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ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS OF A CLASS OF $(k+1)$-ORDER RATIONAL DIFFERENCE EQUATIONS

SVETLIN G. GEORGIEV

ABSTRACT. In this paper we investigate the solutions of a class of $(k+1)$-order rational difference equations. We give conditions for the parameters ensuring that the considered equation has a unique positive equilibrium point that is locally asymptotically stable and every positive solution of the considered equation is bounded and increasing and converges to the unique positive equilibrium point.

1. INTRODUCTION

In this paper are investigated the solutions of the following rational difference equation

$$x_{n+1} = \frac{a + a_0 x_n + \cdots + a_k x_{n-k}}{A + A_0 x_n + \cdots + A_k x_{n-k}}, \quad n \in \mathbb{N} \cup \{0\}, \tag{1.1}$$

where a, A, a_j, A_j, x_{-j}, $j \in \{0, \ldots, k\}$, are non-negative numbers such that the numerator of (1.1) is positive for all n. The global asymptotic analysis of such equation has been initiated in the reference [13] for $k = 1$ and was continued in the monograph [6] for $k = 2$. The case $k = 1$ is different from the case $k > 1$ whenever the right hand side of (1.1) is either increasing in both arguments or decreasing in x_n and increasing in x_{n-1}. This case is covered by the results of Amleh, Ladas and Camouzis [1,5] and Kulenović and Merino [12]- [14]. In these cases every solution breaks into two monotone subsequences and so every bounded solution converges to either an equilibrium solution, period-two solution or to the singular point on the boundary. Thus, in this case the main problem is to determine the basins of attraction of different attractors. This is done in [4] using the results in [12]- [14] which clearly determine the boundaries of these basins of attraction as the stable manifolds of some saddle point or non-hyperbolic equilibrium solutions or period-two solutions.

In this paper we give some conditions for the parameters of (1.1) ensuring that (1.1) has a unique positive equilibrium point that is locally asymptotically stable

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and every positive solutions of (1.1) are bounded and increasing and converge to the unique positive equilibrium point. In this way, we give an answer of an open problem(see [11], Open problem 11.4.3).

The set up of the paper is as follows. In Section 2, we formulate the main assumptions for the parameters of the equation (1.1) and we formulate the main result of this paper and give some comparisons of the main result with some well-known results. In Section 3, we formulate and prove some preliminary results. In Section 4, we prove the main result.

2. The main hypotheses and the main result

In this section, we will formulate the main assumptions for the parameters of the equation (1.1) and we will formulate our main result. Suppose that $c > 0$ and the parameters of the equation (1.1) satisfy the following conditions

$$\text{(H1)} \qquad a \geq 0, \quad A > 0, \quad a_j \geq 0, \quad A_j \geq 0, \quad x_{-j} > 0, \quad j \in \{0, \dots, k\}, \\ (a, a_0, \dots, a_k) \neq (0, 0, \dots, 0),$$

$$\text{(H2)} \qquad Aa_{k-l} - aA_{k-l} \geq 0,$$

$$A_j a_{k-l} - a_j A_{k-l} \geq 0, \quad j, l \in \{0, \dots, k\},$$

$$\frac{a + a_0 x_0 + \cdots + a_k x_{-k}}{A + A_0 x_0 + \cdots + A_k x_{-k}} \geq x_0,$$

$$\text{(H3)} \qquad \frac{a + (a_0 + \cdots + a_k) c}{A} \leq c,$$

$$\text{(H4)} \qquad 0 < x_{-k} \leq x_{-k+1} \leq \ldots \leq x_0 \leq c \ \text{ and}$$

$$\frac{a_0 + \cdots + a_k - A + \sqrt{(A - (a_0 + \cdots + a_k))^2 + 4a(A_0 + \cdots + A_k)}}{2(A_0 + \cdots + A_k)} \leq c.$$

Example 2.1. *Consider the equation*

$$x_{n+1} = \frac{1 + 2(x_n + x_{n-1} + \cdots + x_{n-4})}{1000 + 3(x_n + x_{n-1} + \cdots + x_{n-4})},$$

with

$$x_0 = x_{-1} = \cdots = x_{-4} = \frac{1}{30000}.$$

Take $c = 2$. Here

$$k = 4, \quad a = 1, \quad a_j = 2, \quad A = 1000, \quad A_j = 3, \quad j \in \{0, \dots, 4\}.$$

Then $(H1)$ holds. Also,

$$\begin{aligned} Aa_{4-l} - aA_{4-l} &= 2000 - 3 \\ &= 1997 \\ &> 0, \end{aligned}$$

$$
\begin{aligned}
A_j a_{4-l} - a_j A_{4-l} &= 0, \quad j,l \in \{0,\dots,4\}, \\
\frac{a + a_0 x_0 + \cdots + a_4 x_{-4}}{A + A_0 x_0 + \cdots + A_4 x_{-4}} &= \frac{a + (a_0 + \cdots + a_4) x_0}{A + (A_0 + \cdots A_4) x_0} \\
&= \frac{1 + 10 x_0}{1000 + 15 x_0} \\
&\geq x_0 \quad \Longleftrightarrow \\
15 x_0^2 + 990 x_0 - 1 \leq 0 \quad &\Longleftrightarrow \\
x_0 &\leq \frac{-990 + \sqrt{990^2 + 60}}{30},
\end{aligned}
$$

i.e., $(H2)$ *holds. Next,*

$$
\begin{aligned}
\frac{a + (a_0 + \cdots + a_4)c}{A} &= \frac{1 + 20}{1000} \\
&< c,
\end{aligned}
$$

i.e., $(H3)$ *holds. Moreover,*

$$
0 < x_0 = \ldots = x_{-4} \leq c
$$

and

$$
\begin{aligned}
&\frac{a_0 + \cdots + a_4 - A + \sqrt{(A - (a_0 + \cdots + a_4))^2 + 4a(A_0 + \cdots + A_4)}}{2(A_0 + \cdots + A_4)} \\
&= \frac{10 - 1000 + \sqrt{(10 - 1000)^2 + 60}}{30} \\
&\leq \frac{-990 + 1000}{30} \\
&= \frac{1}{3} \\
&< c,
\end{aligned}
$$

i.e., $(H4)$ *holds.*

Our main result is as follows.

Theorem 2.1. *Suppose* $(H1)$-$(H4)$. *Then the equation* (1.1) *has a unique positive equilibrium point that is locally asymptotically stable and every positive solutions of* (1.1) *are bounded and increasing and converge to the unique positive equilibrium point.*

When $A > 0$ and $\sum_{j=1}^{k} A_j = 0$ or $A > 0$, $\sum_{j=0}^{k} A_j > 0$, $a = 0$, in [3] the authors give conditions for the parameters of the equation (1.1) so that the zero equilibrium point is asymptotically stable or stable or every positive solution of the equation (1.1) is divergent, or there exists a unique positive equilibrium of (1.1) that is globally asymptotically stable. In our investigations we have $A > 0$, $\sum_{j=0}^{k} A_j \geq 0$, $a \geq 0$ and in the case $A > 0$, $\sum_{j=0}^{k} A_j > 0$, $a > 0$ we give conditions for the parameters of the equation (1.1) so that the equation (1.1) has a unique positive equilibrium point that is locally asymptotically stable and every positive solutions of (1.1) converge to the unique positive equilibrium point. When $\sum_{j=0}^{k} A_j > 0$ and

(s1) $\sum_{j=0}^{k} |a_j - \overline{x}A_j| \le A$

or there exist $L, N > 0$ such that for every positive solution $\{x_n\}_{n\in\mathbb{N}}$ of (1.1)

(s2) $x_n \ge L$, $n \ge N$ and $\sum_{j=0}^{k} |a_j - \overline{x}A_j| < A + L\sum_{j=0}^{k} A_j$, where $A \ge 0$, for some $N \in \mathbb{N}$ and $L > 0$,

in [3] it is proved that the positive equilibrium $\overline{x}$ of (1.1) is globally asymptotically stable. Note that the condition $(s1)$ is different than the conditions $(H2)$-$(H4)$ in this paper. Moreover, in this paper we have $x_n \le c$, $n \in \mathbb{N}$, for a positive constant c(see Proposition 3.3 in this paper), which is different than the condition $(s2)$. If $A > 0$, $\sum_{j=0}^{k} A_j > 0$ and

(s3) $a_j = \overline{x}A_j$, $j \in \{0,\ldots,k\}$; or $a_j \ge \overline{x}A_j$, $j \in \{0,\ldots,k\}$; or $a_j \le \overline{x}A_j$, $j \in \{0,\ldots,k\}$; or $a_i > \overline{x}A_i$, $a_j < \overline{x}A_j$ for some $i, j \in \{0,\ldots,k\}$,

in [3] the authors give conditions for the parameters of the equation (1.1) so that the positive equilibrium point $\overline{x}$ of (1.1) is globally asymptotically stable or stable on $[0,\infty)$. Note that the conditions $(s3)$ are more restrictive than the conditions $(H2)$-$(H4)$.

3. Preliminary results

We will start with the following useful remark.

Remark 3.1. Suppose $(H1)$ and $(H2)$. Define

$$f(y_0, y_1, \ldots, y_k) = \frac{a + a_k y_0 + a_{k-1} y_1 + \cdots + a_0 y_k}{A + A_k y_0 + A_{k-1} y_1 + \cdots + A_0 y_k}$$

for $(y_0, y_1, \ldots, y_k) \in [0,\infty)^{k+1}$. We have, using $(H1)$ and $(H2)$,

$$\begin{aligned}
\frac{\partial f}{\partial y_l}(y_0, y_1, \ldots, y_k) &= \frac{1}{(A + A_k y_0 + A_{k-1} y_1 + \cdots + A_0 y_k)^2} \\
&\quad \times \Big(a_{k-l}(A + A_k y_0 + A_{k-1} y_1 + \cdots + A_0 y_k) \\
&\quad - A_{k-l}(a + a_k y_0 + a_{k-1} y_1 + \cdots + a_0 y_k) \Big) \\
&= \frac{1}{(A + A_k y_0 + A_{k-1} y_1 + \cdots + A_0 y_k)^2} \\
&\quad \times \Big((A a_{k-l} - a A_{k-l}) + (A_k a_{k-l} - a_k A_{k-l}) y_0 \\
&\quad + \cdots \\
&\quad + (A_0 a_{k-l} - a_0 A_{k-l}) y_k \Big) \\
&\ge 0, \quad l \in \{0, \ldots, k\},
\end{aligned}$$

for any $(y_0, y_1, \ldots, y_k) \in [0,\infty)^{k+1}$.

Proposition 3.1. *Suppose* $(H1)$, $(H2)$ *and the first condition of* $(H4)$. *Then any positive solution* $\{x_n\}_{n\in\mathbb{N}}$ *of the equation* (1.1) *is increasing.*

Proof. We have

$$\begin{aligned} x_1 - x_0 &= \frac{a + a_0 x_0 + \cdots + a_k x_{-k}}{A + A_0 x_0 + \cdots + A_k x_{-k}} - x_0 \\ &\geq 0 \end{aligned}$$

and

$$\begin{aligned} x_2 - x_1 &= f(x_{1-k}, x_{2-k}, \ldots, x_1) - f(x_{-k}, x_{-k+1}, \ldots, x_0) \\ &= f(x_{1-k}, x_{2-k}, \ldots, x_1) - f(x_{-k}, x_{2-k}, \ldots, x_1) \\ &+ f(x_{-k}, x_{2-k}, \ldots, x_1) - f(x_{-k}, x_{1-k}, \ldots, x_1) \\ &+ \cdots \\ &+ f(x_{-k}, x_{-k+1}, \ldots, x_1) - f(x_{-k}, x_{-k+1}, \ldots, x_0) \\ &= \frac{\partial f}{\partial y_0}(\xi_{1-k}, x_{2-k}, \ldots, x_1)(x_{1-k} - x_{-k}) \\ &+ \frac{\partial f}{\partial y_1}(x_{-k}, \xi_{2-k}, \ldots, x_1)(x_{2-k} - x_{1-k}) \\ &+ \cdots \\ &+ \frac{\partial f}{\partial y_k}(x_{-k}, x_{-k+1}, \ldots, \xi_1)(x_1 - x_0) \\ &\geq 0, \end{aligned}$$

where $\xi_{j-k} \in (x_{j-1-k}, x_{j-k})$, $j \in \{1, \ldots, k+1\}$. Assume that

$$x_j \geq x_{j-1}, \quad j \in \{-k+1, \ldots, n\},$$

for some $n \in \mathbb{N}$. We will prove that $x_{n+1} \geq x_n$. Really, we have

$$\begin{aligned} x_{n+1} - x_n &= f(x_{n-k}, x_{n-k+1}, \ldots, x_n) - f(x_{n-1-k}, x_{n-k}, \ldots, x_{n-1}) \\ &= f(x_{n-k}, x_{n-k+1}, \ldots, x_{n-1}) - f(x_{n-1-k}, x_{n-k+1}, \ldots, x_{n-1}) \\ &+ f(x_{n-1-k}, x_{n-k+1}, \ldots, x_{n-1}) - f(x_{n-1-k}, x_{n-k}, \ldots, x_{n-1}) \\ &+ \cdots \\ &+ f(x_{n-1-k}, x_{n-k}, \ldots, x_n) - f(x_{n-1-k}, x_{n-k}, \ldots, x_{n-1}) \\ &= \frac{\partial f}{\partial y_0}(\zeta_{n-k}, x_{n-k+1}, \ldots, x_{n-1})(x_{n-k} - x_{n-1-k}) \\ &+ \frac{\partial f}{\partial y_1}(x_{n-1-k}, \zeta_{n-k+1}, \ldots, x_{n-1})(x_{n-k+1} - x_{n-k}) \\ &+ \cdots \\ &+ \frac{\partial f}{\partial y_k}(x_{n-1-k}, x_{n-k}, \ldots, \zeta_n)(x_n - x_{n-1}) \\ &\geq 0, \end{aligned}$$

where $\zeta_{j-k} \in (x_{j-k-1}, x_{j-k})$, $j \in \{n, \ldots, n+k\}$. Therefore $\{x_n\}_{n\in\mathbb{N}}$ is increasing. This completes the proof. $\square$

Proposition 3.2. *Suppose* $(H1)$, $(H3)$ *and* $(H4)$. *Then any positive solution* $\{x_n\}_{n\in\mathbb{N}}$ *of the equation* (1.1) *is bounded and*

$$0 < x_n \leq c, \quad n \in \mathbb{N}.$$

Proof. By $(H3)$ and $(H4)$, we have

$$\begin{aligned} x_1 &= \frac{a + a_0 x_0 + \cdots + a_k x_{-k}}{A + A_0 x_0 + \cdots + A_k x_{-k}} \\ &\leq \frac{a + (a_0 + \cdots + a_k)c}{A} \\ &\leq c. \end{aligned}$$

Assume that $x_n \leq c$ for some $n \in \mathbb{N}$. We will prove that $x_{n+1} \leq c$. Really, again by $(H3)$ and $(H4)$, we find

$$\begin{aligned} x_{n+1} &= \frac{a + a_0 x_n + \cdots + a_k x_{n-k}}{A + A_0 x_n + \cdots + A_k x_{n-k}} \\ &\leq \frac{a + (a_0 + \cdots + a_k)c}{A} \\ &\leq c. \end{aligned}$$

Thus, $0 < x_n \leq c$ for any $n \in \mathbb{N}$. This completes the proof. □

Proposition 3.3. *Suppose* $(H1)$-$(H4)$. *Then any positive solution* $\{x_n\}_{n\in\mathbb{N}}$ *of the equation* (1.1) *is convergent and*

$$\lim_{n\to\infty} x_n = \frac{a_0 + \cdots + a_k - A + \sqrt{(A - (a_0 + \cdots + a_k))^2 + 4a(A_0 + \cdots + A_k)}}{2(A_0 + \cdots + A_k)}.$$

Proof. By Proposition 3.1, we have that any positive solution $\{x_n\}_{n\in\mathbb{N}}$ of the equation (1.1) is increasing. By Proposition 3.2, we get that any positive solution of the equation (1.1) is bounded. Therefore any positive solution $\{x_n\}_{n\in\mathbb{N}}$ of the equation (1.1) is convergent. Let

$$\lim_{n\to\infty} x_n = \overline{x}.$$

We have

$$\begin{aligned} \overline{x} &= \lim_{n\to\infty} \frac{a + a_0 x_n + \cdots + a_k x_{n-k}}{A + A_0 x_n + \cdots + A_k x_{n-k}} \\ &= \frac{a + (a_0 + \cdots + a_k)\overline{x}}{A + (A_0 + \cdots + A_k)\overline{x}}. \end{aligned}$$

Thus,

$$(A_0 + \cdots + A_k)\overline{x}^2 + (A - (a_0 + \cdots + a_k))\overline{x} - a = 0$$

and then

$$\overline{x} = \frac{a_0 + \cdots + a_k - A + \sqrt{(A - (a_0 + \cdots + a_k))^2 + 4a(A_0 + \cdots + A_k)}}{2(A_0 + \cdots + A_k)}.$$

This completes the proof. □

Remark 3.2. Note that $\bar{x}$, defined by the proof of Proposition 3.3, is the unique positive equilibrium point of the equation (1.1).

4. Proof of the main result

Let $\varepsilon > 0$ be arbitrarily chosen. Take $\delta \in (0,\varepsilon]$ and suppose that

$$(\bar{x}-x_0)+(\bar{x}-x_{-1})+\cdots+(\bar{x}-x_{-k}) < \delta.$$

Then $\bar{x}-x_0 < \delta$. By Proposition 3.3, we have

$$\bar{x}-x_n \geq 0, \quad n \in \mathbb{N}.$$

By the proof of Proposition 3.1, we have $x_0 \leq x_1$ and

$$x_{n+1} \geq x_n, \quad n \in \mathbb{N}.$$

Then

$$x_n \geq x_0, \quad n \in \mathbb{N},$$

and

$$\begin{aligned} \bar{x}-x_n &\leq \bar{x}-x_0 \\ &< \delta \\ &\leq \varepsilon, \quad n \in \mathbb{N}. \end{aligned}$$

Thus, $\bar{x}$ is locally stable. Since $0 \leq x_{-j} \leq \bar{x} \leq c$, $j \in \{0,\ldots,k\}$, there exists a $\gamma > 0$ so that

$$(\bar{x}-x_0)+(\bar{x}-x_{-1})+\cdots+(\bar{x}-x_{-k}) < \gamma.$$

Hence and $\lim_{n\to\infty} x_n = \bar{x}$, we conclude that $\bar{x}$ is locally asymptotically stable. This completes the proof of the main result.

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Svetlin G, Georgiev
Sorbonne University
Paris
France