log y}$ is the logarithmic mean, $E_{1,1} = (x^x/y^y)^{\frac{1}{x-y}}/e$ is the identric mean, etc. More generally, the r-th power mean $\left(\frac{x^r+y^r}{2}\right)^{1/r}$ is equal to $E_{r,2r}$.

An attempt to define a class of "weighted" extended mean values as a ratio of two Stolarsky means with additional parameters has been given

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by A. Witkowski ([15], [16]). In this article we shall stick to more standard definitions of weighted means. For this cause, we need notions of the weighted arithmetic mean $A = A(p, q; x, y)$ and weighted geometric mean $G = G(p, q; x, y)$, given by

$$A := px + qy; \quad G := x^p y^q,$$

where

$$p, q, x, y \in \mathbb{R}_+; \quad p + q = 1.$$

Note that $A > G$ for $x \neq y$.

1. 2. In [12] and [13] we introduced a class W of weighted two parameters means which *includes* the Stolarsky class E as a particular case. Namely, for $p, q, x, y \in \mathbb{R}_+$, $p + q = 1$, $rs(r - s)(x - y) \neq 0$ we define

$$\begin{aligned} W = W_{r,s}(p, q; x, y) : &= \Big(\frac{r^2}{s^2} \frac{A(p, q; x^s, y^s) - G(p, q; x^s, y^s)}{A(p, q; x^r, y^r) - G(p, q; x^r, y^r)}\Big)^{1/(s-r)} \\ &= \Big(\frac{r^2}{s^2} \frac{px^s + qy^s - x^{ps} y^{qs}}{px^r + qy^r - x^{pr} y^{qr}}\Big)^{1/(s-r)}. \end{aligned} \tag{1}$$

Various identities concerning the means W can be established; some of them are the following

$$W_{r,s}(p, q; x, y) = W_{s,r}(p, q; x, y)$$

$$W_{r,s}(p, q; x, y) = W_{r,s}(q, p; y, x); \; W_{r,s}(p, q; y, x) = xyW_{r,s}(p, q; x^{-1}, y^{-1}); \tag{2}$$

$$W_{ar,as}(p, q; x, y) = (W_{r,s}(p, q; x^a, y^a))^{1/a}, \; a \neq 0. \tag{3}$$

Note that

$$\begin{aligned} W_{2r,2s}(1/2, 1/2; x, y) &= \Big(\frac{r^2}{s^2} \frac{x^{2s} + y^{2s} - 2(\sqrt{xy})^{2s}}{x^{2r} + y^{2r} - 2(\sqrt{xy})^{2r}}\Big)^{1/2(s-r)} \\ &= \Big(\frac{r^2}{s^2} \frac{(x^s - y^s)^2}{(x^r - y^r)^2}\Big)^{1/2(s-r)} = E(r, s; x, y). \end{aligned}$$

Hence $E \subset W$.

The weighted means from the class W can be extended continuously to the domain

$$D = \{(r, s, x, y) | r, s \in \mathbb{R}, \; x, y \in \mathbb{R}_+\}.$$

This extension is given by

$$W_{r,s}(p,q;x,y)$$
$$=\begin{cases}\left(\frac{r^2}{s^2}\frac{px^s+qy^s-x^{ps}y^{qs}}{px^r+qy^r-x^{pr}y^{qr}}\right)^{1/(s-r)}, & rs(r-s)(x-y)\neq 0\\ \left(2\frac{px^s+qy^s-x^{ps}y^{qs}}{pqs^2\log^2(x/y)}\right)^{1/s}, & s(x-y)\neq 0, r=0\\ \exp\left(\frac{-2}{s}+\frac{px^s\log x+qy^s\log y-(p\log x+q\log y)x^{ps}y^{qs}}{px^s+qy^s-x^{ps}y^{qs}}\right), & s(x-y)\neq 0, r=s\\ x^{(p+1)/3}y^{(q+1)/3}, & x\neq y,\ r=s=0\\ x, & y=x.\end{cases}$$

Especially interesting is studying of the *shifted Stolarsky means* E^*, defined by

$$E^*_{r,s}(x,y):=\lim_{p\to 0^+}W_{r,s}(p,q;x,y).$$

Their analytic continuation to the whole (r,s) plane is given by

$$E^*_{r,s}(x,y)$$
$$=\begin{cases}\left(\frac{r^2(x^s-y^s(1+s\log(x/y)))}{s^2(x^r-y^r(1+r\log(x/y)))}\right)^{1/(s-r)}, & rs(r-s)(x-y)\neq 0;\\ \left(\frac{2}{s^2}\frac{x^s-y^s(1+s\log(x/y))}{\log^2(x/y)}\right)^{1/s}, & s(x-y)\neq 0,\ r=0;\\ \exp\left(\frac{-2}{s}+\frac{(x^s-y^s)\log x-sy^s\log y\log(x/y)}{x^s-y^s(1+s\log(x/y))}\right), & s(x-y)\neq 0, r=s;\\ x^{1/3}y^{2/3}, & r=s=0;\\ x, & x=y.\end{cases}$$

1. 3. Several papers are recently produced aiming to define an extension of the class E to $n,\ n>2$ variables (cf [4], [5], [9]).

In [12] we give another attempt to generalize Stolarsky means to the multi-variable case in a simple and applicable way. The proposed task can be accomplished by the formula (1), wherefrom the mentioned generalization follows naturally.

Therefore, taking the means A and G in their multi-variable forms, we get from (1):

$$W_{r,s}(\mathbf{p};\mathbf{x})$$
$$=\begin{cases}\left(\frac{r^2(\sum_1^n p_ix_i^s-(\prod_1^n x_i^{p_i})^s)}{s^2(\sum_1^n p_ix_i^r-(\prod_1^n x_i^{p_i})^r)}\right)^{1/(s-r)}, & rs(s-r)\neq 0;\\ \left(\frac{2}{s^2}\frac{\sum_1^n p_ix_i^s-(\prod_1^n x_i^{p_i})^s}{\sum_1^n p_i\log^2 x_i-(\sum_1^n p_i\log x_i)^2}\right)^{1/s}, & r=0, s\neq 0;\\ \exp\left(\frac{-2}{s}+\frac{\sum_1^n p_ix_i^s\log x_i-(\sum_1^n p_i\log x_i)(\prod_1^n x_i^{p_i})^s}{\sum_1^n p_ix_i^s-(\prod_1^n x_i^{p_i})^s}\right), & r=s\neq 0;\\ \exp\left(\frac{\sum_1^n p_i\log^3 x_i-(\sum_1^n p_i\log x_i)^3}{3(\sum_1^n p_i\log^2 x_i-(\sum_1^n p_i\log x_i)^2)}\right), & r=s=0.\end{cases}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n,\ n \geq 2$ and $\mathbf{p}$ is an arbitrary positive weight sequence associated with $\mathbf{x}$. Also $W_{r,s}(\mathbf{p}; \mathbf{x}_0) = a$ for $\mathbf{x}_0 = (a, a, \ldots, a), a > 0$.

The above formulae are obtained by an appropriate limit process, implying continuity. For instance $W_{s,s}(\mathbf{p}, \mathbf{x}) = \lim_{r\to s} W_{r,s}(\mathbf{p}, \mathbf{x})$ and $W_{0,0}(\mathbf{p}, \mathbf{x}) = \lim_{s\to 0} W_{0,s}(\mathbf{p}, \mathbf{x})$.

Remark 1. Analogously to the former considerations, one can define a class of Stolarsky means in n variables $E_{r,s}(\mathbf{x}; n)$ as

$$E_{r,s}(\mathbf{x}; n) := W_{nr,ns}(\mathbf{p}_0, \mathbf{x}),$$

where $\mathbf{p}_0 = \{1/n\}_1^n$.

Therefore,

$$E_{r,s}(\mathbf{x}; n) = \Big(\frac{r^2}{s^2} \frac{\sum_1^n x_i^{ns} - n \prod_1^n x_i^s}{\sum_1^n x_i^{nr} - n \prod_1^n x_i^r}\Big)^{\frac{1}{n(s-r)}}, \quad rs(r-s) \neq 0.$$

Its extension over the (r, s) plane and other properties are left to the readers.

The following basic theorems are of importance.

Theorem 1. ([13]) *The expressions $W_{r,s}(\mathbf{p}; \mathbf{x})$ are actually means i.e. for arbitrary weight sequence $\mathbf{p}$ we have*

$$\min\{x_1, x_2, \ldots, x_n\} \leq W_{r,s}(\mathbf{p}; \mathbf{x}) \leq \max\{x_1, x_2, \ldots, x_n\}.$$

Theorem 2. ([12]) *The means $W_{r,s}(\mathbf{p}; \mathbf{x})$ are monotone increasing in either r or s.*

1. 4. In this paper we shall investigate further common properties of the means $W_{r,s}(p, q; x, y)$ and the class of Stolarsky means i.e. the case $p = q = 1/2$. Our results are contained in the next

Theorem 3. *The means $W_{r,s}(p, q; x, y)$ are*

a. *symmetric in x, y, $W_{r,s}(p, q; x, y) \equiv W_{r,s}(p, q; y, x)$ if and only if $p = q$ or $x = y$. Moreover, it can be proved that*

$$(p - q)(x - y)(W_{r,s}(p, q; x, y) - W_{r,s}(p, q; y, x)) \geq 0;$$

b. *homogenous of order one, $W_{r,s}(p, q; \lambda x, \lambda y) = \lambda W_{r,s}(p, q; x, y)$;*

c. *monotone increasing in both variables x and y;*

d. *monotone increasing in both variables r and s;*

e. *logarithmically convex (concave) for $r, s \in I \subseteq \mathbb{R}$ if and only if $p = q = 1/2$.*

From this theorem one can see that the properties a. and e. exclusively belong to the class E of Stolarsky means, while other properties are in common for the wider class W of weighted Stolarsky means.

2. Proofs

Proof of Theorem 3. Proofs of the parts a. and b. are left to the readers.

c. We shall prove monotonicity in x. Because $W_{r,s}(p,q;x,y) = W_{r,s}(q,p;y,x)$, the same is valid for the variable y. Note that the property (2) allows us to assume that $s \geq r$. Now in the case $s > r$, an easy proof can be produced applying the following (cf [14]).

Proposition 1. *Let $f, g : [a,b] \to \mathbb{R}$ be two continuous and differentiable functions. Further, let $g' \neq 0$ on (a,b). If f'/g' is increasing (decreasing) on (a,b), then the functions $(f(x)-f(a))/(g(x)-g(a))$ and $(f(x)-f(b))/(g(x)-g(b))$ are also increasing (resp. decreasing) on (a,b).*

Let $f_s(x) := (px^s - x^{ps}y^{qs})/s^2,\ s \neq 0$. We have

$$\frac{f_s'(x)}{f_r'(x)} = x^{p(s-r)}\frac{r}{s}\frac{x^{qs}-y^{qs}}{x^{qr}-y^{qr}} = a(x)b(x),$$

with $a(x) := x^{p(s-r)};\ b(x) := \frac{qr}{qs}\frac{x^{qs}-y^{qs}}{x^{qr}-y^{qr}}$.

Now $a(x)$ is evidently increasing in x and $b(x)$ is also increasing by the well known property of Stolarsky means ([1], [2]). Since

$$\frac{f_s(x)-f_s(y)}{f_r(x)-f_r(y)} = \frac{r^2}{s^2}\frac{px^s+qy^s-x^{ps}y^{qs}}{px^r+qy^r-x^{pr}y^{qr}},$$

by Proposition 1 we conclude that the required monotonicity property holds true for $x \geq y$. That it also holds for $y \geq x$ follows from the other part of Proposition 1, and the proof is done.

The same argument is valid if $r \vee s = 0$.

In the case $r = s \neq 0$, we have to prove that the function

$$F_s(p,q;x,y) = \frac{px^s\log x + qy^s\log y - (p\log x + q\log y)x^{ps}y^{qs}}{px^s+qy^s-x^{ps}y^{qs}}$$

is, for fixed p, q, s, y, monotone increasing in $x \in \mathbb{R}_+$.

For this purpose we need the following

Lemma 1. *For each real $s \neq 0, p \neq 1$, the function*

$$U(p,s;u) = \frac{u^s\log u}{u^s-u^{ps}},$$

is monotone increasing in $u \in \mathbb{R}_+$.

Indeed, since

$$U_u' = \frac{u^{qs-1}}{(u^{qs}-1)^2}(u^{qs}-1-qs\log u),$$

with $q = 1-p$, the proof of lemma follows.

Denoting

$$f(x)=f(p,q,s;x):=px^s\log x-(p\log x+q\log y)x^{ps}y^{qs};$$
$$g(x)=g(p,q,s;x):=px^s-x^{ps}y^{qs},$$

we have

$$\begin{aligned}\frac{f'(x)}{g'(x)}&=\frac{p(s\log x+1)x^{s-1}-p(1+s(p\log x+q\log y))x^{ps-1}y^{qs}}{ps(x^{s-1}-x^{ps-1}y^{qs})}\\&=\frac{1}{s}+\log y+p\log\frac{x}{y}+q\frac{x^s}{x^s-x^{ps}y^{qs}}\log\frac{x}{y}\\&=\frac{1}{s}+\log y+p\log u+qU(p,s;u),\end{aligned}$$

where $u:=x/y$. Since,

$$\frac{f(x)-f(y)}{g(x)-g(y)}=F_s(p,q;x,y),$$

applying Proposition 1 with Lemma 1, it follows that $F_s(p,q;x,y)$ is monotone increasing in x, as required.

Finally, the case $r=s=0$ is trivial.

d. The proof of this assertion for the means $W_{r,s}(\mathbf{p};\mathbf{x})$ is given in [12], applying the method originated from [11].

e. Suppose that $r,s\in I$, where I is some sub-interval of $\mathbb{R}$. Then there exist $r\neq 0, a>0$ such that $r, r\pm a\in I$. If the means $W_{r,s}(p,q;x,y)$ are log-convex (log-concave) for all $r,s\in I$, then the expression

$$A:=W_{r,r-a}(p,q;x,y)W_{r,r+a}(p,q;x,y)-(W_{r,r}(p,q;x,y))^2$$

should be of constant sign for all values of $x,y\in\mathbb{R}^+$. Putting there $y=1$, $x^a=e^t$, $r/a=b$, the form A reduces to an equivalent form B,

$$B:=\Big(\frac{b-1}{b+1}\Big)^2\frac{pe^{(b+1)t}+q-e^{p(b+1)t}}{pe^{(b-1)t}+q-e^{p(b-1)t}}-\exp\Big(\frac{-4}{b}+\frac{2pt(e^{bt}-e^{pbt})}{pe^{bt}+q-e^{pbt}}\Big).$$

Developing in $t\in\mathbb{R}$, somewhat laborious calculation gives

$$B=\frac{1}{810}(p-2)(p+1)(2p-1)t^3+O(t^4).$$

Therefore the form B (hence A) can be of constant sign for sufficiently small $t\in\mathbb{R}$ only if $p=1/2(=q)$. In this case i.e. the case of Stolarsky means, the famous Feng Qi theorem [10] asserts that those means are log-concave for $r,s\in\mathbb{R}^+$ and log-convex for $r,s\in\mathbb{R}^-$. □

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