SARAJEVO JOURNAL OF MATHEMATICS
Vol.7 (19) (2011), 11–17

This paper is withdrawn by the request of the Editorial Board on the basis of the paper entitled "Remarks on two recent results about polynomials with prescribed zeros" by M. A. Qazi published in Sarajevo J. Math., Vol.9, No.2 (2013), 217–220. The Editorial Board are expressing their gratitude to Professor M. A. Qazi.

GROWTH OF THE MAXIMUM MODULUS OF POLYNOMIALS WITH PRESCRIBED ZEROS

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Abstract. If $p(z)$ be a polynomial of degree n which does not vanish in the disk $|z| < k$, then for $k = 1$, it is well known that

$$\max_{|z|=r<1} |p(z)| \geq \left(\frac{r+1}{2}\right)^n \max_{|z|=1} |p(z)|,$$

and

$$\max_{|z|=R>1} |p(z)| \leq \frac{R^n+1}{2} \max_{|z|=1} |p(z)|.$$

In this paper, we consider a class of lacunary polynomials and present certain generalizations as well as improvements of the above inequalities for the two cases $k \geq 1$ and $k < 1$.

1. Introduction and statement of results

Let $p(z)$ be a polynomial of degree n and let $M(p, R) = \max_{|z|=R} |p(z)|$. Then it is a simple consequence of maximum modulus principle (for reference see [6, vol. I, p. 137, prob. III, 269]) that

$$M(p, R) \leq R^n M(p, 1) \text{ for } R \geq 1. \tag{1.1}$$

The result is best possible and equality holds for $p(z) = \alpha z^n$, where $|\alpha| = 1$.

It was shown by Ankeny and Rivlin [1] that if $p(z) \neq 0$ in $|z| < 1$, then inequality (1.1) can be replaced by

$$M(p, R) \leq \frac{R^n + 1}{2} M(p, 1) \text{ for } R \geq 1. \tag{1.2}$$

2000 *Mathematics Subject Classification.* 30A10, 30C10, 30C15.
Key words and phrases. Polynomials, maximum modulus, zeros, extremal problems.

The above inequality is best possible and equality holds for $p(z) = \alpha + \beta z^n$, where $|\alpha| = |\beta|$.

For the polynomials of degree n and $r < 1$, applying (1.1) to the polynomial $p(rz)$, taking $R = 1/r$, gives $\left|p(e^{i\theta})\right| \le r^{-n} \max_{|z|=1} |p(rz)|$ for $0 \le \theta < 2\pi$. Hence,

$$M(p,r) \ge r^n M(p,1) \text{ for } 0 \le r < 1, \tag{1.3}$$

where equality holds if and only if $p(z) = \alpha z^n$.

If $p(z) \ne 0$ in $|z| < 1$, then Rivlin [7] proved

$$M(p,r) \ge \left(\frac{r+1}{2}\right)^n M(p,1) \quad \text{for} \quad 0 \le r < 1. \tag{1.4}$$

Here equality is attained if $p(z) = \alpha(z-\beta)^n$, $|\beta| = 1$.

Inequality (1.2) was generalized by Aziz [2] for the polynomials having no zeros in $|z| < k$, $k \ge 1$ and proved the following result.

Theorem A. *If $p(z)$ is a polynomial of degree n, which does not vanish in $|z| < k$, $k \ge 1$, then*

$$M(p,R) \le \frac{R^n + k^n}{1 + k^n} M(p,1) \textit{ for } R > k^2, \tag{1.5}$$

provided $\left|p'(k^2 z)\right|$ and $|p'(z)|$ attains the maximum at the same point on $|z| = 1$. The result is best possible and equality holds for $p(z) = z^n + k^n$.

On the other hand for the polynomials not vanishing in $|z| < k$, $k < 1$, the following inequality has been recently proved by Dewan and Hans [3].

Theorem B. *If $p(z)$ is a polynomial of degree n, which does not vanish in $|z| < k$, $k < 1$, then*

$$M(p,r) \ge \frac{r^n + k^n}{1 + k^n} M(p,1) \textit{ for } 0 < k < r < 1, \tag{1.6}$$

provided $|p'(z)|$ and $|q'(z)|$ attains the maximum at the same point on $|z| = 1$, where $q(z) = z^n \overline{p(1/\overline{z})}$. The result is best possible and equality holds for $p(z) = z^n + k^n$.

In this paper, we firstly generalize Theorem A to the lacunary type of polynomials $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu < n$, by proving the following result.

Theorem 1. *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu < n$ is a polynomial of degree n, which does not vanish in $|z| < k$, $k \ge 1$, then*

$$M(p,R) \le \frac{R^n + k^n(1 + k^{n-\mu+1}) - k^{2n}}{1 + k^{n-\mu+1}} M(p,1) \textit{ for } R > k^2, \tag{1.7}$$

provided $\left|p'(k^2z)\right|$ *and* $|p'(z)|$ *attains the maximum at the same point on* $|z| = 1$.

Remark 1. For $\mu = 1$, Theorem 1 reduces to inequality (1.5) due to Aziz [2].

Our next result is an extension as well as generalization of Theorem B. More precisely, we prove

Theorem 2. *If* $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu < n$ *is a polynomial of degree* n, *which does not vanish in* $|z| < k$, $k < 1$, *then*

$$M(p,r) \geq \left(\frac{r^{n-\mu+1} + k^{n-\mu+1}}{\lambda^{n-\mu+1} + k^{n-\mu+1}}\right)^{\frac{n}{n-\mu+1}} M(p,\lambda) \quad \text{for} \quad 0 < k < r < \lambda \leq 1, \tag{1.8}$$

provided $|p'(z)|$ *and* $|q'(z)|$ *attain maximums at the same point on* $|z| = 1$, *where* $q(z) = z^n \overline{p(1/\overline{z})}$. *The result is best possible and equality holds for* $p(z) = \left(z^{n-\mu+1} + k^{n-\mu+1}\right)^{\frac{n}{n-\mu+1}}$.

If we take $\lambda = 1$ in Theorem 2, then inequality (1.8) reduces to the following result.

Corollary 1. *If* $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu < n$ *is a polynomial of degree* n, *which does not vanish in* $|z| < k$, $k < 1$, *then*

$$M(p,r) \geq \left(\frac{r^{n-\mu+1} + k^{n-\mu+1}}{1 + k^{n-\mu+1}}\right)^{\frac{n}{n-\mu+1}} M(p,1) \quad \text{for} \quad 0 < k < r \leq 1, \tag{1.9}$$

provided $|p'(z)|$ *and* $|q'(z)|$ *attain maximums at the same point on* $|z| = 1$, *where* $q(z) = z^n \overline{p(1/\overline{z})}$. *The result is best possible with equality holds for* $p(z) = \left(z^{n-\mu+1} + k^{n-\mu+1}\right)^{\frac{n}{n-\mu+1}}$.

Remark 2. For $\mu = 1$, Corollary 1 reduces to inequality (1.6) due to Dewan and Hans [3].

2. Lemmas

We need the following lemmas for the proofs of these theorems.

Lemma 1. *Let* $p(z) = a_n z^n + \sum_{\nu=\mu}^{n} a_{n-\nu} z^{n-\nu}$, $1 \leq \mu < n$ *is a polynomial of degree* n, *having no zero in* $|z| < k$, $k \leq 1$ *and* $q(z) = z^n \overline{p(1/\overline{z})}$. *If* $|p'(z)|$ *and* $|q'(z)|$ *attain maximums at the same point on* $|z| = 1$, *then*

$$M(p',1) \leq \frac{n}{1 + k^{n-\mu+1}} M(p,1). \tag{2.1}$$

The above lemma is due to Dewan and Hans [4].

Lemma 2. *If $p(z)$ is a polynomial of degree n, then for $|z| = 1$*

$$\left|p'(z)\right| + \left|q'(z)\right| \leq nM(p,1), \tag{2.2}$$

where $q(z) = z^n\overline{p(1/\overline{z})}$.

The above lemma is a special case of a result due to Govil and Rahman [5].

3. Proofs of the theorems

Proof of Theorem 1. Since $p(z)$ has all its zeros in $|z| \geq k \geq 1$, it follows that the polynomial $H(z) = p(kz)$ has all its zeros in $|z| \geq 1$. Now if $q(z) = z^n\overline{p(1/\overline{z})}$, then the polynomial $G(z) \equiv z^n\overline{H(1/\overline{z})} \equiv z^n\overline{p(k/\overline{z})} \equiv k^n q(z/k)$ has all its zeros in $|z| < 1$. Moreover $|H(z)| = |G(z)|$ for $|z| = 1$, it follows by the maximum modulus principle that

$$|H(z)| \leq |G(z)| \text{ for } |z| \geq 1. \tag{3.1}$$

Hence for every complex number λ with $|\lambda| > 1$, it follows by Rouché's theorem that the polynomial $H(z) - \lambda G(z)$ has all its zeros in $|z| < 1$. By the Gauss Lucas theorem the polynomial $H'(z) - \lambda G'(z)$ has all its zeros in $|z| < 1$, which implies

$$\left|H'(z)\right| \leq \left|G'(z)\right| \text{ for } |z| \geq 1. \tag{3.2}$$

Substituting for $H(z)$ and $G(z)$ in (3.2), we get

$$k\left|p'(kz)\right| \leq k^{n-1}\left|q'(z/k)\right|. \tag{3.3}$$

Since $a_1 = a_2 = \cdots = a_{\mu-1} = 0$, from (3.3), we get

$$k^{\mu}\left|\sum_{\nu=\mu}^{n} \nu a_{\nu}(kz)^{\nu-\mu}\right| \leq k^{n-1}\left|q'(z/k)\right| \text{ for } |z| \geq 1. \tag{3.4}$$

In fact (3.4) holds for $|z| = 1$. But $q'(z/k) \neq 0$ in $|z| > 1$, by the maximum modulus principle it also holds for $|z| > 1$. Taking kz instead of z in (3.4), we have

$$k^{\mu}\left|\sum_{\nu=\mu}^{n} \nu a_{\nu}(k^2z)^{\nu-\mu}\right| \leq k^{n-1}\left|q'(z)\right| \text{ for } |z| \geq 1/k.$$

In particular,

$$k^{\mu}\left|\sum_{\nu=\mu}^{n} \nu a_{\nu}(k^2z)^{\nu-\mu}\right| \leq k^{n-1}\left|q'(z)\right| \text{ for } |z| = 1,$$

this implies

$$k^{2-\mu}\left|\sum_{\nu=\mu}^{n} \nu a_{\nu}(k^2z)^{\nu-1}\right| \leq k^{n-1}\left|q'(z)\right| \text{ for } |z| = 1.$$

Consequently

$$k^{3-\mu}\left|p'(k^2z)\right| \le k^n\left|q'(z)\right| \text{ for } |z|=1,$$

which gives with the help of Lemma 2

$$k^{3-\mu}\left|p'(k^2z)\right| + k^n\left|p'(z)\right| \le nk^nM(p,1) \text{ for } |z|=1.$$

This, by hypothesis, implies that

$$k^{3-\mu}\max_{|z|=1}\left|p'(k^2z)\right| + k^n\max_{|z|=1}\left|p'(z)\right| \le nk^nM(p,1). \tag{3.5}$$

Now $p'(z)$ is a polynomial of degree $n-1$ and $k \ge 1$, therefore, by (1.1), it follows that

$$\max_{|z|=1}\left|p'(k^2z)\right| \le k^{2(n-1)}\max_{|z|=1}\left|p'(z)\right|.$$

Using this in (3.5) we get

$$(k^{n-\mu+1}+1)k^2\max_{|z|=1}\left|p'(k^2z)\right| \le nk^{2n}M(p,1).$$

Applying (1.1) again to the polynomial $p'(k^2z)$, we obtain for all $t \ge 1$ and $0 \le \theta < 2\pi$

$$k^2\left|p'(k^2te^{i\theta})\right| \le \frac{nk^{2n}t^{n-1}}{1+k^{n-\mu+1}}M(p,1). \tag{3.6}$$

Now for each θ, $0 \le \theta < 2\pi$ and $R \ge 1$, we have

$$p(k^2Re^{i\theta}) - p(k^2e^{i\theta}) = \int_1^R k^2e^{i\theta}p'(k^2te^{i\theta})dt.$$

This gives with the help of (3.6)

$$\begin{aligned}\left|p(k^2Re^{i\theta}) - p(k^2e^{i\theta})\right| &\le \int_1^R k^2\left|p'(k^2te^{i\theta})\right|dt\\ &\le \frac{k^{2n}}{1+k^{n-\mu+1}}\left\{\int_1^R nt^{n-1}dt\right\}M(p,1)\\ &= \frac{k^{2n}(R^n-1)}{1+k^{n-\mu+1}}M(p,1). \qquad (3.7)\end{aligned}$$

Since by (3.1), we have in particular

$$|H(kz)| \le |G(kz)| \text{ for } |z|=1,$$

this implies

$$\left|p(k^2e^{i\theta})\right| \le k^n\left|q(e^{i\theta})\right| = k^n\left|p(e^{i\theta})\right|,$$

it follows from (3.7) that for each θ, $0 \le \theta < 2\pi$ and $R \ge 1$,

$$\left|p(k^2Re^{i\theta})\right| \le \left\{\frac{k^{2n}(R^n-1)}{1+k^{n-\mu+1}} + k^n\right\} M(p,1)$$
$$= \frac{k^{2n}R^n - k^{2n} + k^n + k^{2n-\mu+1}}{1+k^{n-\mu+1}} M(p,1).$$

This implies

$$\max_{|z|=R\ge k^2} |p(z)| \le \frac{R^n + k^n(1+k^{n-\mu+1}) - k^{2n}}{1+k^{n-\mu+1}} M(p,1),$$

which completes the proof of Theorem 1. □

Proof of Theorem 2. If $p(z) \ne 0$ in $|z| < k$, $k < 1$ and $0 < t < 1$, $k < t$, then $P(z) = p(tz)$ has no zero in $|z| < k/t$, $k/t < 1$. Applying Lemma 1 to the polynomial $P(z)$, we get

$$M(P',1) \le \frac{n}{1+(k/t)^{n-\mu+1}} M(P,1),$$

which is equivalent to

$$M(p',t) \le \frac{nt^{n-\mu}}{k^{n-\mu+1}+t^{n-\mu+1}} M(p,t). \tag{3.8}$$

For $0 < r < \lambda \le 1$ and $0 < \theta \le 2\pi$, we have

$$p(\lambda e^{i\theta}) - p(re^{i\theta}) = \int_r^\lambda e^{i\theta} p'(te^{i\theta})dt.$$

This implies

$$\left|p(\lambda e^{i\theta}) - p(re^{i\theta})\right| \le \int_r^\lambda \left|p'(te^{i\theta})\right| dt,$$

which gives

$$\left|p(\lambda e^{i\theta})\right| \le \left|p(re^{i\theta})\right| + \int_r^\lambda \left|p'(te^{i\theta})\right| dt,$$

which further implies

$$M(p,\lambda) \le M(p,r) + \int_r^\lambda M(p',t)dt.$$

Combining the above inequality with (3.8), we get

$$M(p,\lambda) \le M(p,r) + \int_r^\lambda \frac{nt^{n-\mu}}{k^{n-\mu+1}+t^{n-\mu+1}} M(p,t)dt. \tag{3.9}$$

If we choose

$$\phi(\lambda) = M(p,r) + \int_r^\lambda \frac{nt^{n-\mu}}{k^{n-\mu+1}+t^{n-\mu+1}} M(p,t)dt,$$

then

$$\phi'(\lambda) = \frac{n\lambda^{n-\mu}}{k^{n-\mu+1} + \lambda^{n-\mu+1}} M(p, \lambda),$$

and inequality (3.9), gives

$$\phi'(\lambda) - \frac{n\lambda^{n-\mu}}{k^{n-\mu+1} + \lambda^{n-\mu+1}} \phi(\lambda) \leq 0.$$

Multiplying the above inequality by $(k^{n-\mu+1} + \lambda^{n-\mu+1})^{-\frac{n}{n-\mu+1}}$, we get

$$\frac{d}{d\lambda} \left\{ (k^{n-\mu+1} + \lambda^{n-\mu+1})^{-\frac{n}{n-\mu+1}} \phi(\lambda) \right\} \leq 0,$$

which implies that $(k^{n-\mu+1} + \lambda^{n-\mu+1})^{-\frac{n}{n-\mu+1}} \phi(\lambda)$ is a non-increasing function of λ in (0,1). Therefore for $0 < k < r < \lambda \leq 1$, we have

$$\phi(r) \geq \left(\frac{k^{n-\mu+1} + r^{n-\mu+1}}{k^{n-\mu+1} + \lambda^{n-\mu+1}} \right)^{\frac{n}{n-\mu+1}} \phi(\lambda).$$

Now since $\phi(r) = M(p, r)$ and $\phi(\lambda) \geq M(p, \lambda)$, we get

$$M(p, r) \geq \left(\frac{k^{n-\mu+1} + r^{n-\mu+1}}{k^{n-\mu+1} + \lambda^{n-\mu+1}} \right)^{\frac{n}{n-\mu+1}} M(p, \lambda).$$

Which completes the proof of Theorem 2. □

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(Received: May 17, 2010)
(Revised: August 25, 2010)

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