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ON THE BOUNDEDNESS OF SOLUTIONS OF A NON-AUTONOMOUS DIFFERENTIAL EQUATION OF SECOND ORDER

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ABSTRACT. We study the boundedness of the solutions to a non-autonomous differential equation of second order. With this work, we improve some stability results in the literature of boundedness of the solutions. We give two examples to illustrate the theoretical analysis in this work.

1. INTRODUCTION AND MAIN RESULTS

In 1974, Baker [2] considered the following second order nonlinear differential equation

$$(r(t)u')' + \phi(t, u, u')u' + p(t)f(u) = 0. \tag{1}$$

This equation can be written as the system

$$\begin{aligned} x' &= y, \\ y' &= -\frac{1}{r(t)}[r'(t)y + \phi(t, x, y)y + p(t)f(x)], \end{aligned} \tag{2}$$

where we assume that $f : \Re \to \Re$, $\phi : I \times \Re^2 \to \Re$, and $p : I \to \Re$ are continuous, and $r : I \to \Re$ is differentiable, and $r(t) > 0$ for all $t \in I$, $I = [0, \infty)$. We assume in what follows that $p : I \to \Re$ is differentiable. In [2], the author established two theorems which include some sufficient conditions and guarantee that the zero solution of (2) is uniformly stable.

In this paper, instead of Eq. (1), we consider the following non-autonomous second order differential equation

$$(r(t)x')' + \phi(t, x, x')x' + p(t)f(x) = q(t, x, x'). \tag{3}$$

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We write Eq. (3) as the system

$$\begin{aligned} x' &= y, \\ y' &= -\frac{1}{r(t)}[r'(t)y + \phi(t,x,y)y + p(t)f(x)] + \frac{1}{r(t)}q(t,x,y), \end{aligned} \tag{4}$$

where $q : I \times \Re^2 \to \Re$ is continuous.

It should be noted that in [2] Baker also derived sufficient conditions for the global attractivity of the zero solution of (4).

This work is motivated by the paper of Baker [2]. It is worth mentioning that the author in [2] discussed the stability of the solutions of Eq. (1) without giving any example on the topic. We here establish two new results on the boundedness of the solutions of Eq. (3) and also give two examples to show the applicability of our results. On the other hand, in particular, for some results performed on the boundedness of solutions of some certain second order nonlinear differential equations, the reader can refer to the papers of Cheng and Xu [3], Kato ([4], [5]), Malyseva [6], Muresan [7], Nápoles Valdés [8], Saker [9], Tunç ([10], [11], [12], [13]), C. Tunç and E. Tunç [14], Zhao [15], Wang [16] and the references cited in these papers. It should be noted that Eq. (3) and the assumptions that will be established here are different from those in the papers mentioned above and literature.

Our first main problem is the following theorem.

Theorem 1. *In addition to the basic assumptions imposed on the functions* r, ϕ, p, f *and* q, *we assume that the following conditions hold:*

i)

$$f(0) = 0, \int_0^x f(s)ds > 0 \textit{ for all } x \in \Re, (x \neq 0),$$
$$\int_0^x f(s)ds \to \infty \textit{ as } |x| \to \infty,$$
$$\phi(t,x,y) \geq 0 \textit{ for all } (t,x,y) \in I \times \Re^2,$$

ii) pr *is monotone on* I *and there are positive constants* p_0, p_1, r_0 *and* r_1 *such that*

$$p_0 \leq p(t) \leq p_1 \textit{ and } r_0 \leq r(t) \leq r_1 \textit{ for all } t \geq 0,$$
$$|q(t,x,y)| \leq |e(t)|, \int_0^\infty |e(s)|\, ds < \infty.$$

Then, every solution of Eq. (3), *together with its derivative, is bounded as* $t \to \infty$.

Proof. We first assume that pr is monotone increasing on I. Define the Liapunov function

$$V_1(t,x,y) = 2\int_0^x f(s)ds + \frac{r(t)}{p(t)}y^2.$$

In view of the assumptions of Theorem 1, it follows that

$$V_1(t,0,0) = 0$$

and

$$V_1(t,x,y) \geq 2\int_0^x f(s)ds + \frac{r_0}{p_1}y^2 > 0$$

for all $x \neq 0$ and $y \neq 0$.

Let $(x,y) = (x(t),y(t))$ be a solution of (4). Along with this solution, calculating the time derivative of Liapunov function $V_1(t,x,y)$, we find

$$\begin{aligned} V_1'(t,x(t),y(t)) &= \frac{r'(t)p(t) - r(t)p'(t)}{p^2(t)}y^2(t) - \frac{2r'(t)}{p(t)}y^2(t) \\ &\quad - \frac{2}{p(t)}\phi(t,x(t),y(t))y^2(t) + \frac{2}{p(t)}y(t)q(t,x(t),y(t)) \\ &= -\frac{[p(t)r(t)]'}{p^2(t)}y^2(t) - \frac{2}{p(t)}\phi(t,x(t),y(t))y^2(t) \\ &\quad + \frac{2}{p(t)}y(t)q(t,x(t),y(t)). \end{aligned}$$

In view of the assumptions that pr is monotone increasing, $\phi(t,x,y) \geq 0$, $|q(t,x,y)| \leq |e(t)|$, $0 < p_0 \leq p(t)$ and the inequality $2|y| \leq 1 + y^2$, it follows that

$$V_1'(t,x(t),y(t)) \leq \frac{2}{p(t)}\,|y(t)|\ \ |q(t,x(t),y(t))| \leq \frac{1}{p_0}\ |e(t)| + \frac{1}{p_0}\ |e(t)|\,y^2(t).$$

On the other hand, we have $\frac{p_1}{r_0}V_1(t,x,y) \geq y^2$. Hence, the last two inequalities imply

$$V_1'(t,x(t),y(t)) \leq \frac{1}{p_0}\ |e(t)| + \frac{p_1}{p_0 r_0}\ |e(t)|\ V_1(t,x(t),y(t)).$$

Integrating this inequality from 0 to t, we get

$$\begin{aligned} V_1(t,x(t),y(t)) \leq V_1(0,x_0,y_0) &+ \frac{1}{p_0}\int_0^t |e(s)|\ ds \\ &+ \frac{p_1}{p_0 r_0}\int_0^t |e(s)|\ V_1(s,x(s),y(s))ds. \end{aligned}$$

Using the Gronwall-Reid-Bellman inequality (see Ahmad and Rama Mohana Rao [1]), one can obtain

$$V_1(t,x(t),y(t)) \le \{V_1(0,x_0,y_0) + \frac{1}{p_0}\int_0^t |e(s)| \ ds\} \exp(\frac{p_1}{p_0 r_0}\int_0^t |e(s)| \, ds).$$

Since the integrals in the last inequality converge when $t \to \infty$ by the assumption $\int_0^\infty |e(s)| \, ds < \infty$, we can conclude that $V_1(t,x(t),y(t))$ is bounded for all $t \ge 0$. This shows that every solution of Eq. (3) together with its derivative is bounded as $t \to \infty$.

Now we assume that pr is monotone decreasing on I. For this case, define the Liapunov function

$$V_2(t,x,y) = 2p(t)r(t)\int_0^x f(s)ds + r^2(t)y^2.$$

In view of the assumptions of Theorem 1, it follows that

$$V_2(t,0,0) = 0$$

and

$$V_2(t,x,y) \ge 2p_0 r_0 \int_0^x f(s)ds + r_0^2 y^2 > 0$$

for all $x \ne 0$ and $y \ne 0$.

Let $(x,y) = (x(t),y(t))$ be a solution of (4). By a direct calculation using this solution, the time derivative of Liapunov function $V_2(t,x,y)$ yields

$$\begin{aligned} V_2'(t,x(t),y(t)) &= 2[p(t)r(t)]' \int_0^x f(s)ds \\ &\quad - 2r(t)\phi(t,x,y)y^2 + 2r(t)q(t,x(t),y(t))y(t). \end{aligned}$$

Subject to the assumptions of $[p(t)r(t)]' \le 0$, $\phi(t,x,y) \ge 0$, $|q(t,x,y)| \le |e(t)|$, $0 < r(t) \le r_1$ and the inequality $2|y| \le 1 + y^2$, we get

$$\begin{aligned} V_2'(t,x(t),y(t)) &\le 2r(t)\,|y(t)| \ \ |q(t,x(t),y(t))| \\ &\le r_1 \ |e(t)| + r_1 \ |e(t)|\, y^2(t). \end{aligned}$$

The remaining part of the proof is similar to the proof of first part of Theorem 1. Therefore, we omit the details. The proof of Theorem 1 is now completed. □

Example 1. Consider the following non-autonomous second order differential equation

$$\left(3 - \frac{1}{1+t^2}\right)x'' + \frac{2t}{(1+t^2)^2}x' + \left(3 + \frac{1}{1+t^2}\right)(2x+4x^3) = \frac{1}{1+t^2+x^2+x'^2}.$$

We state this equation as the system

$$x' = y,$$
$$y' = -\frac{2t}{(1+t^2)(2+3t^2)}y - \frac{4+3t^2}{2+3t^2}(2x+4x^3) + \frac{1+t^2}{(1+t^2+x^2+y^2)(2+3t^2)}.$$

Hence, it follows the existence of the following expressions:

$$\begin{gathered}
r(t) = 3 - \frac{1}{1+t^2}, t \geq 0, \\
2 \leq 3 - \frac{1}{1+t^2} \leq 3, \\
r_0 = 2, r_1 = 3, \\
p(t) = 3 + \frac{1}{1+t^2}, \\
3 \leq 3 + \frac{1}{1+t^2} \leq 4, \\
p_0 = 3, p_1 = 4, \\
p(t)r(t) = \frac{(4+3t^2)(2+3t^2)}{(1+t^2)^2} = 9 - \frac{1}{(t^2+1)^2}, \\
[p(t)r(t)]' = \frac{4t}{(1+t^2)^3} \geq 0, t \geq 0,
\end{gathered}$$

that is, pr is monotone increasing on $I = [0, \infty)$,

$$\begin{gathered}
\phi(t,x,y) = 0, \\
f(x) = 2x + 4x^3, f(0) = 0, \\
\int_0^x f(s)ds = x^2 + x^4 > 0, (x \neq 0), (x^2 + x^4) \to \infty \text{ as } |x| \to \infty, \\
|q(t,x,y)| = \frac{1}{1+t^2+x^2+y^2} \leq \frac{1}{1+t^2} = |e(t)|, \\
\int_0^\infty |e(s)|\, ds = \int_0^\infty \frac{1}{1+s^2} ds = \frac{\pi}{2} < \infty.
\end{gathered}$$

It is also seen that

$$\begin{aligned}
V_1(t,x,y) &= 2(x^2+x^4) + \left(\frac{2+3t^2}{4+3t^2}\right) y^2 \\
&\geq 2(x^2+x^4) + \frac{1}{2}y^2 > 0
\end{aligned}$$

for all $x \neq 0$ and $y \neq 0$, $V_1(t,0,0) = 0$, and

$$\begin{aligned}\frac{d}{dt}V_1(t,x(t),y(t)) &= -\frac{4t}{(1+t^2)(4+3t^2)^2}y^2(t) + \frac{2(1+t^2)}{(4+3t^2)(1+t^2+x^2+y^2)}y(t)\\ &\leq \frac{2}{(4+3t^2)}\,|y(t)|\\ &\leq \frac{1}{4+3t^2} + \frac{1}{4+3t^2}y^2(t)\\ &\leq \frac{1}{4+3t^2} + \frac{2}{4+3t^2}V_1(t,x(t),y(t)).\end{aligned}$$

Integrating the last inequality on $[0,\infty)$ and using the Gronwall-Reid-Bellman inequality, one can conclude that $V_1(t,x(t),y(t))$ is bounded for all $t \geq 0$. This shows that every solution of the above equation, together with its derivative, is bounded as $t \to \infty$. In view of the discussion made above, it also follows that all the assumptions of Theorem 1 hold for the case pr is monotone increasing on $I = [0,\infty)$.

The second main result is the following theorem.

Theorem 2. *Suppose that all the assumptions of Theorem 1 hold except pr is monotone on I. If p and r are both monotone on I, then every solution of Eq.* (3) *together with its derivative, is bounded as $t \to \infty$.*

Proof. If both p and r are monotone increasing or both monotone decreasing on I, then Theorem 2 is a corollary of Theorem 1.

We now assume that $p'(t) \geq 0$ and $r'(t) \leq 0$ for all $t \in I$. Define the Liapunov function

$$V_3(t,x,y) = 2r(t)\int_0^x f(s)ds + \frac{1}{p(t)}[r(t)y]^2.$$

In view of the assumptions of Theorem 2, it follows that

$$V_3(t,0,0) = 0$$

and

$$V_3(t,x,y) \geq 2r_0\int_0^x f(s)ds + \frac{r_0^2}{p_1}y^2 > 0$$

for all $x \neq 0$ and $y \neq 0$.

Let $(x,y) = (x(t),y(t))$ be a solution of (4). Using this solution, the time derivative of Liapunov function $V_3(t,x,y)$ yields that

$$\begin{aligned}V_3'(t,x(t),y(t)) = 2r'(t)\int_0^x f(s)ds - \frac{p'(t)r^2(t)}{p^2(t)}y^2(t)\\ - \frac{2r(t)}{p(t)}\phi(t,x(t),y(t))y^2(t) + \frac{2r(t)}{p(t)}y(t)q(t,x(t),y(t)).\end{aligned}$$

Making use of the assumptions $p'(t) \geq 0$, $r'(t) \leq 0$, $\phi(t,x,y) \geq 0$, $|q(t,x,y)| \leq |e(t)|$, $0 < p_0 \leq p(t) \leq p_1$, $0 < r_0 \leq r(t) \leq r_1$ and the inequality $2|y| \leq 1 + y^2$, we have

$$\begin{aligned} V_3'(t,x(t),y(t)) &\leq \frac{2r_1}{p_0} |y(t)| \;\; |q(t,x(t),y(t))| \\ &\leq \frac{r_1}{p_0} \;\; |e(t)| + \frac{r_1}{p_0} \;\; |e(t)| \, y^2(t). \end{aligned}$$

On the other hand, it is clear that $\frac{p_1}{r_0^2} V_3(t,x,y) \geq y^2$. Hence, it follows that

$$V_3'(t,x(t),y(t)) \leq \frac{r_1}{p_0} \;\; |e(t)| + \frac{p_1 r_1}{p_0 r_0^2} \;\; |e(t)| \;\; V_3(t,x(t),y(t)).$$

Integrating the last inequality on $[0, \infty)$ and using the Gronwall-Reid-Bellman inequality (see Ahmad and Rama Mohana Rao [1]), we get

$$V_3(t,x(t),y(t)) \leq \{V_3(0,x_0,y_0) + \frac{r_1}{p_0} \int_0^t |e(s)| \;\; ds\} \exp(\frac{p_1 r_1}{p_0 r_0^2} \int_0^t |e(s)| \, ds).$$

By the hypotheses $\int_0^\infty |e(s)| \, ds < \infty$, it follows that the integrals in the last inequality converge when $t \to \infty$. Hence, $V_3(t,x(t),y(t))$ is bounded for all $t \geq 0$. Therefore, we conclude that every solution of Eq. (3) together with its derivative, is bounded as $t \to \infty$.

We finally assume that $p'(t) \leq 0$ and $r'(t) \geq 0$ for all $t \in I$. For this case, define Liapunov function

$$V_4(t,x,y) = 2p(t) \int_0^x f(s)ds + r(t)y^2.$$

In view of the assumptions of Theorem 2, we have

$$V_4(t,0,0) = 0$$

and

$$V_4(t,x,y) \geq 2p_0 \int_0^x f(s)ds + r_0 y^2 > 0$$

for all $x \neq 0$ and $y \neq 0$.

Let $(x,y) = (x(t),y(t))$ be a solution of (4). Using this solution, the time derivative of Liapunov function $V_4(t,x,y)$ implies that

$$\begin{aligned} V_4'(t,x(t),y(t)) = 2p'(t) \int_0^x & f(s)ds - r'(t)y^2 \\ & - 2\phi(t,x(t),y(t))y^2(t) + 2q(t,x(t),y(t))y(t). \end{aligned}$$

Subject to the assumptions of Theorem 2, one can obtain

$$\begin{aligned} V_4'(t,x(t),y(t)) &\leq 2\,|y(t)| \;\; |q(t,x(t),y(t))| \\ &\leq \;\; |e(t)| + \;\; |e(t)| \, y^2. \end{aligned}$$

It is also obvious that $\frac{1}{r_0}V_4(t,x,y) \geq y^2$. Hence, we get

$$V_4'(t,x(t),y(t)) \leq |e(t)| \ + r_0^{-1}\,|e(t)| \ V_4(t,x(t),y(t)).$$

Integrating this inequality on $[0,\ \infty)$, we get

$$V_4(t,x(t),y(t)) \leq V_4(0,x_0,y_0) + \int_0^t |e(s)| \ ds + r_0^{-1}\int_0^t |e(s)| \ V_4(s,x(s),y(s))ds.$$

Using the Gronwall-Reid-Bellman inequality (see Ahmad and Rama Mohana Rao [1]), we can obtain

$$V_4(t,x(t),y(t)) \leq \{V_4(0,x_0,y_0) + \int_0^t |e(s)| \ ds\}\exp(r_0^{-1}\int_0^t |e(s)|\,ds).$$

Since, by the hypotheses $\int_0^\infty |e(s)|\,ds < \infty$, the integrals in the last inequality converge at $t \to \infty$. Therefore, we can conclude $V_4(t,x(t),y(t))$ is bounded for all $t \geq 0$. Thus, every solution of Eq. (3) together with its derivative is bounded as $t \to \infty$. □

Example 2. Consider non-linear differential equation of second order

$$\left(3 + \frac{1}{1+t^2}\right)x'' - \frac{2t}{(1+t^2)^2}x' + 6[2-\exp(-t)]x^5 = \frac{\exp(t)}{1+\exp(2t)+x^2+x'^2}.$$

We state this equation as the following system

$$\begin{aligned} x' &= y, \\ y' &= -\frac{2t}{(1+t^2)(4+3t^2)}y - \frac{6[2-\exp(-t)](1+t^2)}{4+3t^2}x^5 \\ &\quad + \frac{(1+t^2)\exp(t)}{(1+\exp(2t)+x^2+y^2)(4+3t^2)}. \end{aligned}$$

Hence, we have the following:

$$r(t) = 3 + \frac{1}{1+t^2}, t \geq 0,$$
$$3 \leq 3 + \frac{1}{1+t^2} \leq 4,$$
$$r_0 = 3, r_1 = 4,$$
$$r'(t) = -\frac{2t}{(1+t^2)^2} \leq 0,$$
$$p(t) = 6[2-\exp(-t)],$$
$$1 \leq 2-\exp(-t) \leq 2,$$

$$p_0 = 6, p_1 = 12,$$
$$p'(t) = 6\exp(-t) \geq 0,$$
$$\phi(t,x,y) = 0,$$
$$f(x) = 6x^5,$$
$$f(0) = 0,$$
$$\int_0^x f(s)ds = x^6 > 0, (x \neq 0), x^6 \to \infty \text{ as } |x| \to \infty,$$
$$|q(t,x,y)| = \frac{\exp(t)}{(1+\exp(2t)+x^2+y^2)} \leq \frac{\exp(t)}{1+\exp(2t)} = |e(t)|,$$
$$\int_0^\infty |e(s)|\, ds = \int_0^\infty \frac{\exp(s)}{1+\exp(2s)} ds = \frac{\pi}{4} < \infty.$$

It is also seen that

$$\begin{aligned} V_3(t,x,y) &= 2\left(\frac{4+3t^2}{1+t^2}\right)x^6 + \frac{1}{6[2-\exp(-t)]}\left(\frac{4+3t^2}{1+t^2}\right)^2 y^2 \\ &\geq 6x^6 + \frac{9}{12}y^2 > 0 \end{aligned}$$

for all $x \neq 0$ and $y \neq 0$, $V_3(t,0,0) = 0$, and

$$\begin{aligned} \frac{d}{dt}V_3(t,x(t),y(t)) &= -\frac{4t}{(1+t^2)^2}x^6(t) - \frac{\exp(-t)}{6[2-\exp(-t)]^2}\left(\frac{4+3t^2}{1+t^2}\right)^2 y^2(t) \\ &\quad + \frac{2(4+3t^2)\exp(t)}{6[2-\exp(-t)](1+t^2)[1+\exp(2t)+x^2+y^2]}y(t) \\ &\leq \frac{3\exp(t)}{2+2\exp(2t)]}\,|y(t)| \\ &\leq \frac{3\exp(t)}{2+2\exp(2t)} + \frac{3\exp(t)}{2+2\exp(2t)}y^2(t) \\ &\leq \frac{2\exp(t)}{1+\exp(2t)} + \frac{2\exp(t)}{1+\exp(2t)}V_3(t,x(t),y(t)). \end{aligned}$$

by $|y| \leq 1+y^2$ and $y^2 \leq \frac{4}{3}V_3(t,x(t),y(t))$.

Integrating the last inequality from 0 to t and using the Gronwall-Reid-Bellman inequality (see Ahmad and Rama Mohana Rao [1]), it follows that

$$\begin{aligned} V_3(t,x(t),y(t)) \leq V_3(0,x_0,y_0) &+ 2\int_0^t \frac{\exp(s)}{1+\exp(2s)}\, ds \\ &+ 2\int_0^t \frac{\exp(s)}{1+\exp(2s)}\, V_3(s,x(s),y(s))ds \end{aligned}$$

$$\leq \left\{ V_3(0, x_0, y_0) + 2\int_0^t \frac{\exp(s)}{1+\exp(2s)}\, ds \right\} \exp\left\{ 2\int_0^t \frac{\exp(s)}{1+\exp(2s)}\, ds \right\}.$$

It is clear that the integrals in the above inequality converge as $t \to \infty$ since $\int_0^\infty \frac{\exp(s)}{1+\exp(2s)} ds = \frac{\pi}{4} < \infty$. Hence, we conclude that $V_3(t, x(t), y(t))$ is bounded for all $t \geq 0$. This verifies that every solution of the above equation together with its derivative is bounded as $t \to \infty$ for the case $p'(t) \geq 0$ and $r'(t) \leq 0$ on $I = [0, \infty)$.

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References

[1] S. Ahmad and M. Rama Mohana Rao, *Theory of Ordinary Differential Equations. With Applications in Biology and Engineering*, Affiliated East-West Press Pvt. Ltd., New Delhi, 1999.

[2] John W. Baker, *Stability properties of a second order damped and forced nonlinear differential equation*, SIAM J. Appl. Math., 27 (1974), 159–166.

[3] C. Cheng and J. Xu, *Boundedness of solutions for a class of second-order differential equations*, Nonlinear Anal., 68 (7) (2008), 1993–2004.

[4] J. Kato, *On a boundedness condition for solutions of a generalized Li énard equation*, J. Differ. Equations, 65 (2) (1986), 269–286.

[5] J. Kato, *A simple boundedness theorem for a Liénard equation with damping*, Ann. Pol. Math., 51 (1990), 183–188.

[6] I. A. Malyseva, *Boundedness of solutions of a Liénard differential equation*, Differ. Uravn., 15 (8) (1979), 1420–1426

[7] M. Mureşan, *Boundedness of solutions for Liénard type equations*, Mathematica, 40(63) (2) (1998), 243–257.

[8] Juan E. Nápoles Valdés, *Boundedness and global asymptotic stability of the forced Liénard equation* (Spanish), Rev. Union Mat. Argent., 41 (4) (2000), 47–59.

[9] S. H. Saker, *Boundedness of solutions of second-order forced nonlinear dynamic equations*, Rocky Mt. J. Math., 36 (6) (2006), 2027–2039.

[10] C. Tunç, *Some new stability and boundedness results on the solutions of the nonlinear vector differential equations of second order*, Iran. J. Sci. Technol. Trans. A Sci., 30 (2) (2006), 213–221.

[11] C. Tunç, *A new boundedness theorem for a class of second order differential equations*, Arab. J. Sci. Eng. Sect. A Sci., 33 (1) (2008), 83–92.

[12] C. Tunç, *Boundedness analysis for certain two-dimensional differential systems via a Lyapunov approach*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.), 53(101) (1) (2010), 1–8.

[13] C. Tunç, *A note on boundedness of solutions to a class of non-autonomous differential equations of second order*, Appl. Anal. Discrete Math., 4 (2010), 361–372.

[14] C. Tunç, and E. Tunç, *On the asymptotic behavior of solutions of certain second-order differential equations*, J. Franklin Inst., 344 (5) (2007), 391–398.

[15] L. Zhao, *Boundedness and convergence for the non-Liénard type differential equation*, Acta Math. Sci. Ser. B Eng. Ed., 27 (2) (2007), 338–346.

[16] L. Wang, *Boundedness of nonlinear second order differential equations,* Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal., 16 (2009), Differential Equations and Dynamical Systems, suppl. S1, 88–93.

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