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DYNAMICS OF AN ANTI-COMPETITIVE TWO DIMENSIONAL RATIONAL SYSTEM OF DIFFERENCE EQUATIONS

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Abstract. We investigate the global asymptotic behavior of solutions of the following anti-competitive system of rational difference equations

$$x_{n+1}=\frac{\gamma_1 y_n}{A_1+x_n}+h, \quad y_{n+1}=\frac{\beta_2 x_n}{A_2+y_n}, \quad n=0,1,\dots,$$

where the parameters $\gamma_1, \beta_2, A_1, A_2$ and h are positive numbers and the initial conditions x_0, y_0 are arbitrary nonnegative numbers.

1. Introduction

A first order system of difference equations

$$\left\{\begin{array}{l} x_{n+1}=f\left(x_n, y_n\right) \\ y_{n+1}=g\left(x_n, y_n\right) \end{array}\right., \quad n=0,1,\dots,\left(x_0, y_0\right)\in\mathcal{R}, \tag{1}$$

where $\mathcal{R}\subset\mathbb{R}^2, (f,g):\mathcal{R}\to\mathcal{R}, f, g$ are continuous function is *competitive* if $f(x,y)$ is non-decreasing in x and non-increasing in y, and $g(x,y)$ is non-increasing in x and non-decreasing in y.

System (1) where the functions f and g have monotonic character opposite of the monotonic character in competitive system will be called *anti-competitive.*

We consider the following system of difference equations

$$x_{n+1}=\frac{\gamma_1 y_n}{A_1+x_n}+h, \quad y_{n+1}=\frac{\beta_2 x_n}{A_2+y_n}, \quad n=0,1,\dots, \tag{2}$$

where the parameters $A_1, \gamma_1, A_2, \beta_2$ and h are positive numbers and the initial conditions x_0, y_0 are arbitrary nonnegative numbers. It is obvious that System (2) is anti-competitive.

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Competitive systems of the form (2) were studied by many authors such as Clark and Kulenović [3], Hirsch and Smith [8], Kulenović and Merino [12], Kulenović and Nurkanović [16], [17], Garić-Demirović, Kulenović and Nurkanović [1], [6], Smith [20], [21] and others.

We now give some basic notions about systems and maps in the plane of the form:

$$x_{n+1} = f\left(x_n, y_n\right), \quad y_{n+1} = g\left(x_n, y_n\right), \quad n = 0, 1, 2, \ldots, \tag{3}$$

where f and g are continuous functions and $f(x,y)$ is non-decreasing in x and non-increasing in y and $g(x,y)$ is non-increasing in x and non-decreasing in y in some domain A.

Consider the map $T = (f, g)$ on a set $\mathcal{R} \subset \mathbb{R}^2$, and let $E \in \mathcal{R}$. The point $E \in \mathcal{R}$ is called a *fixed point* if $T(E) = E$. An *isolated* fixed point is a fixed point that has a neighborhood with no other fixed points in it. A fixed point $E \in \mathcal{R}$ is an *attractor* if there exists a neighborhood $\mathcal{U}$ of E such that $T^n(\mathbf{x}) \to E$ as $n \to \infty$ for $\mathbf{x} \in \mathcal{U}$; the *basin of attraction* is the set of all $\mathbf{x} \in \mathcal{R}$ such that $T^n(\mathbf{x}) \to E$ as $n \to \infty$. A fixed point E is a global attractor on a set $\mathcal{K}$ if E is an attractor and $\mathcal{K}$ is a subset of the basin of attraction of E. If T is differentiable at a fixed point E, and if the Jacobian $J_T(E)$ has one eigenvalue with modulus less than one and a second eigenvalue with modulus greater than one, E is said to be a *saddle*. See [18] for additional definitions.

Definition 1. *Let $T = (f, g)$ be a continuously differentiable vector function and let U be a neighborhood of a saddle point $(\bar{x}, \bar{y})$ of (3). The local stable manifold $\mathcal{W}^s_{loc}$ is the set*

$$\mathcal{W}^s_{loc}((\bar{x}, \bar{y})) = \left\{(x, y) : T^n\left(x, y\right) \in U \ \ (\forall n \geq 0) \ \ \wedge \ \lim_{n\to\infty} T^n\left(x, y\right) = (\bar{x}, \bar{y})\right\}.$$

The global stable manifold W^s of a saddle point $(\bar{x}, \bar{y})$ is the set

$$\mathcal{W}^s((\bar{x}, \bar{y})) = \left\{(x, y) : \quad \lim_{n\to\infty} T^n\left(x, y\right) = (\bar{x}, \bar{y})\right\}.$$

The main result in linearized stability analysis is the following result [12], [18].

Theorem 2. ***(Linearized Stability Theorem)*** *Let $T = (f, g)$ be a continuously differentiable function defined on an open set W in $\mathbb{R}^2$, and let $E = (\bar{x}, \bar{y})$ in W be a fixed point of T.*

(a) *If all the eigenvalues of the Jacobian matrix $J_T(E)$ have modulus less than one, then the equilibrium point E of (3) is asymptotically stable.*

(b) *If at least one of the eigenvalues of the Jacobian matrix* $J_T(E)$ *has modulus greater than one, then the equilibrium point* E *of* (3) *is unstable.*

(c) *All the eigenvalues of the Jacobian matrix* $J_T(E)$ *have modulus less than one if and only if every solution of the characteristic equation*

$$\lambda^2 - TrJ_T(E)\lambda + DetJ_T(E) = 0 \tag{4}$$

lies inside unit circle, that is if and only if

$$|\,TrJ_T(E)\,| < 1 + DetJ_T(E) < 2. \tag{5}$$

(d) *If the characteristic equation* (4) *has one root that lies inside the unit circle and one root that lies outside the unit circle, then the equilibrium point* E *of* (3) *is a saddle point.*

(e) *The characteristic equation* (4) *has one root that lies inside the unit circle and one root that lies outside the unit circle if and only if*

$$|\,TrJ_T(E)\,| > |1 + DetJ_T(E)| \quad \textit{and} \quad (TrJ_T(E))^2 - 4DetJ_T(E) > 0. \tag{6}$$

(f) *If at least one of the eigenvalues of the Jacobian matrix* $J_T(E)$ *has modulus equal one, then the equilibrium point* E *of* (3) *is non-hyperbolic.*

(g) *The characteristic equation* (4) *has at least one root that lies on the unit circle if and only if*

$$|\,TrJ_T(E)\,| = |1 + DetJ_T(E)|$$

or

$$DetJ_T(E) = 1 \quad \textit{and} \quad TrJ_T(E) \leq 2.$$

Here we give some basic facts about the monotone maps in the plane, see [4], [5] and [20]. Now, we write system (2) in the form:

$$\begin{pmatrix} x \\ y \end{pmatrix}_{n+1} = T \begin{pmatrix} x \\ y \end{pmatrix}_n,$$

where the map T is given as

$$T : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \frac{\gamma_1 y}{A_1 + x} + h \\ \frac{\beta_2 x}{A_2 + y} \end{pmatrix} = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}.$$

The map T may be viewed as a monotone map if we define a partial order on $\mathbb{R}^2$ so that the positive cone in this new partial order is the fourth quadrant. Specifically, for $\mathbf{v} = (v_1, v_2), \mathbf{w} = (w_1, w_2) \in \mathbb{R}^2$ we say that $\mathbf{v} \preceq \mathbf{w}$ if $v_1 \leq w_1$ and $w_2 \leq v_2$. Two points $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2_+$ are said to be *related* if $\mathbf{v} \preceq \mathbf{w}$ or $\mathbf{w} \preceq \mathbf{v}$. Also, a strict inequality between points may be defined as $\mathbf{v} \prec \mathbf{w}$ if $\mathbf{v} \preceq \mathbf{w}$ and $\mathbf{v} \neq \mathbf{w}$. A stronger inequality may be defined as $\mathbf{v} \prec\prec \mathbf{w}$ if $v_1 < w_1$ and $w_2 < v_2$. A map $f : \operatorname{Int}\mathbb{R}^2_+ \to \operatorname{Int}\mathbb{R}^2_+$ is *strongly monotone* if $\mathbf{v} \prec \mathbf{w}$

implies that $f(\mathbf{v}) \prec\prec f(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in \operatorname{Int} \mathbb{R}^2_+$. Clearly, being related is an invariant under iteration of a strongly monotone map. Differentiable strongly monotone maps have Jacobian with constant sign configuration

$$\begin{bmatrix} + & - \\ - & + \end{bmatrix}.$$

The mean value theorem and the convexity of $\mathbb{R}^2_+$ may be used to show that T is monotone, as in [4].

For $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, define $Q_l(\mathbf{x})$ for $l = 1, \ldots, 4$ to be the usual four quadrants based at $\mathbf{x}$ and numbered in a counterclockwise direction, for example, $Q_1(\mathbf{x}) = \left\{\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2 : x_1 \le y_1, x_2 \le y_2\right\}$.We now state three results for competitive maps in the plane.

The following definition is from [20].

Definition 3. *Let $\mathcal{S}$ be a nonempty subset of $\mathbb{R}^2$. A competitive map $T : \mathcal{S} \to \mathcal{S}$ is said to satisfy condition (O+) if for every x, y in $\mathcal{S}$, $T(x) \preceq_{ne} T(y)$ implies $x \preceq_{ne} y$, and T is said to satisfy condition (O−) if for every x, y in $\mathcal{S}$, $T(x) \preceq_{ne} T(y)$ implies $y \preceq_{ne} x$.*

The following theorem was proved by de Mottoni-Schiaffino for the Poincaré map of a periodic competitive Lotka-Volterra system of differential equations. Smith generalized the proof to competitive and cooperative maps [20, 21].

Theorem 4. *Let $\mathcal{S}$ be a nonempty subset of $\mathbb{R}^2$. If T is a competitive map for which (O+) holds then for all $x \in \mathcal{S}$, $\{T^n(x)\}$ is eventually componentwise monotone. If the orbit of x has compact closure, then it converges to a fixed point of T. If instead (O−) holds, then for all $x \in \mathcal{S}$, $\{T^{2n}\}$ is eventually componentwise monotone. If the orbit of x has compact closure in $\mathcal{S}$, then its omega limit set is either a period-two orbit or a fixed point.*

The following result is from [20], with the domain of the map specialized to be the cartesian product of intervals of real numbers. It gives a sufficient condition for conditions $(O+)$ and $(O-)$.

Theorem 5. *Let $\mathcal{R} \subset \mathbb{R}^2$ be the cartesian product of two intervals in $\mathbb{R}$. Let $T : \mathcal{R} \to \mathcal{R}$ be a C^1 competitive map. If T is injective and $\det J_T(x) > 0$ for all $x \in \mathcal{R}$ then T satisfies (O+). If T is injective and $\det J_T(x) < 0$ for all $x \in \mathcal{R}$ then T satisfies (O−).*

The next result is from [10].

Theorem 6. *Let T be an anti-competitive map on an rectangle $\mathcal{R} = [a_1, a_2] \times [b_1, b_2]$ and $\overline{\mathbf{x}} \in int(\mathcal{R})$, where $-\infty \le a_1 < a_2 \le \infty$ and $-\infty \le b_1 < b_2 \le \infty$. Suppose*

a) *$T(int(\mathcal{R})) \subset int(\mathcal{R})$ and T is strongly anti-competitive on $int(\mathcal{R})$.*

b) *T has a unique fixed pint $\overline{\mathbf{x}}$ on $(Q_1(\overline{\mathbf{x}}) \cup Q_3(\overline{\mathbf{x}})) \cap int(\mathcal{R})$.*

c) *Either*

  I) *T has no prime period two orbits in $(Q_2(\overline{\mathbf{x}}) \cup Q_4(\overline{\mathbf{x}})) \cap int(\mathcal{R})$ or,*

  II) *$\det J_T(\overline{\mathbf{x}}) \neq 0$ and $T(\mathbf{x}) = \overline{\mathbf{x}}$ only if $\mathbf{x} = \overline{\mathbf{x}}$.*

d) *$\overline{\mathbf{x}}$ is a saddle point.*

Then,

1. *The global stable manifold $\mathcal{W}^s(\overline{\mathbf{x}})$ is the graph of a continuous, strictly increasing function, and $\mathcal{W}^s(\overline{\mathbf{x}})$ has endpoints in $\partial\mathcal{R}$.*
2. *Let*

$$\mathcal{W}_- := \left\{\mathbf{x} \in \mathcal{R}\backslash\mathcal{W}^s(\overline{\mathbf{x}}) : \exists \mathbf{x}' \in \mathcal{W}^s(\overline{\mathbf{x}}) \text{ with } \mathbf{x}' \preceq_{se} \mathbf{x}\right\},$$
$$\mathcal{W}_+ := \left\{\mathbf{x} \in \mathcal{R}\backslash\mathcal{W}^s(\overline{\mathbf{x}}) : \exists \mathbf{x}' \in \mathcal{W}^s(\overline{\mathbf{x}}) \text{ with } \mathbf{x} \preceq_{se} \mathbf{x}'\right\}.$$

Then ,

2.i) *If $\mathbf{x} \in \mathcal{W}_+$, then $T^{2n}(\mathbf{x})$ eventually enters into $int(Q_4(\overline{\mathbf{x}}) \cap \mathcal{R})$ and $T^{2n+1}(\mathbf{x})$ eventually enters into $int(Q_2(\overline{\mathbf{x}}) \cap \mathcal{R})$.*

2.ii) *If $\mathbf{x} \in \mathcal{W}_-$, then $T^{2n}(\mathbf{x})$ eventually enters into $int(Q_2(\overline{\mathbf{x}}) \cap \mathcal{R})$ and $T^{2n+1}(\mathbf{x})$ eventually enters into $int(Q_4(\overline{\mathbf{x}}) \cap \mathcal{R})$.*

The following result gives a convergence result for a system in $\mathcal{R}^2$ when there exist an invariant rectangle and the map of the system satisfies certain monotonicity and algebraic conditions. See [11] and [15].

Theorem 7. *Let $\mathcal{R} = [a,b] \times [c,d]$ and*

$$f : \mathcal{R} \to [a,b], \quad g : \mathcal{R} \to [c,d]$$

be a continuous function such that:

(a) *$f(x,y)$ is non-increasing in the first and non-increasing in the second variable and $g(x,y)$ is non-decreasing in the first and non-increasing in the second variable for each $(x,y) \in \mathcal{R}$;*

(b) *If $(m_1, M_1, m_2, M_2) \in \mathcal{R}^2$ is a solution of*

$$\begin{cases} M_1 = f(m_1, M_2), m_1 = f(M_1, m_2) \\ M_2 = g(M_1, m_2), m_2 = g(m_1, M_2) \end{cases}, \tag{7}$$

then $m_1 = M_1$ and $m_2 = M_2$.

Then System (1) has a unique equilibrium $(\overline{x}, \overline{y})$ and every solution of (1) with $(x_0, y_0) \in \mathcal{R}$ converges to the unique equilibrium $(\overline{x}, \overline{y})$.

Variations on Theorem 7 have appeared and have been used very effectively to completely settle global attractivity issues of many systems of rational equations, see [15], [16].

The following result gives the rate of convergence of solutions of a system of difference equations

$$\mathbf{x}_{n+1} = [A + B(n)]\,\mathbf{x}_n, \tag{8}$$

where $\mathbf{x}_n$ is k-dimensional vector, $A \in \mathbb{C}^{k\times k}$ is a constant matrix, and $B : \mathbb{Z}^+ \to \mathbb{C}^{k\times k}$ is a matrix function satisfying

$$\|B(n)\| \to 0 \quad \text{when } n \to \infty, \tag{9}$$

where $\|\cdot\|$ denotes any matrix norm which is associated with the vector norm; $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^2$ given by

$$\|(x, y)\| = \sqrt{x^2 + y^2}.$$

Theorem 8. *Suppose that condition* (9) *holds. If* $\mathbf{x}$ *is a solution of* (8), *then either* $\mathbf{x}_n = 0$ *for all large* n *or*

$$\rho = \lim_{n\to\infty} \sqrt[n]{\|\mathbf{x}_n\|} \tag{10}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Theorem 9. *Suppose that condition* (9) *holds. If* $\mathbf{x}$ *is a solution of* (8), *then either* $\mathbf{x}_n = 0$ *for all large* n *or*

$$\rho = \lim_{n\to\infty} \frac{\|\mathbf{x}_{n+1}\|}{\|\mathbf{x}_n\|} \tag{11}$$

exists and is equal to the modulus of one the eigenvalues of matrix A.

2. Linearized stability analysis

In this Section we prove the following local stability result.

Theorem 10. *System* (2) *has the unique positive equilibrium point* $E = (\overline{x}, \overline{y})$.

i) *If* $\beta_2\gamma_1 < (A_1 + h)\,A_2$, *then* E *is locally asimptotically stable.*
ii) *If* $\beta_2\gamma_1 = (A_1 + h)\,A_2$, *then* E *is non-hyperbolic.*
iii) *If* $\beta_2\gamma_1 > (A_1 + h)\,A_2$, *then* E *is a saddle point.*

Proof. The equilibrium point of System (2) satisfies the following system of equations

$$\overline{x} = \frac{\gamma_1\overline{y}}{A_1 + \overline{x}} + h, \overline{y} = \frac{\beta_2\overline{x}}{A_2 + \overline{y}}. \tag{12}$$

It is easy to see that the unique positive equilibrium $E = (\overline{x}, \overline{y})$ is an intersection of the following two parabolas

$$\overline{y} = \frac{1}{\gamma_1}\left(\overline{x}^2 + (A_1 - h)\,\overline{x} - A_1 h\right), \overline{x} = \frac{1}{\beta_2}\left(\overline{y}^2 + A_2\overline{y}\right). \tag{13}$$

The map T associated to System (2) is

$$T(x,y) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} \frac{\gamma_1 y}{A_1+x} + h \\ \frac{\beta_2 x}{A_2+y} \end{pmatrix}. \tag{14}$$

The Jacobian matrix of T at the equilibrium point $E = (\overline{x}, \overline{y})$ is

$$J_T(\overline{x},\overline{y}) = \begin{pmatrix} -\frac{\gamma_1 \overline{y}}{(A_1+\overline{x})^2} & \frac{\gamma_1}{A_1+\overline{x}} \\ \frac{\beta_2}{A_2+\overline{y}} & -\frac{\beta_2 \overline{x}}{(A_2+\overline{y})^2} \end{pmatrix}. \tag{15}$$

The characteristic equation has the following form

$$\lambda^2 - p\lambda + q = 0,$$

where

$$p = Tr J_T(E) = -\frac{\gamma_1 \overline{y}}{(A_1+\overline{x})^2} - \frac{\beta_2 \overline{x}}{(A_2+\overline{y})^2},$$
$$q = Det J_T(E) = \frac{\gamma_1 \beta_2 \overline{x}\overline{y}}{(A_1+\overline{x})^2 (A_2+\overline{y})^2} - \frac{\gamma_1 \beta_2}{(A_1+\overline{x})(A_2+\overline{y})}.$$

i) Now we check conditions (5).

(1) It is clear that $p < 0$. Using (12) we have
$|p| < 1+q \Leftrightarrow \frac{\gamma_1 \overline{y}}{(A_1+\overline{x})^2} + \frac{\beta_2 \overline{x}}{(A_2+\overline{y})^2} < 1 + \frac{\gamma_1\beta_2\overline{x}\overline{y}}{(A_1+\overline{x})^2(A_2+\overline{y})^2} - \frac{\gamma_1\beta_2}{(A_1+\overline{x})(A_2+\overline{y})}$
$\Leftrightarrow \frac{\overline{x}-h}{A_1+\overline{x}} + \frac{\overline{y}}{A_2+\overline{y}} < 1 + \frac{\overline{y}}{A_2+\overline{y}} \cdot \frac{\overline{x}-h}{A_1+\overline{x}} - \frac{\gamma_1\beta_2}{(A_1+\overline{x})(A_2+\overline{y})}$
$\Leftrightarrow (\overline{x}-h)(A_2+\overline{y}) + \overline{y}(A_1+\overline{x}) < (A_1+\overline{x})(A_2+\overline{y}) + \overline{y}(\overline{x}-h) - \gamma_1\beta_2$
$\Leftrightarrow \beta_2\gamma_1 < (A_1+h)A_2.$

(2) It is easy to check that $q < 0$, which implies that $q < 1$. Indeed

$$\begin{aligned} q &= \frac{\gamma_1\beta_2\overline{x}\overline{y}}{(A_1+\overline{x})^2(A_2+\overline{y})^2} - \frac{\gamma_1\beta_2}{(A_1+\overline{x})(A_2+\overline{y})} \\ &= \frac{\gamma_1\beta_2}{(A_1+\overline{x})(A_2+\overline{y})}\left(\frac{\overline{x}\overline{y}}{(A_1+\overline{x})(A_2+\overline{y})} - 1\right) < 0. \end{aligned}$$

Therefore, the conditions (5) are satisfied if $\beta_2\gamma_1 < (A_1+h)A_2$ and the equilibrium point $E = (\overline{x},\overline{y})$ is locally asimptotically stable.

ii) Now we check conditions (6).

1. $|p| > |1+q| \Leftrightarrow -p > |1+q| \Leftrightarrow (p-1-q<0) \wedge (p+1+q<0)$
1° $p+1+q < 0 \Leftrightarrow -p > 1+q \Leftrightarrow |p| > 1+q \Leftrightarrow \beta_2\gamma_1 > (A_1+h)A_2$,
2° $p-1-q < 0 \Leftrightarrow \cdots \Leftrightarrow \gamma_1\beta_2 < (2\overline{x}+A_1-h)(2\overline{y}+A_2)$

which is satisfied, because from (12) we have

$$\gamma_1\beta_2 = \frac{(\overline{x}-h)(A_1+\overline{x})(A_2+\overline{y})}{\overline{x}} < \frac{(\overline{x}-h)(A_1+\overline{x})(A_2+2\overline{y})}{\overline{x}}$$
$$< (2\overline{x}+A_1-h)(2\overline{y}+A_2)$$

2. $p^2-4q>0 \Leftrightarrow \left(\frac{\gamma_1\overline{y}}{(A_1+\overline{x})^2}+\frac{\beta_2\overline{x}}{(A_2+\overline{y})^2}\right)^2-\frac{4\gamma_1\beta_2\overline{x}\overline{y}}{(A_1+\overline{x})^2(A_2+\overline{y})^2}+\frac{4\gamma_1\beta_2}{(A_1+\overline{x})(A_2+\overline{y})}>0 \Leftrightarrow \left(\frac{\gamma_1\overline{y}}{(A_1+\overline{x})^2}-\frac{\beta_2\overline{x}}{(A_2+\overline{y})^2}\right)^2+\frac{4\gamma_1\beta_2}{(A_1+\overline{x})(A_2+\overline{y})}>0$,
which is always satisfied.

iii) Since $\beta_2\gamma_1=(A_1+h)A_2 \Leftrightarrow p+1+q=0 \Rightarrow |p|=|1+q|$, we see that under this condition the equilibrium point $E=(\overline{x},\overline{y})$ is non-hyperbolic. □

3. Global results

In this section we present the results on the global dynamics of System (2).

Case 1: $\beta_2\gamma_1<(A_1+h)A_2$

Theorem 11. *Assume that $\beta_2\gamma_1<(A_1+h)A_2$. Then the equilibrium point $E=(\overline{x},\overline{y})$ is globally asimptotically stable.*

Proof. Taking in account that $f(x,y)$ is decreasing in x and increasing in y, while $g(x,y)$ is increasing in x and decreasing in y, we have

$$h \le x_{n+1} = \frac{\gamma_1 y_n}{A_1+x_n}+h \le \frac{\gamma_1 U_2}{A_1+h}+h \le U_1,$$
$$0 \le y_{n+1} = \frac{\beta_2 x_n}{A_2+y_n} \le \frac{\beta_2 U_1}{A_2} \le U_2,$$

which implies

$$U_1 \ge \frac{\gamma_1 U_2}{A_1+h}+h \ge \frac{\gamma_1\frac{\beta_2 U_1}{A_2}}{A_1+h}+h$$
$$\Rightarrow \left(1-\frac{\gamma_1\beta_2}{A_2(A_1+h)}\right)U_1 \ge h.$$

The last inequality implies

$$U_1 \ge \frac{A_2(A_1+h)h}{A_2(A_1+h)-\beta_2\gamma_1} \quad \text{and} \quad U_2 \ge \frac{\beta_2(A_1+h)h}{A_2(A_1+h)-\beta_2\gamma_1}. \tag{16}$$

This shows that the following box

$$[h,U_1]\times[0,U_2]$$

is an attractive box.

Now we show that $[h, U_1^*] \times [0, U_2^*]$, where $U_1^* = \dfrac{A_2(A_1+h)h}{A_2(A_1+h) - \beta_2\gamma_1}$ and $U_2^* = \dfrac{\beta_2(A_1+h)h}{A_2(A_1+h) - \beta_2\gamma_1}$, is an invariant box, i.e.

$$(x, y) \in [h, U_1^*] \times [0, U_2^*] \Rightarrow T(x, y) \in [h, U_1^*] \times [0, U_2^*].$$

Indeed

$$h \le T_1(x, y) = \frac{\gamma_1 y}{A_1 + x} + h \le \frac{\gamma_1 U_2^*}{A_1 + h} + h = \frac{A_2(A_1+h)h}{A_2(A_1+h) - \beta_2\gamma_1} = U_1^*,$$

$$0 \le T_2(x, y) = \frac{\beta_2 x}{A_2 + y} \le \frac{\beta_2 U_1^*}{A_2} = \frac{\beta_2(A_1+h)h}{A_2(A_1+h) - \beta_2\gamma_1} = U_2^*.$$

The second condition of Theorem 7 has the following form

$$M_1 = \frac{\gamma_1 M_2}{A_1 + m_1} + h, \qquad m_1 = \frac{\gamma_1 m_2}{A_1 + M_1} + h,$$

$$M_2 = \frac{\beta_2 M_1}{A_2 + m_2}, \qquad m_2 = \frac{\beta_2 m_1}{A_2 + M_2}.$$

These equations imply:

$$m_1 M_1 - A_1 h = \gamma_1 M_2 + m_1 h - A_1 M_1 = \gamma_1 m_2 + M_1 h - A_1 m_1,$$

$$m_2 M_2 = \beta_2 M_1 - A_2 M_2 = \beta_2 m_1 - A_2 m_2.$$

Using algebraic manipulations, we obtain

$$(A_1 + h)(m_1 - M_1) = \gamma_1 (m_2 - M_2),$$

$$\beta_2 (m_1 - M_1) = A_2 (m_2 - M_2).$$

This implies

$$(\beta_2\gamma_1 - (A_1+h) A_2)(M_1 - m_1) = 0,$$

that is $M_1 = m_1$.

Similarly, we obtain $M_2 = m_2$.

The conclusion follows from Theorem 7 and the first statement of Theorem 10. □

Case 2: $\beta_2\gamma_1 > (A_1 + h) A_2$

Lemma 12. *If $\beta_2\gamma_1 > (A_1 + h) A_2$, then System (2) has no prime period-two solutions.*

Proof. System (2) can be reduced to the following second-order difference equation:

$$x_{n+2} = \frac{\gamma_1^2 \beta_2 x_n}{(\gamma_1 A_2 + (x_{n+1} - h)(A_1 + x_n))(A_1 + x_{n+1})} + h, \tag{17}$$

or to the following second-order difference equation:

$$y_{n+2} = \beta_2 \frac{\gamma_1 \beta_2 y_n + h\left(A_1 \beta_2 + y_{n+1}\left(A_2 + y_n\right)\right)}{\left(A_2 + y_{n+1}\right)\left(A_1 \beta_2 + y_{n+1}\left(A_2 + y_n\right)\right)}. \tag{18}$$

Now it is sufficient to prove that both of the difference equations (17) and (18) have no prime period-two solutions. Assume that this is not true for (17), that is, that

$$\varphi, \psi, \varphi, \psi, \ldots, (\varphi \neq \psi)$$

is a prime period-two solution of (17). Then we have

$$\begin{aligned} \varphi &= \frac{\gamma_1^2 \beta_2 \varphi}{\left(\gamma_1 A_2 + (\psi - h)\left(A_1 + \varphi\right)\right)\left(A_1 + \psi\right)} + h, \\ \psi &= \frac{\gamma_1^2 \beta_2 \psi}{\left(\gamma_1 A_2 + (\varphi - h)\left(A_1 + \psi\right)\right)\left(A_1 + \varphi\right)} + h. \end{aligned} \tag{19}$$

This implies

$$\begin{aligned} \gamma_1 A_2 (\varphi - h)\left(A_1 + \psi\right) + (\varphi - h)(\psi - h)\left(A_1 + \varphi\right)\left(A_1 + \psi\right) &= \gamma_1^2 \beta_2 \varphi, \\ \gamma_1 A_2 (\psi - h)\left(A_1 + \varphi\right) + (\psi - h)(\varphi - h)\left(A_1 + \psi\right)\left(A_1 + \varphi\right) &= \gamma_1^2 \beta_2 \psi. \end{aligned}$$

By subtraction, we obtain

$$\gamma_1 (\varphi - \psi)\left(\left(A_1 + h\right) A_2 - \beta_2 \gamma_1\right) = 0,$$

and this implies that $\varphi = \psi$ (because $\beta_2 \gamma_1 > \left(A_1 + h\right) A_2$), which is a contradiction.

Now assume that

$$\chi, \phi, \chi, \phi, \ldots, (\chi \neq \phi)$$

is a prime period-two solution of (18). Then we have

$$\begin{aligned} \chi &= \beta_2 \frac{\gamma_1 \beta_2 \chi + h\left(A_1 \beta_2 + \phi\left(A_2 + \chi\right)\right)}{\left(A_2 + \phi\right)\left(A_1 \beta_2 + \phi\left(A_2 + \chi\right)\right)}, \\ \phi &= \beta_2 \frac{\gamma_1 \beta_2 \phi + h\left(A_1 \beta_2 + \chi\left(A_2 + \phi\right)\right)}{\left(A_2 + \chi\right)\left(A_1 \beta_2 + \chi\left(A_2 + \phi\right)\right)}, \end{aligned}$$

from which

$$\begin{aligned} \chi\left(A_2 + \phi\right)\left(A_1 \beta_2 + \phi\left(A_2 + \chi\right)\right) &= \gamma_1 \beta_2^2 \chi + A_1 \beta_2^2 h + \beta_2 h \phi\left(A_2 + \chi\right), \\ \phi\left(A_2 + \chi\right)\left(A_1 \beta_2 + \chi\left(A_2 + \phi\right)\right) &= \gamma_1 \beta_2^2 \phi + A_1 \beta_2^2 h + \beta_2 h \chi\left(A_2 + \phi\right). \end{aligned}$$

By subtracting, we have

$$\beta_2 (\chi - \phi)\left(\left(A_1 + h\right) A_2 - \beta_2 \gamma_1\right) = 0,$$

and this implies that $\chi = \phi$, which is a contradiction. □

Lemma 13. *The map T associated to System (2) satisfies the following:*

$$T(x, y) = (\overline{x}, \overline{y}) \quad \textit{only for } (x, y) = (\overline{x}, \overline{y}).$$

Proof. By using (12), we have

$$T(x,y)=(\overline{x},\overline{y})\Leftrightarrow\begin{cases}\frac{\gamma_1 y}{A_1+x}+h=\frac{\gamma_1\overline{y}}{A_1+\overline{x}}+h\\ \frac{\beta_2 x}{A_2+y}=\frac{\beta_2\overline{x}}{A_2+\overline{y}}\end{cases}$$
$$\Leftrightarrow\begin{cases}y(A_1+\overline{x})=\overline{y}(A_1+x)\\ x(A_2+\overline{y})=\overline{x}(A_2+y)\end{cases}$$
$$\Leftrightarrow\begin{cases}A_1(y-\overline{y})+\overline{x}y-x\overline{y}=0\\ A_2(x-\overline{x})+x\overline{y}-\overline{x}y=0\end{cases}. \tag{20}$$

Adding these two equations yields

$$A_1(y-\overline{y})=-A_2(x-\overline{x}). \tag{21}$$

By using (21), the first equation in (20) becomes

$$(x-\overline{x})\left(-A_2-\frac{A_2}{A_1}\overline{x}-\overline{y}\right)=0,$$

which implies $x=\overline{x}$. From (21) we obtain $y=\overline{y}$. □

Theorem 14. *Assume that $\beta_2\gamma_1>(A_1+h)A_2$. Then there exists a set $\mathcal{C}\subset\mathcal{R}$ which is invariant subset of the basin of attraction of E. The set $\mathcal{C}$ is a graph of a strictly increasing continuous function of the first variable on an interval (and so is a manifold) and separates $\mathcal{R}$ into two connected and invariant components, namely*

$$\mathcal{W}_-:=\{x\in\mathcal{R}\backslash\mathcal{C}:\exists y\in\mathcal{C}\ \text{with}\ x\preceq_{se}y\},$$
$$\mathcal{W}_+:=\{x\in\mathcal{R}\backslash\mathcal{C}:\exists y\in\mathcal{C}\ \text{with}\ y\preceq_{se}x\}.$$

which satisfy:

i) *If $(x_0,y_0)\in\mathcal{W}_+$, then*

$$\lim_{n\to\infty}(x_{2n},y_{2n})=(\infty,0)\quad\text{and}\quad\lim_{n\to\infty}(x_{2n+1},y_{2n+1})=(h,\infty).$$

ii) *If $(x_0,y_0)\in\mathcal{W}_-$,*

$$\lim_{n\to\infty}(x_{2n},y_{2n})=(h,\infty)\quad\text{and}\quad\lim_{n\to\infty}(x_{2n+1},y_{2n+1})=(\infty,0).$$

Proof. System (2) can be decomposed into the system of the even-indexed and odd-indexed terms as follows

$$\begin{cases}x_{2n+1}=\frac{\gamma_1 y_{2n}}{A_1+x_{2n}}+h,\\ x_{2n}=\frac{\gamma_1 y_{2n-1}}{A_1+x_{2n-1}}+h,\\ y_{2n+1}=\frac{\beta_2 x_{2n}}{A_2+y_{2n}},\\ y_{2n}=\frac{\beta_2 x_{2n-1}}{A_2+y_{2n-1}}.\end{cases}$$

The existence of the set $\mathcal{C}$ with the stated properties follows from Lemmas 12 and 13 and Theorem 6. $\square$

Case 3: $\beta_2\gamma_1 = (A_1 + h) A_2$

The following result holds.

Theorem 15. *Assume that $\beta_2\gamma_1 = (A_1 + h) A_2$. Then System (2) has an infinite number of period-two solutions located along the graph of a curve of third order:*

$$\begin{cases} \beta_2 A_1 A_2 h + \beta_2 A_2 h x + h\left(A_2^2 + \beta_2 A_1\right) y + \\ + A_2 h y^2 - \left(A_2^2 - \beta_2 h + \beta_2 A_1\right) xy - \beta_2 x^2 y - A_2 x y^2 \\ = 0 \end{cases} \tag{22}$$

Proof. The second iterate of T is

$$T^2(x,y) = \left(\tfrac{\beta_2\gamma_1 x (A_1+x)}{(A_2+y)(\gamma_1 y + (A_1+h)(A_1+x))} + h, \tfrac{\beta_2(\gamma_1 y + (A_1+x)h)(A_2+y)}{(A_2(A_2+y)+\beta_2 x)(A_1+x)}\right).$$

Period-two solution is a fixed point of a map T^2, i.e.

$$T^2(x,y) = (x,y),$$

which is equivalent to the following system

$$\begin{cases} \tfrac{\beta_2\gamma_1 x (A_1+x)}{(A_2+y)(\gamma_1 y + (A_1+h)(A_1+x))} + h = x, \\ \tfrac{\beta_2(\gamma_1 y + (A_1+x)h)(A_2+y)}{(A_2(A_2+y)+\beta_2 x)(A_1+x)} = y, \end{cases}$$

i.e.

$$\begin{aligned} \beta_2\gamma_1 x (A_1 + x) &= (x-h)(A_2+y)(\gamma_1 y + h(A_1+x)) \\ &\quad + (x-h)(A_2+y) A_1 (A_1+x), \\ (\gamma_1 y + (A_1+x) h)(A_2+y) &= \frac{1}{\beta_2} y (A_2(A_2+y) + \beta_2 x)(A_1+x), \end{aligned}$$

which implies

$$\begin{aligned} \beta_2\gamma_1 x (A_1+x) = (x-h) \frac{1}{\beta_2} y (A_2(A_2+y) + \beta_2 x)(A_1+x) \\ + (x-h)(A_2+y) A_1 (A_1+x). \end{aligned}$$

Since $A_1 + x > 0$ and $\beta_2\gamma_1 = (A_1+h) A_2$, we obtain

$$\beta_2(A_1+h) A_2 x = (x-h)\left[y(A_2(A_2+y) + \beta_2 x) + \beta_2(A_2+y) A_1\right],$$

i.e. (22).

This means that System (2) has infinitely many period-two solutions which belong to the third order curve given by (22). $\square$

Theorem 16. *Assume that* $\beta_2\gamma_1 = (A_1+h)\,A_2$. *Then every solution of System* (2) *converges to a period-two solution of System* (2) *that belongs to a curve* (22).

Proof. Here we prove that the map T is injective, which implies that T^2 is injective. Indeed, $T(x_1,y_1) = T(x_2,y_2)$ implies that

$$A_1(y_1-y_2) = x_1y_2 - x_2y_1 \tag{23}$$

and

$$A_2(x_1-x_2) = x_2y_1 - x_1y_2. \tag{24}$$

Now

$$y_1 \gtrless y_2 \overset{(23)}{\Rightarrow} x_1y_2 \gtrless x_2y_1 \Rightarrow x_1 \gtrless x_2 \overset{(24)}{\Rightarrow} x_2y_1 \gtrless x_1y_2,$$

which is a contradiction. Consequently

$$y_1 = y_2 \overset{(23)}{\Rightarrow} x_1y_2 = x_2y_1 \Rightarrow x_1 = x_2 \overset{(24)}{\Rightarrow} (x_1,y_1) = (x_2,y_2),$$

i.e. T is injective.
On the other hand for the map $T^2(x,y) = (T_1(x,y), T_2(x,y))$ we have

$$\frac{\partial T_1}{\partial x} = \frac{\frac{\gamma_1\beta_2}{A_2+y}\left(A_1+\frac{\gamma_1 y}{A_1+x}+h\right)+\frac{\gamma_1^2\beta_2 xy}{(A_1+x)^2(A_2+y)}}{\left(A_1+\frac{\gamma_1 y}{A_1+x}+h\right)^2}$$

$$\frac{\partial T_1}{\partial y} = \frac{\frac{-\gamma_1\beta_2 x}{(A_2+y)^2}\left(A_1+\frac{\gamma_1 y}{A_1+x}+h\right)-\frac{\gamma_1^2\beta_2 x}{(A_1+x)(A_2+y)}}{\left(A_1+\frac{\gamma_1 y}{A_1+x}+h\right)^2}$$

$$\frac{\partial T_2}{\partial x} = \frac{\frac{-\gamma_1\beta_2 y}{(A_1+x)^2}\left(A_2+\frac{\beta_2 x}{A_2+y}\right)-\frac{\gamma_1\beta_2^2 y}{(A_1+x)(A_2+y)}}{\left(A_2+\frac{\beta_2 x}{A_2+y}\right)^2}$$

$$\frac{\partial T_2}{\partial y} = \frac{\frac{\gamma_1\beta_2}{A_1+x}\left(A_2+\frac{\beta_2 x}{A_2+y}\right)+\frac{\gamma_1\beta_2^2 xy}{(A_1+x)(A_2+y)^2}}{\left(A_2+\frac{\beta_2 x}{A_2+y}\right)^2}$$

and

$$\begin{aligned}\det J_{T^2}(x,y) &= \begin{vmatrix} \frac{\partial T_1}{\partial x} & \frac{\partial T_1}{\partial y} \\ \frac{\partial T_2}{\partial x} & \frac{\partial T_2}{\partial y} \end{vmatrix} = \frac{\partial T_1}{\partial x}\frac{\partial T_2}{\partial y} - \frac{\partial T_2}{\partial x}\frac{\partial T_1}{\partial y} \\ &= \frac{\gamma_1\beta_2}{\left(A_1+\frac{\gamma_1 y}{A_1+x}+h\right)^2\left(A_2+\frac{\beta_2 x}{A_2+y}\right)^2}(AB-CD),\end{aligned}$$

where

$$A=\left[\frac{\gamma_1\beta_2}{A_2+y}\left(A_1+\frac{\gamma_1 y}{A_1+x}+h\right)+\frac{\gamma_1^2\beta_2 xy}{(A_1+x)^2(A_2+y)}\right],$$
$$B=\left[\frac{\gamma_1\beta_2}{A_1+x}\left(A_2+\frac{\beta_2 x}{A_2+y}\right)+\frac{\gamma_1\beta_2^2 xy}{(A_1+x)(A_2+y)^2}\right],$$
$$C=\left[\frac{-\gamma_1\beta_2 y}{(A_1+x)^2}\left(A_2+\frac{\beta_2 x}{A_2+y}\right)-\frac{\gamma_1\beta_2^2 y}{(A_1+x)(A_2+y)}\right],$$
$$D=\left[\frac{-\gamma_1\beta_2 x}{(A_2+y)^2}\left(A_1+\frac{\gamma_1 y}{A_1+x}+h\right)-\frac{\gamma_1^2\beta_2 x}{(A_1+x)(A_2+y)}\right].$$

Now we obtain

$$\begin{aligned}
\Gamma-\Lambda &= \frac{1}{(A_1+x)(A_2+y)}\left(A_1+\frac{\gamma_1 y}{A_1+x}+h\right)\left(A_2+\frac{\beta_2 x}{A_2+y}\right)\\
&\quad+\frac{\beta_2 xy}{(A_1+x)(A_2+y)^3}\left(A_1+\frac{\gamma_1 y}{A_1+x}+h\right)+\frac{\gamma_1 xy}{(A_1+x)^3(A_2+y)}\left(A_2+\frac{\beta_2 x}{A_2+y}\right)\\
&\quad+\frac{\gamma_1\beta_2 x^2y^2}{(A_1+x)^3(A_2+y)^3}-\frac{xy}{(A_1+x)^2(A_2+y)^2}\left(A_1+\frac{\gamma_1 y}{A_1+x}+h\right)\left(A_2+\frac{\beta_2 x}{A_2+y}\right)\\
&\quad-\frac{\gamma_1 xy}{(A_1+x)^3(A_2+y)}\left(A_2+\frac{\beta_2 x}{A_2+y}\right)-\frac{\beta_2 xy}{(A_1+x)(A_2+y)^3}\left(A_1+\frac{\gamma_1 y}{A_1+x}+h\right)\\
&\quad-\frac{\gamma_1\beta_2 xy}{(A_1+x)^2(A_2+y)^2}\\
&= \frac{1}{(A_1+x)(A_2+y)}\left(A_1+\frac{\gamma_1 y}{A_1+x}+h\right)\left(A_2+\frac{\beta_2 x}{A_2+y}\right)+\frac{\gamma_1\beta_2 x^2y^2}{(A_1+x)^3(A_2+y)^3}\\
&\quad-\frac{xy}{(A_1+x)^2(A_2+y)^2}\left(A_1+\frac{\gamma_1 y}{A_1+x}+h\right)\left(A_2+\frac{\beta_2 x}{A_2+y}\right)-\frac{\gamma_1\beta_2 xy}{(A_1+x)^2(A_2+y)^2}\\
&\quad-\frac{xy}{(A_1+x)^2(A_2+y)^2}\left(A_1+\frac{\gamma_1 y}{A_1+x}+h\right)\left(A_2+\frac{\beta_2 x}{A_2+y}\right)-\frac{\gamma_1\beta_2 xy}{(A_1+x)^2(A_2+y)^2},
\end{aligned}$$

that is

$$\begin{aligned}
\Gamma-\Lambda &= \frac{A_1A_2}{(A_1+x)(A_2+y)}+\frac{\gamma_1 A_2 y}{(A_1+x)^2(A_2+y)}+\frac{hA_2}{(A_1+x)(A_2+y)}\\
&\quad+\frac{\beta_2 A_1 x}{(A_1+x)(A_2+y)^2}+\frac{\gamma_1\beta_2 xy}{(A_1+x)^2(A_2+y)^2}+\frac{\beta_2 hx}{(A_1+x)(A_2+y)^2}\\
&\quad+\frac{\gamma_1\beta_2 x^2y^2}{(A_1+x)^3(A_2+y)^3}-\frac{A_1A_2xy}{(A_1+x)^2(A_2+y)^2}-\frac{\gamma_1 A_2 xy^2}{(A_1+x)^3(A_2+y)^2}\\
&\quad-\frac{hA_2xy}{(A_1+x)^2(A_2+y)^2}-\frac{\beta_2 A_1 x^2 y}{(A_1+x)^2(A_2+y)^3}-\frac{\gamma_1\beta_2 x^2y^2}{(A_1+x)^3(A_2+y)^3}\\
&\quad-\frac{\beta_2 hx^2y}{(A_1+x)^2(A_2+y)^3}-\frac{\gamma_1\beta_2 xy}{(A_1+x)^2(A_2+y)^2}\\
&= \frac{A_1A_2}{(A_1+x)(A_2+y)}+\frac{\gamma_1 A_2 y}{(A_1+x)^2(A_2+y)}+\frac{hA_2}{(A_1+x)(A_2+y)}+\frac{\beta_2 A_1 x}{(A_1+x)(A_2+y)^2}\\
&\quad+\frac{\beta_2 hx}{(A_1+x)(A_2+y)^2}-\frac{A_1A_2xy}{(A_1+x)^2(A_2+y)^2}-\frac{\gamma_1 A_2 xy^2}{(A_1+x)^3(A_2+y)^2}\\
&\quad-\frac{hA_2xy}{(A_1+x)^2(A_2+y)^2}-\frac{\beta_2 A_1 x^2 y}{(A_1+x)^2(A_2+y)^3}-\frac{\beta_2 hx^2y}{(A_1+x)^2(A_2+y)^3}\\
&= G\left(1-\frac{xy}{(A_1+x)(A_2+y)}\right),
\end{aligned}$$

where

$$G = \frac{A_1 A_2}{(A_1+x)(A_2+y)} + \frac{\gamma_1 A_2 y}{(A_1+x)^2(A_2+y)} + \frac{hA_2}{(A_1+x)(A_2+y)} + \frac{\beta_2 A_1 x}{(A_1+x)(A_2+y)^2} + \frac{\beta_2 hx}{(A_1+x)(A_2+y)^2}.$$

Since $xy < (A_1 + x)(A_2 + y)$, that is $1 - \frac{xy}{(A_1+x)(A_2+y)} > 0$, we have that $\det J_{T^2}(x, y) > 0$ for all nonnegative x and y.

By Theorem 5 this means that the condition $(O+)$ is satisfied. The conclusion follows from Theorems 4 and 16 . $\square$

4. Rate of convergence

In this section we will determine the rate of convergence of a solution of System (2) that converges to the equilibrium $E = (\overline{x}, \overline{y})$, in case described Theorem 11.

Theorem 17. *Assume that $\beta_2\gamma_1 > (A_1 + h) A_2$. Every solution $\{(x_n, y_n)\}$ of System (2) then converges to the equilibrium $E = (\overline{x}, \overline{y})$ and E is globally asymptotically stable. The error vector*

$$e_n = \begin{pmatrix} e_n^1 \\ e_n^2 \end{pmatrix} = \begin{pmatrix} x_n - \overline{x} \\ y_n - \overline{y} \end{pmatrix}$$

of every solution of (2) satisfies both of the following asymptotic relations:

$$\lim_{n\to\infty} \sqrt[n]{\|e_n\|} = |\lambda_i (J_T (E))| \quad \text{for some} \quad i = 1, 2, \tag{25}$$

and

$$\lim_{n\to\infty} \frac{\|e_{n+1}\|}{\|e_n\|} = |\lambda_i (J_T (E))| \quad \text{for some} \quad i = 1, 2, \tag{26}$$

where $|\lambda_i (J_T (E))|$ is equal to the modulus of one of the eigenvalues of the Jacobian matrix evaluated at the equilibrium E.

Proof. First we will find a system satisfied by the error terms. The error terms are given as (using (2) and (12))

$$\begin{aligned} x_{n+1} - \overline{x} &= \frac{\gamma_1 y_n}{A_1 + x_n} + h - \frac{\gamma_1 \overline{y}}{A_1 + \overline{x}} - h \\ &= \gamma_1 \frac{A_1 y_n + \overline{x} y_n - A_1 \overline{y} - x_n \overline{y}}{(A_1 + x_n)(A_1 + \overline{x})} \\ &= \gamma_1 \frac{(A_1 + \overline{x})(y_n - \overline{y}) - \overline{y}(x_n - \overline{x})}{(A_1 + x_n)(A_1 + \overline{x})} \\ &= \frac{-\gamma_1 \overline{y}}{(A_1 + x_n)(A_1 + \overline{x})}(x_n - \overline{x}) + \frac{\gamma_1}{A_1 + x_n}(y_n - \overline{y}) \end{aligned}$$

$$\begin{aligned}
y_{n+1}-\overline{y} &= \frac{\beta_2 x_n}{A_2+y_n}-\frac{\beta_2\overline{x}}{A_2+\overline{y}} \\
&= \beta_2\frac{A_2x_n+x_n\overline{y}-A_2\overline{x}-\overline{x}y_n}{(A_2+y_n)(A_2+\overline{y})} \\
&= \beta_2\frac{(A_2+\overline{y})(x_n-\overline{x})-\overline{x}(y_n-\overline{y})}{(A_2+y_n)(A_2+\overline{y})} \\
&= \frac{\beta_2}{A_2+y_n}(x_n-\overline{x})-\frac{\beta_2\overline{x}}{(A_2+y_n)(A_2+\overline{y})}(y_n-\overline{y}),
\end{aligned}$$

that is

$$\left\{\begin{array}{l}
x_{n+1}-\overline{x}=\frac{-\gamma_1\overline{y}}{(A_1+x_n)(A_1+\overline{x})}(x_n-\overline{x})+\frac{\gamma_1}{A_1+x_n}(y_n-\overline{y}) \\
y_{n+1}-\overline{y}=\frac{\beta_2}{A_2+y_n}(x_n-\overline{x})-\frac{\beta_2\overline{x}}{(A_2+y_n)(A_2+\overline{y})}(y_n-\overline{y}).
\end{array}\right. \tag{27}$$

Set $e_n^1=x_n-\overline{x}$, $e_n^2=y_n-\overline{y}$. Then system (27) can be represented as

$$e_{n+1}^1=a_ne_n^1+b_ne_n^2,\quad e_{n+1}^2=c_ne_n^1+d_ne_n^2,$$

where

$$a_n=\frac{-\gamma_1\overline{y}}{(A_1+x_n)(A_1+\overline{x})},\quad b_n=\frac{\gamma_1}{A_1+x_n},$$
$$c_n=\frac{\beta_1}{A_2+y_n},\quad d_n=\frac{-\beta_1\overline{x}}{(A_2+y_n)(A_2+\overline{y})}.$$

Taking the limits of a_n, b_n, c_n and d_n, we obtain

$$\lim_{n\to\infty}a_n=\frac{-\gamma_1\overline{y}}{(A_1+\overline{x})^2}=\frac{h-\overline{x}}{A_1+\overline{x}},\quad \lim_{n\to\infty}b_n=\frac{\gamma_1}{A_1+\overline{x}}=\frac{\overline{x}-h}{\overline{y}},$$
$$\lim_{n\to\infty}c_n=\frac{\beta_2}{A_2+\overline{y}}=\frac{\overline{y}}{\overline{x}},\quad \lim_{n\to\infty}d_n=\frac{-\beta_2\overline{x}}{(A_2+\overline{y})^2}=\frac{-\overline{y}}{A_2+\overline{y}},$$

that is

$$a_n=\frac{h-\overline{x}}{A_1+\overline{x}}+\alpha_n,\quad b_n=\frac{\overline{x}-h}{\overline{y}}+\beta_n,$$
$$c_n=\frac{\overline{y}}{\overline{x}}+\gamma_n,\quad d_n=\frac{-\overline{y}}{A_2+\overline{y}}+\delta_n,$$

where

$$\alpha_n\to 0,\quad \beta_n\to 0,\quad \gamma_n\to 0,\quad \delta_n\to 0,\quad (n\to\infty).$$

Now we have system of the form (8)

$$e_{n+1}=[A+B(n)]\,e_n,$$

where

$$A=\begin{pmatrix}\frac{h-\overline{x}}{A_1+\overline{x}} & \frac{\overline{x}-h}{\overline{y}} \\ \frac{\overline{y}}{\overline{x}} & \frac{-\overline{y}}{A_2+\overline{y}}\end{pmatrix},\quad B(n)=\begin{pmatrix}\alpha_n & \beta_n \\ \gamma_n & \delta_n\end{pmatrix}$$

and

$$\|B(n)\| \to 0 \quad (n \to \infty).$$

Thus, the limiting system of error terms can be written as

$$\begin{pmatrix} e^1_{n+1} \\ e^2_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{h-\overline{x}}{A_1+\overline{x}} & \frac{\overline{x}-h}{\overline{y}} \\ \frac{\overline{y}}{\overline{x}} & \frac{-\overline{y}}{A_2+\overline{y}} \end{pmatrix} \begin{pmatrix} e^1_n \\ e^2_n \end{pmatrix}.$$

This system is exactly the linearized system of (2) evaluated at the equilibrium $E = (\overline{x}, \overline{y})$. Then Theorems 8 and 9 imply the result. $\square$

References

[1] A. Brett, M. Garić-Demirović, M. R. S. Kulenović and M. Nurkanović, *Global behavior of two competitive rational systems of difference equations in the plane*, Commun. Appl. Nonlinear Anal., 16 (2009), 1–18.

[2] E. Camouzis, M. R. S. Kulenović, G. Ladas and O. Merino, *Rational systems in the plane - open problems and conjectures*, J. Difference Equ. Appl., 15 (2009), 303–323.

[3] D. Clark and M. R. S. Kulenović, *On a coupled system of rational difference equations*, Comput. Math. Appl., 43 (2002), 849–867.

[4] D. Clark, M. R. S. Kulenović and J. F. Selgrade, *Global asymptotic behavior of a two dimensional difference equation modelling competition*, Nonlinear Anal., TMA, 52 (2003), 1765–1776.

[5] C. A. Clark, M. R. S. Kulenović and J.F. Selgrade, *On a system of rational difference equations*, J. Difference Equ. Appl., 11 (2005), 565 – 580.

[6] M. Garic-Demirović, M. R. S. Kulenović and M. Nurkanović, *Global behavior of four competitive rational systems of difference equations in the plane*, Discrete Dyn. Nat. Soc., (2009), Article ID 153058, 34 pages .

[7] M. P. Hassell and H. N. Comins, *Discrete time models for two-species competition*, Theor. Popul. Biol., 9 (1976), 202–221.

[8] M. Hirsch and H. Smith, *Monotone Dynamical Systems, Handbook of Differential Equations, Ordinary Differential Equations*, (second volume), 239–357, Elsevier B. V., Amsterdam, 2005.

[9] S. Kalabušić and M.R.S. Kulenović, *Dynamics of certain anti-competitive systems of rational difference equations in the plane*, J. Difference Equ. Appl., (2010) (to appear).

[10] S. Kalabušić, M.R.S. Kulenović and E. Pilav, *Anti-competitive systems. General results*, (submited).

[11] M. R. S. Kulenović and G. Ladas, *Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures,* Chapman and Hall/CRC, Boca Raton, London, 2001.

[12] M. R. S. Kulenović and O. Merino, *Discrete Dynamical Systems and Difference Equations with Mathematica,* Chapman and Hall/CRC, Boca Raton, London, 2002.

[13] M. R. S. Kulenović and O. Merino, *Competitive-exclusion versus competitive-coexistence for systems in the plane*, Discrete Contin. Dyn. Syst. Ser. B, 6 (2006), 1141–1156.

[14] M. R. S. Kulenović and O. Merino, *A global attractivity result for maps with invariant boxes*, Discrete Contin. Dyn. Syst. Ser. B, 6 (2006), 97–110.

[15] M. R. S. Kulenović and M. Nurkanović, *Asymptotic behavior of two dimensional linear fractional system of difference equations*, Rad. Mat., 11 (2002), 59–78.

[16] M. R. S. Kulenović and M. Nurkanović, *Asymptotic behavior of a competitive system of linear fractional difference equations*, J. Ineq. Appl., (2005), 127–144.

[17] M. R. S. Kulenović and M. Nurkanović, *Asymptotic behavior of a system of linear fractional difference equations*, Adv. Difference Equ., (2006), 1–13.

[18] C. Robinson, *Stability, Symbolic Dynamics, and Chaos*, CRC Press, Boca Raton, 1995.

[19] J. F. Selgrade and M. Ziehe, *Convergence to equilibrium in a genetic model with differential viability between the sexes*, J. Math. Biol., 25 (1987), 477–490.

[20] H. L. Smith, *Planar competitive and cooperative difference equations*, J. Difference Equ. Appl., 3 (1998), 335–357.

[21] H. L. Smith, *The discrete dynamics of monotonically decomposable maps*, J. Math. Biology, 53 (2006), 747–758.

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