GROWTH OF THE MAXIMUM MODULUS OF POLYNOMIALS WITH PRESCRIBED ZEROS

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Abstract. If \( p(z) \) be a polynomial of degree \( n \) which does not vanish in the disk \( |z| < k \), then for \( k = 1 \), it is well known that
\[
\max_{|z|=r<1} |p(z)| \geq \left( \frac{r+1}{2} \right)^n \max_{|z|=1} |p(z)|.
\]
and
\[
\max_{|z|=R>1} |p(z)| \leq \frac{R^n+1}{2} \max_{|z|=1} |p(z)|.
\]
In this paper, we consider a class of lacunary polynomials and present certain generalizations as well as improvements of the above inequalities for the two cases \( k \geq 1 \) and \( k < 1 \).

1. Introduction and Statement of Results

Let \( p(z) \) be a polynomial of degree \( n \) and let \( M(p, R) = \max_{|z|=R} |p(z)| \). Then it is a simple consequence of maximum modulus principle (for reference see [6, vol. I, p. 137, prob. III, 269]) that
\[
M(p, R) \leq R^n M(p, 1) \text{ for } R \geq 1. \tag{1.1}
\]
The result is best possible and equality holds for \( p(z) = \alpha z^n \), where \( |\alpha| = 1 \).

It was shown by Ankeny and Rivlin [1] that if \( p(z) \neq 0 \) in \( |z| < 1 \), then inequality (1.1) can be replaced by
\[
M(p, R) \leq \frac{R^n+1}{2} M(p, 1) \text{ for } R \geq 1. \tag{1.2}
\]

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The above inequality is best possible and equality holds for \( p(z) = \alpha + \beta z^n \), where \(|\alpha| = |\beta|\).

For the polynomials of degree \( n \) and \( r < 1 \), applying (1.1) to the polynomial \( p(rz) \), taking \( R = 1/r \), gives \( |p(e^{i\theta})| \leq r^{-n} \max_{|z|=1} |p(rz)| \) for \( 0 \leq \theta < 2\pi \). Hence,

\[
M(p, r) \geq r^n M(p, 1) \quad \text{for} \quad 0 \leq r < 1, \tag{1.3}
\]

where equality holds if and only if \( p(z) = \alpha z^n \).

If \( p(z) \neq 0 \) in \( |z| < 1 \), then Rivlin [7] proved

\[
M(p, r) \geq \left( \frac{r + 1}{2} \right)^n M(p, 1) \quad \text{for} \quad 0 \leq r < 1. \tag{1.4}
\]

Here equality is attained if \( p(z) = \alpha(z - \beta)^n \), \( |\beta| = 1 \).

Inequality (1.2) was generalized by Aziz [2] for the polynomials having no zeros in \( |z| < k \), \( k \geq 1 \) and proved the following result.

**Theorem A.** If \( p(z) \) is a polynomial of degree \( n \), which does not vanish in \( |z| < k \), \( k \geq 1 \), then

\[
M(p, R) \leq \frac{R^n + k^n}{1 + k^n} M(p, 1) \quad \text{for} \quad R > k^2, \tag{1.5}
\]

provided \(|p'(k^2z)|\) and \(|p'(z)|\) attains the maximum at the same point on \(|z| = 1\). The result is best possible and equality holds for \( p(z) = z^n + k^n \).

On the other hand for the polynomials not vanishing in \( |z| < k \), \( k < 1 \), the following inequality has been recently proved by Dewan and Hans [3].

**Theorem B.** If \( p(z) \) is a polynomial of degree \( n \), which does not vanish in \( |z| < k \), \( k < 1 \), then

\[
M(p, r) \geq \frac{r^n + k^n}{1 + k^n} M(p, 1) \quad \text{for} \quad 0 < k < r < 1, \tag{1.6}
\]

provided \(|p'(z)|\) and \(|q'(z)|\) attains the maximum at the same point on \(|z| = 1\), where \( q(z) = z^n p(1/z) \). The result is best possible and equality holds for \( p(z) = z^n + k^n \).

In this paper, we firstly generalize Theorem A to the lacunary type of polynomials \( p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^{\nu}, 1 \leq \mu < n \), by proving the following result.

**Theorem 1.** If \( p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^{\nu}, 1 \leq \mu < n \) is a polynomial of degree \( n \), which does not vanish in \( |z| < k \), \( k \geq 1 \), then

\[
M(p, R) \leq \frac{R^n + k^n(1 + k^{n-\mu+1}) - k^{2n}}{1 + k^{n-\mu+1}} M(p, 1) \quad \text{for} \quad R > k^2, \tag{1.7}
\]
provided $|p'(k^2z)|$ and $|p'(z)|$ attains the maximum at the same point on $|z| = 1$.

**Remark 1.** For $\mu = 1$, Theorem 1 reduces to inequality (1.5) due to Aziz [2].

Our next result is an extension as well as generalization of Theorem B. More precisely, we prove

**Theorem 2.** If $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu < n$ is a polynomial of degree $n$, which does not vanish in $|z| < k$, $k < 1$, then

$$M(p, r) \geq \left( \frac{r^{n-\mu+1} + k^{n-\mu+1}}{\lambda^{n-\mu+1} + k^{n-\mu+1}} \right)^{\frac{n}{n-\mu+1}} M(p, \lambda) \quad \text{for} \quad 0 < k < r < \lambda \leq 1,$$

(1.8)

provided $|p'(z)|$ and $|q'(z)|$ attain maximums at the same point on $|z| = 1$, where $q(z) = z^n p(1/z)$. The result is best possible and equality holds for $p(z) = (z^{n-\mu+1} + k^{n-\mu+1})^{\frac{n}{n-\mu+1}}$.

If we take $\lambda = 1$ in Theorem 2, then inequality (1.8) reduces to the following result.

**Corollary 1.** If $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu < n$ is a polynomial of degree $n$, which does not vanish in $|z| < k$, $k < 1$ and $q(z) = z^n p(1/z)$. If $|p'(z)|$ and $|q'(z)|$ attain maximums at the same point on $|z| = 1$, then

$$M(p', r) \leq \left( \frac{r^{n-\mu+1} + k^{n-\mu+1}}{1 + k^{n-\mu+1}} \right)^{\frac{n}{n-\mu+1}} M(p', 1) \quad \text{for} \quad 0 < k < r \leq 1,$$

(1.9)

provided $|p'(z)|$ and $|q'(z)|$ attain maximums at the same point on $|z| = 1$, where $q(z) = z^n p(1/z)$. The result is best possible with equality holds for $p(z) = (z^{n-\mu+1} + k^{n-\mu+1})^{\frac{n}{n-\mu+1}}$.

**Remark 2.** For $\mu = 1$, Corollary 1 reduces to inequality (1.6) due to Dewan and Hans [3].

2. **Lemmas**

We need the following lemmas for the proofs of these theorems.

**Lemma 1.** Let $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu < n$ is a polynomial of degree $n$, having no zero in $|z| < k$, $k \leq 1$ and $q(z) = z^n p(1/z)$. If $|p'(z)|$ and $|q'(z)|$ attain maximums at the same point on $|z| = 1$, then

$$M(p', 1) \leq \frac{n}{1 + k^{n-\mu+1}} M(p, 1).$$

(2.1)

The above lemma is due to Dewan and Hans [4].
Lemma 2. If \( p(z) \) is a polynomial of degree \( n \), then for \( |z| = 1 \)
\[
|p'(z)| + |q'(z)| \leq nM(p, 1), \tag{2.2}
\]
where \( q(z) = z^n p(1/z) \).

The above lemma is a special case of a result due to Govil and Rahman [5].

3. Proofs of the theorems

Proof of Theorem 1. Since \( p(z) \) has all its zeros in \( |z| \geq k \geq 1 \), it follows that the polynomial \( H(z) = p(kz) \) has all its zeros in \( |z| \geq 1 \). Now if \( q(z) = z^n p(1/z) \), then the polynomial \( G(z) \equiv z^n \frac{H(1/z)}{H(z)} \equiv z^n p(k/z) \equiv k^n q(z/k) \) has all its zeros in \( |z| < 1 \). Moreover \( |H(z)| = |G(z)| \) for \( |z| = 1 \), it follows by the maximum modulus principle that
\[
|H(z)| \leq |G(z)| \text{ for } |z| \geq 1. \tag{3.1}
\]

Hence for every complex number \( \lambda \) with \( |\lambda| > 1 \), it follows by Rouché’s theorem that the polynomial \( H'(z) - \lambda G'(z) \) has all its zeros in \( |z| < 1 \). By the Gauss Lucas theorem the polynomial \( H'(z) - \lambda G'(z) \) has all its zeros in \( |z| < 1 \), which implies
\[
|H'(z)| \leq |G'(z)| \text{ for } |z| \geq 1. \tag{3.2}
\]

Substituting for \( H(z) \) and \( G(z) \) in (3.2), we get
\[
k |p'(kz)| \leq k^{n-1} |q'(z/k)|. \tag{3.3}
\]

Since \( a_1 = a_2 = \cdots = a_{\mu-1} = 0 \), from (3.3), we get
\[
k^{\mu} \sum_{\nu=\mu}^{n} \nu a_{\nu}(kz)^{\nu-\mu} \leq k^{n-1} |q'(z/k)| \text{ for } |z| \geq 1. \tag{3.4}
\]

In fact (3.4) holds for \( |z| = 1 \). But \( q'(z/k) \neq 0 \) in \( |z| > 1 \), by the maximum modulus principle it also holds for \( |z| > 1 \). Taking \( kz \) instead of \( z \) in (3.4), we have
\[
k^{\mu} \sum_{\nu=\mu}^{n} \nu a_{\nu}(k^2 z)^{\nu-\mu} \leq k^{n-1} |q'(z)| \text{ for } |z| \geq 1/k.
\]

In particular,
\[
k^{\mu} \sum_{\nu=\mu}^{n} \nu a_{\nu}(k^2 z)^{\nu-\mu} \leq k^{n-1} |q'(z)| \text{ for } |z| = 1,
\]
this implies
\[
k^{2-\mu} \sum_{\nu=\mu}^{n} \nu a_{\nu}(k^2 z)^{\nu-1} \leq k^{n-1} |q'(z)| \text{ for } |z| = 1.
\]
Consequently
\[ k^{3-\mu} |p'(k^2 z)| \leq k^n |q'(z)| \text{ for } |z| = 1, \]
which gives with the help of Lemma 2
\[ k^{3-\mu} |p'(k^2 z)| + k^n |p'(z)| \leq nk^n M(p, 1) \text{ for } |z| = 1. \]

This, by hypothesis, implies that
\[ k^{3-\mu} \max_{|z|=1} |p'(k^2 z)| + k^n \max_{|z|=1} |p'(z)| \leq nk^n M(p, 1). \] (3.5)

Now \( p'(z) \) is a polynomial of degree \( n-1 \) and \( k \geq 1 \), therefore, by (1.1), it follows that
\[ \max_{|z|=1} |p'(k^2 z)| \leq k^{2(n-1)} \max_{|z|=1} |p'(z)|. \]

Using this in (3.5) we get
\[ (k^{2n-\mu+1} + 1)k^2 \max_{|z|=1} |p'(k^2 z)| \leq nk^{2n} M(p, 1). \]

Applying (1.1) again to the polynomial \( p'(k^2 z) \), we obtain for all \( t \geq 1 \) and \( 0 \leq \theta < 2\pi \)
\[ k^2 \left| p'(k^2 t e^{i\theta}) \right| \leq \frac{nk^{2n} t^{n-1}}{1 + k^{n-\mu+1}} M(p, 1). \] (3.6)

Now for each \( \theta, 0 \leq \theta < 2\pi \) and \( R \geq 1 \), we have
\[ p(k^2 Re^{i\theta}) - p(k^2 e^{i\theta}) = \int_1^R k^2 e^{i\theta} p'(k^2 t e^{i\theta}) dt. \]

This gives with the help of (3.6)
\[
\left| p(k^2 Re^{i\theta}) - p(k^2 e^{i\theta}) \right| \leq \int_1^R k^2 \left| p'(k^2 t e^{i\theta}) \right| dt \\
\leq \frac{k^{2n} t^{n-1}}{1 + k^{n-\mu+1}} M(p, 1) \\
= \frac{k^{2n}(R^n - 1)}{1 + k^{n-\mu+1}} M(p, 1). \] (3.7)

Since by (3.1), we have in particular
\[ |H(kz)| \leq |G(kz)| \text{ for } |z| = 1, \]
this implies
\[ \left| p(k^2 e^{i\theta}) \right| \leq k^n \left| q(e^{i\theta}) \right| = k^n \left| p(e^{i\theta}) \right|. \]
it follows from (3.7) that for each \( \theta \), \( 0 \leq \theta < 2\pi \) and \( R \geq 1 \),
\[
\left| p(k^2 \text{Re}^i\theta) \right| \leq \left\{ \frac{k^{2n}(R^n - 1)}{1 + k^{n-\mu+1}} + k^n \right\} M(p, 1) = \frac{k^{2n}R^n - k^{2n} + k^n + k^{2n-\mu+1}}{1 + k^{n-\mu+1}} M(p, 1)
\]
This implies
\[
\max_{|z|=R \geq k^2} |p(z)| \leq \frac{R^n + k^n(1 + k^{n-\mu+1}) - k^{2n}}{1 + k^{n-\mu+1}} M(p, 1),
\]
which completes the proof of Theorem 1. \( \square \)

**Proof of Theorem 2.** If \( p(z) \neq 0 \) in \( |z| < k, k < 1 \) and \( 0 < t < 1, k < t \), then \( P(z) = p(tz) \) has no zero in \( |z| < k/t, k/t < 1 \). Applying Lemma 1 to the polynomial \( P(z) \), we get
\[
M(P', 1) \leq \frac{n}{1 + (k/t)^{n-\mu+1}} M(P, 1),
\]
which is equivalent to
\[
M(p', t) \leq \frac{nt^{n-\mu}}{k^{n-\mu+1} + t^{n-\mu+1}} M(p, t). \quad (3.8)
\]
For \( 0 < r < \lambda \leq 1 \) and \( 0 < \theta \leq 2\pi \), we have
\[
p(\lambda e^{i\theta}) - p(re^{i\theta}) = \int_r^\lambda e^{i\theta} p'(te^{i\theta}) dt.
\]
This implies
\[
\left| p(\lambda e^{i\theta}) - p(re^{i\theta}) \right| \leq \int_r^\lambda \left| p'(te^{i\theta}) \right| dt,
\]
which gives
\[
\left| p(\lambda e^{i\theta}) \right| \leq \left| p(re^{i\theta}) \right| + \int_r^\lambda \left| p'(te^{i\theta}) \right| dt,
\]
which further implies
\[
M(p, \lambda) \leq M(p, r) + \int_r^\lambda M(p', t) dt.
\]
Combining the above inequality with (3.8), we get
\[
M(p, \lambda) \leq M(p, r) + \int_r^\lambda \frac{nt^{n-\mu}}{k^{n-\mu+1} + t^{n-\mu+1}} M(p, t) dt. \quad (3.9)
\]
If we choose
\[
\phi(\lambda) = M(p, r) + \int_r^\lambda \frac{nt^{n-\mu}}{k^{n-\mu+1} + t^{n-\mu+1}} M(p, t) dt,
\]
then
\[ \phi'(\lambda) = \frac{n\lambda^{n-\mu}}{k^{n-\mu+1} + \lambda^{n-\mu+1}} M(p, \lambda), \]
and inequality (3.9), gives
\[ \phi'(\lambda) - \frac{n\lambda^{n-\mu}}{k^{n-\mu+1} + \lambda^{n-\mu+1}} \phi(\lambda) \leq 0. \]
Multiplying the above inequality by \((k^{n-\mu+1} + \lambda^{n-\mu+1})^{-\frac{n}{n-\mu+1}}\), we get
\[ \frac{d}{d\lambda} \left\{ (k^{n-\mu+1} + \lambda^{n-\mu+1})^{-\frac{n}{n-\mu+1}} \phi(\lambda) \right\} \leq 0, \]
which implies that \((k^{n-\mu+1} + \lambda^{n-\mu+1})^{-\frac{n}{n-\mu+1}} \phi(\lambda)\) is a non-increasing function of \(\lambda\) in \((0,1)\). Therefore for \(0 < k < r < \lambda \leq 1\), we have
\[ \phi(r) \geq \left( \frac{k^{n-\mu+1} + r^{n-\mu+1}}{k^{n-\mu+1} + \lambda^{n-\mu+1}} \right)^{-\frac{n}{n-\mu+1}} \phi(\lambda). \]
Now since \(\phi(r) = M(p, r)\) and \(\phi(\lambda) \geq M(p, \lambda)\), we get
\[ M(p, r) \geq \left( \frac{k^{n-\mu+1} + r^{n-\mu+1}}{k^{n-\mu+1} + \lambda^{n-\mu+1}} \right)^{-\frac{n}{n-\mu+1}} M(p, \lambda). \]
Which completes the proof of Theorem 2. \(\Box\)

References


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