SARAJEVO JOURNAL OF MATHEMATICS
Vol.7 (19) (2011), 31–38

ON THE TOPOLOGICAL NATURE OF THE STABLE SETS ASSOCIATED TO THE SECOND INVARIANT OF THE ORDER q STANDARD LYNESS' EQUATION

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ABSTRACT. We prove a conjecture asserted in a previous paper (see [2]) about order q Lyness difference equation in $\mathbb{R}_*^+$: $u_{n+q}\, u_n = a + u_{n+q-1} + \ldots + u_{n+1}$, with $a > 0$. It is known that the function on $\mathbb{R}_*^{+q}$ defined by

$$H(x) = \frac{(1+x_1+x_2)(1+x_2+x_3)\ldots}{x_1 \ldots x_q} \qquad \frac{(1+x_{q-1}+x_q)(a+x_1x_q+x_1+x_2+\ldots+x_q)}{x_1\ldots x_q}$$

is an *invariant* for this equation. It is conjectured in [2] (and proved for $q = 3$ only) that if $M > M_a := \min H$ is *sufficiently near* to M_a, then the set $S(M) := \{x \,|\, H(x) = M\}$ is homeomorphic to the sphere $\mathbb{S}^{q-1}$. Here we prove this conjecture for every $q \geq 3$, and deduce from it that if the equilibrium L of the map T associated to the Lyness' equation, where H attains its minimum, is for some q *the only critical point* of H, then the sets $S(M)$ are, for this q, homeomorphic to $\mathbb{S}^{q-1}$ for every $M > M_a$, and that this is the case for $q = 3$, 4 or 5.

1. Introduction

We recall some facts about the order q Lyness' difference equation in $\mathbb{R}_*^+$

$$u_{n+q} = \frac{a + u_{n+q-1} + u_{n+q-2} + \ldots + u_{n+1}}{u_n}, \;\; a > 0, \tag{1}$$

which is studied in some papers: see [1] and [9] for the case $q = 2$, [2] and [3] for $q = 3$, [2] and [4] for the general case for q; and see [6] for the origins and the interest of this difference equation.

2000 *Mathematics Subject Classification.* 37E, 39A10, 58F20.
Key words and phrases. Dynamical systems, difference equations, Lyness' sequence.

We associate to Lyness' difference equation (1) its dynamical system in $\mathbb{R}_*^{+q}$:

$$T(x_1, x_2, \ldots, x_q) := \Big(\frac{a + x_1 + \ldots + x_{q-1}}{x_q}, x_1, x_2, \ldots, x_{q-1}\Big), \qquad (2)$$

so that, if $M_n = (u_{n+q-1}, u_{n+q-2}, \ldots, u_n)$, then $T(M_n) = M_{n+1}$. The equilibrium of (2) is the point $L := (\ell, \ell, \ldots, \ell)$ where ℓ is the positive solution of the equation $X^2 - (q-1)X - a = 0$, and satisfies the inequality

$$\ell > q - 1. \qquad (3)$$

It is known that the function on $\mathbb{R}_*^{+q}$

$$G(x) := \frac{(1 + x_1)(1 + x_2)\ldots(1 + x_q)(a + x_1 + x_2 + \ldots + x_q)}{x_1 x_2 \ldots x_q} \qquad (4)$$

is an *invariant* for T, that is $G \circ T = G$ (see [2], [6], [7]).

The function G tends to $+\infty$ at the point of infinity of the locally compact set $\mathbb{R}_*^{+q}$, that is the sets $\widetilde{\Sigma}(K) := \{G \leq K\}$ are compact in $\mathbb{R}_*^{+q}$. It is proved in [2] that G attains its absolute minimum K_a at the point L, and that *L is the only critical point of G*. From this result (see [2]), for $K > K_a$ the invariant sets $\Sigma(K) := \{x \,|\, G(x) = K\}$ are differential manifolds which are homeomorphic to the $(q-1)$-dimensional sphere $\mathbb{S}^{q-1}$ (and $\Sigma(K)$ is the boundary of $\widetilde{\Sigma}(K)$ which is homeomorphic to the corresponding ball).

Now in [5] it is showed that for $q \geq 3$ there is a second invariant for T, the function

$$H(x) = \frac{(1 + x_1 + x_2)(1 + x_2 + x_3)\ldots}{x_1 \ldots x_q} \frac{(1 + x_{q-1} + x_q)(a + x_1 x_q + x_1 + x_2 + \ldots + x_q)}{x_1 \ldots x_q} \qquad (5)$$

This invariant tends to $+\infty$ at the point of infinity of $\mathbb{R}_*^{+q}$, and attains its absolute minimum M_a at the point L, but in [2] we proved the following conjecture only for $q = 3$, $q = 4$ and $q = 5$.

Conjecture 1. *The point L is the only critical point of H.*

In [2], we also thought that if $M > M_a$ the topological nature of the sets

$$S(M) := \{x \,|\, H(x) = M\} \text{ and } \widetilde{S}(M) := \{x \,|\, H(x) \leq M\} \qquad (6)$$

is the same as for the $\Sigma(K)$ and $\widetilde{\Sigma}(K)$ for $K > K_a$; in [2] this assertion was stated as "Conjecture 4":

Conjecture 4. *For $M > M_a$ the set $S(M)$ is homeomorphic to the sphere $\mathbb{S}^{q-1}$, and $S(M)$ is the boundary of the set $\widetilde{S}(M)$ which is homeomorphic to the ball.*

One can hope that Conjecture 4 follows from Conjecture 1 (which implies already that $S(M)$ is a differential manifold), but in [2] we proved only that this is true if we add another conjecture, which proves that Conjecture 4 is true *for M sufficiently near to M_a*.

In Section 2 we will recall how to pass from Conjecture 1 to Conjecture 4 with the aid of this added conjecture, and we will reduce this last conjecture to a result (denoted Conjecture 5 in [2]) on some quadratic form. We will prove this result in Section 3.

This implies from [2], where Conjecture 1 is proved for $q = 3$, 4 and 5, that Conjecture 4 is true for these values of q (for $q = 3$ another proof of Conjecture 4 is in [3]).

2. How to pass from Conjecture 1 to Conjecture 4

First the following result is in [2]:

Proposition 1. *If the Conjecture* 1 *is true, and if the Conjecture* 4 *is true for M sufficiently near to M_a, then the Conjecture* 4 *is true for every $M > M_a$.*

Proof. For the convenience of the reader we repeat the arguments of [2]. The result will be an easy consequence of the classical following assertion in differential geometry (see [8], Theorem 50, pg. 109-110).

Fact. Let be V a differential manifold, and $f : V \mapsto \mathbb{R}$ a smooth function, such that

(i) f has a unique critical point α, at its absolute minimum $m = f(\alpha)$;
(ii) for every λ, μ such that $m \leq \lambda < \mu$, the subset $V_\lambda^\mu := \{x \mid \lambda \leq f(x) \leq \mu\}$ is compact.

Then there is a diffeomorphism of V which maps V_m^λ onto V_m^μ.

We apply this fact with $V = \mathbb{R}_*^{+q}$, $f = H$, $\alpha = L$, and $m = M_a$. If M is sufficiently near to M_a, $\widetilde{S}(M)$ is homeomorphic to the ball, and $S(M)$ is its boundary and is homeomorphic to the sphere. Moreover for $M' > M$ the set $\{M \leq H \leq M'\}$ is compact. So for every $M' > M$ the set $\widetilde{S}(M')$ is homeomorphic to a ball, and the set $S_a(M')$ is its boundary and is homeomorphic to the unit sphere $\mathbb{S}^{q-1}$: this is Conjecture 4. $\square$

Now we assert a new result, which was the Conjecture 5 in [2]:

Assertion A. Every eigenvalue λ of the quadratic form

$$Q(u) = \sum_{2\leq i<j\leq q-1} u_i u_j + \ell(u_1 + u_q) \sum_{i=2}^{q-1} u_i + \ell(u_1^2 + u_q^2)$$

$$- (\ell^2 - \ell - 1)u_1u_q + \ell^2 \sum_{j=1}^{q-1} u_j u_{j+1} \quad (7)$$

satisfies the inequality $\lambda < \ell^2 + 2\ell$.

The statement in Conjecture 5 [2] was: "the quadratic form $R(u) := (2\ell^2 + 4\ell)(\sum_{i=1}^q u_i^2) - 2Q(u)$ is definite positive"; it is clearly equivalent to Assertion A. The role of Assertion A is given by the following:

Proposition 2. *If Assertion A is true, then the Conjecture* 4 *is true for every M sufficiently near to M_a.*

Proof. We extend the proof that is given for $q = 3$ in [2]. Precisely we will prove that if $M > M_a$ is sufficiently near to M_a, then the set $\widetilde{S}(M)$ is starlike with respect to the point L, and a ray from L cuts $S(M)$ at exactly one point. So $S(M)$ will be the boundary of $\widetilde{S}(M)$ and of the set $\{H < M\}$. Hence $\widetilde{S}(M)$ will be homeomorphic to an euclidean ball, and $S(M)$ will be homeomorphic to a sphere, if M is sufficiently near to M_a, that is Conjecture 4 will be true for M sufficiently near to M_a.

Put, for $\rho > 0$, $\rho < \ell$ and $\vec{u}$ any *unitary vector* of $\mathbb{R}^q$, $\delta_{\vec{u}}(\rho) := \ln H(L+\rho\vec{u})$. It is sufficient to prove the relation

$$\lim_{\rho \to 0} \frac{\delta'_{\vec{u}}(\rho)}{\rho} \geq D > 0, \text{ uniformly with respect to } \vec{u}, \quad (8)$$

in order to get the starlikeness of $\widetilde{S}(M)$, and the fact that a ray from L cuts $S(M)$ at exactly a point, if ρ is sufficiently small. This results from the fact that δ will be increasing on a neighborhood of 0 independent of $\vec{u}$, and so the equation $\ln H(L + \rho\vec{u}) = \ln M$ will have a unique solution ρ if M is sufficiently near to M_a, because the sets $\{x \,|\, H(x) < M\}$ are fundamental neighborhoods of L.

Easy calculations give for the value of $\delta'_{\vec{u}}(\rho)$:

$$\frac{2\rho u_1 u_q + \ell(u_1 + u_q) + \sum_{i=1}^q u_i}{2\ell^2 + \ell + \rho\big(\ell(u_1 + u_q) + \sum_{i=1}^q u_i\big) + \rho^2 u_1 u_q} + \sum_{j=1}^{q-1} \frac{u_j + u_{j+1}}{1 + 2\ell + \rho(u_j + u_{j+1})} - \sum_{i=1}^{q} \frac{u_i}{\ell + \rho u_i}.$$

Then the asymptotic expansion of $\delta'_{\vec{u}}(\rho)$ at 0 has the form $\rho B(\vec{u}) + o(\rho)$, where it is easy to see that $o(\rho)$ is *uniform* with respect to $\vec{u}$. With the relation $\sum_{i=1}^q u_i^2 = 1$, and puting $R(\vec{u}) := \ell^2(2\ell+1)^2 B(\vec{u})$, we have

$$R(\vec{u}) = (2\ell+1)^2 + 2\ell(2\ell+1)u_1u_q - \left[\ell(u_1+u_q) + \sum_{i=1}^q u_i\right]^2 - \ell^2 \sum_{j=1}^{q-1} (u_j + u_{j+1})^2,$$

that is, after easy calculations

$$R(\vec{u}) = 2\ell^2 + 4\ell - 2Q(\vec{u}), \tag{9}$$

where Q is the quadratic form of Assertion A.

So it is clear that the Assertion A will prove (8), and so the Proposition 2, because the maximum of Q over the unit sphere is its largest eigenvalue. □

So the main result of this paper, which will prove the implication

$$\text{Conjecture 1} \;\Rightarrow\; \text{Conjecture 4} \tag{10}$$

is that Assertion A is true.

3. The truthfulness of Assertion A

So we prove in this section the following result (which was proved in [2] for $q = 3$ only).

Theorem 1. *The assertion A is true for every $q \geq 3$.*

Proof. First, denote $\mathcal{M}$ the matrix of the quadratic form Q, λ one of its eigenvalues, root of the equation $\det(\mathcal{M} - \lambda I_d) = 0$. We write easily the matrix $2(\mathcal{M} - \lambda I_d)$:

$$\begin{pmatrix}
2\ell-2\lambda & \ell+\ell^2 & \ell & \ell & \dots & \dots & \ell & \ell & -(\ell^2-\ell-1) \\
\ell+\ell^2 & -2\lambda & 1+\ell^2 & 1 & \dots & \dots & 1 & 1 & \ell \\
\ell & 1+\ell^2 & -2\lambda & 1+\ell^2 & \dots & \dots & 1 & 1 & \ell \\
\ell & 1 & 1+\ell^2 & -2\lambda & \ddots & \dots & 1 & 1 & \ell \\
\ell & 1 & 1 & 1+\ell^2 & \ddots & \dots & 1 & 1 & \ell \\
\dots & \dots & \dots & \dots & \ddots & \ddots & \ddots & \dots & \dots \\
\ell & 1 & 1 & 1 & \dots & 1+\ell^2 & -2\lambda & 1+\ell^2 & \ell \\
\ell & 1 & 1 & 1 & \dots & 1 & 1+\ell^2 & -2\lambda & \ell+\ell^2 \\
-(\ell^2-\ell-1) & \ell & \ell & \ell & \dots & \ell & \ell & \ell+\ell^2 & 2\ell-2\lambda
\end{pmatrix} \tag{11}$$

Now we will transform the matrix without changing its determinant. First, we substract row 1 from row q, row 2 from row $q-1$, row 3 from row $q-2$, and so on. Then we see that in each of the $[\frac{q}{2}]$ last new rows the terms of columns 1 and q are opposite, the terms of columns 2 and $q-1$ are also opposite, and also are the terms of columns 3 and $q-2$, and so on. Hence we add column 1 to column q, column 2 to column $q-1$, column 3 to column $q-2$, and so on. We obtain the following form

$$J = \begin{pmatrix} (T) & (U) \\ (V) & 0 \end{pmatrix}, \tag{12}$$

where U and V are two matrices $m \times m$ if $q = 2m$ and U is $(m+1) \times (m+1)$ and V is $m \times m$ if $q = 2m + 1$. So an eigenvalue of Q is solution of the equations $\det(U) = 0$ or $\det(V) = 0$. Our goal is to prove that such an eigenvalue λ is majorized by $\ell^2 + 2\ell$. In order to see this fact, we will use the idea that if a $p \times p$ matrix $(a_{i,j})_{i,j}$ is not invertible, then its second diagonal is not dominating in one of the rows, that is: there exists i such that $|a_{i,p-i}| \le \sum_{j \ne p-i} |a_{i,j}|$. It is to be noticed that the variant of this idea with the principal diagonal does not work for the matrix given in (13).

First we apply this idea to the matrix V, which is:

$$\begin{pmatrix}
0 & 0 & \dots & \dots & \dots & \dots & \dots & 0 & -\ell^2 & 1+2\lambda \\
0 & 0 & \dots & \dots & \dots & \dots & 0 & -\ell^2 & 1+2\lambda & -\ell^2 \\
\dots & \dots & \dots & \dots & \dots & \dots & -\ell^2 & 1+2\lambda & -\ell^2 & 0 \\
\dots & \dots & \dots & \dots & \dots & \iddots & \iddots & \iddots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \iddots & \iddots & \iddots & \dots & \dots & \dots & \dots & \dots \\
0 & -\ell^2 & 1+2\lambda & -\ell^2 & 0 & \dots & \dots & \dots & 0 & 0 \\
-\ell^2 & 1+2\lambda & -\ell^2 & 0 & \dots & \dots & \dots & \dots & 0 & 0 \\
2\lambda-(\ell^2-3\ell-1) & -\ell^2 & 0 & 0 & 0 & \dots & \dots & \dots & 0 & 0
\end{pmatrix} \tag{13}$$

So if λ is a solution of $\det(V) = 0$, then we have

$$|1 + 2\lambda| \le \ell^2 \text{ or } |1 + 2\lambda| \le 2\ell^2 \text{ or } \big|2\lambda - (\ell^2 - 3\ell - 1)\big| \le \ell^2,$$

and in each of this three cases we obtain $2\lambda < 2\ell^2 + 4\ell$, and this was our goal.

Now we look at the case where λ is a solution of the equation $\det(U) = 0$, where U is

$$\begin{pmatrix}
\ell & 2\ell & \dots & 2\ell & 2\ell & 2\ell & 2\ell+\ell^2 & -(2\lambda+\ell^2-3\ell-1) \\
1 & 2 & \dots & 2 & 2 & \ell^2+2 & 1-2\lambda & 2\ell+\ell^2 \\
1 & 2 & \dots & 2 & \ell^2+2 & 1-2\lambda & \ell^2+2 & 2\ell \\
1 & 2 & \dots & \ell^2+2 & 1-2\lambda & \ell^2+2 & 2 & 2\ell \\
\dots & \dots & \dots & \iddots & \iddots & \iddots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \iddots & \iddots & \iddots & \dots & \dots & \dots \\
1 & 2 & \ell^2+2 & 1-2\lambda & \ell^2+2 & \dots & 2 & 2\ell \\
1 & \ell^2+2 & 1-2\lambda & \ell^2+2 & 2 & \dots & 2 & 2\ell \\
\ell^2+1 & 1-2\lambda & \ell^2+2 & 2 & 2 & \dots & 2 & 2\ell \\
-2\lambda & 2\ell^2+2 & 2 & 2 & 2 & \dots & 2 & 2\ell
\end{pmatrix} \tag{14}$$

in the case where $q = 2m + 1$ is odd.

So λ satisfies

$$|1 - 2\lambda| \le 2(\ell^2 + 2) + 2\ell + 1 + 2(m + 1 - 5), \quad \text{or}$$

$$|1-2\lambda| \le (\ell^2+2)+(2\ell+\ell^2)+1+2(m+1-4), \quad \text{or}$$
$$|1-2\lambda| \le (\ell^2+2)+(\ell^2+1)+2\ell+2(m+1-4), \quad \text{or}$$
$$|2\lambda| \le (2\ell^2+2)+2\ell+2(m+1-3), \quad \text{or}$$
$$\left|2\lambda+2\ell^2-3\ell-1\right| \le (\ell^2+2\ell)+\ell+2\ell(m+1-3).$$

We obtain, with the aid of the relation $2m = q-1 < \ell$,

* $2\lambda \le 2\ell^2+3\ell-2$, in the first four cases,
* $2\lambda \le -(\ell^2-3\ell-1)+\ell^2-\ell+2\ell^2 = 2\ell^2+2\ell+1$ in the fifth case.

These majorizations are less than $2\ell^2+4\ell$, and this was our goal.

If $q=2m$, then the matrix of U is (14) without its first column and its last row, and where we replace in this new matrix the first term of the last row: $1-2\lambda$ by $1+\ell^2-2\lambda$. So the first $m-1$ inequalities are strengthened, and the last becomes $\left|1+\ell^2-2\lambda\right| \le (\ell^2+2)+2\ell+2(m-3) \le \ell^2+3\ell-3$, and so $2\lambda \le 2\ell^2+3\ell-2 < 2\ell^2+4\ell$. And this achieves the proof. □

Corollary. *For $q = 3$, 4 or 5 the sets $S(M) = \{H = M\}$ are homeomorphic to $\mathbb{S}^{q-1}$ for every $M > M_a$.*

This corollary comes from the fact, proved in [2], that Conjecture 1 is true for these three values of q.

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(Received: May 26, 2010)
(Revised: August 6, 2010)

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