SARAJEVO JOURNAL OF MATHEMATICS
Vol.7 (19) (2011), 57–66

CLASSIFICATIONS OF SOLUTIONS OF SECOND ORDER NONLINEAR NEUTRAL DELAY DIFFERENCE EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

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ABSTRACT. In this paper the authors classified all solutions of the following equation

$$\Delta\Big(a_n\Big(\Delta(x_n+c_nx_{n-k})\Big)^{\alpha}\Big)+p_nf(x_{n-l})-q_ng(x_{n-m})=0$$

into four classes and obtain conditions for the existence/nonexistence of solutions in these classes. Examples are provided to illustrate the results.

1. Introduction

The qualitative behavior of solutions of nonlinear neutral type delay difference equations is of particular interest since such equations arise in the problems dealing with vibrating masses attached to an elastic bar and as an Euler equations in some variational problems, see for example [1,2,5,7,8,18] and the references cited therin.

In this paper, we are concerned with a quasilinear second order neutral type difference equation of the form

$$\Delta\Big(a_n\Big(\Delta(x_n+c_nx_{n-k})\Big)^{\alpha}\Big)+p_nf(x_{n-l})-q_ng(x_{n-m})=0,\ n\in\mathbb{N},\qquad (1)$$

subject to the following conditions:

(c_1) α is a ratio of odd positive integers and k,l,m are fixed nonnegative integers,
(c_2) $\{a_n\}$ is a positive real sequence, $\{c_n\},\{p_n\}$ and $\{q_n\}$ are real sequences with p_n and q_n not identically zero for large n,
(c_3) f and $g:\mathbb{R}\to\mathbb{R}$ are continuous with $uf(u)>0, ug(u)>0$ for $u\neq 0$.

2000 *Mathematics Subject Classification.* 39A11.
Key words and phrases. Second order, quasilinear, neutral equation, positive and negative coefficients.

Let $\theta = \max\{k, l, m\}$ and N_0 be a fixed nonnegative integer. By a solution of equation (1), we mean a function $\{x_n\}$ which is defined for all $n \geq N_0 - \theta$ and satisfies equation (1) for all $n \geq N_0 \in \mathbb{N}$. As usual, a solution of equation (1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is called nonoscillatory. A nonoscillatory solution $\{x_n\}$ of equation (1) is said to be weakly oscillatory if $\{\Delta x_n\}$ is oscillatory.

Let S denote the set of all nontrivial solutions of equation (1). In view of their asymptotic behavior, all solutions of equation (1) may be a priori divided into the following classes:

$$\begin{aligned}
M^+ =& \Big\{\{x_n\} \in S: \text{ there exists an integer } N \in \mathbb{N} \text{ such that } x_n \Delta x_n \geq 0 \\
& \quad \text{for all } n \geq N\Big\}; \\
M^- =& \Big\{\{x_n\} \in S: \{x_n\} \text{ is nonoscillatory and there exists an integer} \\
& \quad N \in \mathbb{N} \text{ such that } x_n \Delta x_n \leq 0 \text{ for all } n \geq N\Big\}; \\
OS =& \Big\{\{x_n\} \in S: \text{ for every integer } N \in \mathbb{N}, \text{ there exists an integer} \\
& \quad n \geq N \text{ such that } x_n x_{n+1} \leq 0\Big\}; \\
WOS =& \Big\{\{x_n\} \in S: \{x_n\} \text{ is nonoscillatory and for every } N \in \mathbb{N} \text{ there} \\
& \quad \text{exists an integer } n \geq N \text{ such that } \Delta x_n \Delta x_{n+1} \leq 0\Big\}.
\end{aligned}$$

In [6-8,14] the authors considered equation (1) with $q_n \equiv 0$ for all n and studied the oscillatory and asymptotic behavior of all solutions of equation (1). In [3] and [4] the authors consider equation (1) with $\alpha = 1$ and $q_n \equiv 0$ for all n and classified all solutions into the above said four classes and obtain conditions for the existence/ nonexistence of solutions in these classes. Similar classification have been discussed for the equation (1) when $\alpha = 1$ and $p_n \equiv 0$ for all n in [13].

In [11,12,15,16] the authors considered equation (1) with $\alpha = 1$ and $q_n \equiv c_n \equiv 0$ and discussed the existence and nonexistence of solutions of equation in the above said classes. Following this trend, in this paper we shall provide sufficient conditions which ensure the existence and nonexistence of equation (1) in the OS and WOS classes. We shall also study the asymptotic behavior of nonoscillatory solutions of equation (1). The motivation of the present work is stimulated by the recent works of [9,10,17].

2. Main results

First we examine the problem of existence of solutions of equation (1) in class M^+.

Theorem 1. *Assume that*

i) $c_n \geq 0$ *for all* $n \in \mathbb{N}$ *and* $\{c_n\}$ *is nondecreasing;*
ii) $m \geq l$ *and* $0 < \frac{g(u)}{f(u)} \leq M$ *for* $u \neq 0$;
iii) $q_n \geq 0$ *for all* $n \in \mathbb{N}$.

If f *is nondecreasing and*

$$\lim_{n\to\infty} \sup \sum_{s=n_0}^{\infty} (p_s - Mq_s) = \infty, \ n_0 \in \mathbb{N}, \tag{2}$$

then for the equation (1), *we have* $M^+ = \emptyset$.

Proof. Suppose that equation (1) has a solution $\{x_n\} \in M^+$. Without loss of generality, we may assume that there exists an integer $n_1 \geq n_0$ such that $x_n > 0, \Delta x_n \geq 0, x_{n-\theta} > 0, \Delta x_{n-\theta} \geq 0$ for all $n \geq n_1$ since the proof is similar if $x_n < 0$ and $\Delta x_n \leq 0$ for all large n. Let

$$z_n = x_n + c_n x_{n-k}, \ n \geq n_1.$$

Then by condition (i), $z_n > 0$ and $\Delta z_n \geq 0$ for all $n \geq n_1$. Now

$$\begin{aligned}\Delta\Big(\frac{a_n(\Delta z_n)^\alpha}{f(x_{n-l})}\Big) &= \frac{\Delta(a_n(\Delta z_n)^\alpha)}{f(x_{n-l})} - \frac{a_{n+1}(\Delta z_{n+1})^\alpha \Delta f(x_{n-l})}{f(x_{n-l})f(x_{n+1-l})} \\ &\leq -p_n + q_n \frac{g(x_{n-m})}{f(x_{n-l})} \leq -p_n + Mq_n \end{aligned} \tag{3}$$

where we have used (ii) and f is nondecreasing. Summing (3) from n_1 to $n-1$, we obtain

$$\frac{a_n(\Delta z_n)^\alpha}{f(x_{n-l})} - \frac{a_{n_1}(\Delta z_{n_1})^\alpha}{f(x_{n_1-l})} \leq \ -\sum_{s=n_1}^{n-1} (p_s - Mq_s).$$

From (2), we obtain

$$\lim_{n\to\infty} \inf \frac{a_n(\Delta z_n)^\alpha}{f(x_{n-l})} = -\infty$$

which contradicts the assumption $\Delta z_n \geq 0$ for all large n. This completes the proof. $\square$

The following example shows that the assumptions in Theorem 1 cannot be relaxed.

Example 1. Consider the neutral difference equation

$$\Delta(\frac{1}{n}\Delta(x_n + x_{n-1})) + \frac{1}{n(n+1)}x_n - \frac{n-1}{n^2(n+1)}x_{n-1} = 0, \ n \geq 2. \quad (4)$$

Here $a_n = \frac{1}{n}, c_n = 1, \alpha = 1, k = 1, l = 0, m = 1, p_n = \frac{1}{n(n+1)}, q_n = \frac{n-1}{n^2(n+1)}$, $f(u) = u$ and $g(u) = u$. Further $\frac{g(u)}{f(u)} = \frac{u}{u} = 1$, and

$$\lim_{n\to\infty} \sup \sum_{s=2}^{n} (p_s - Mq_s) = \sum_{n=2}^{\infty} \frac{1}{n^2(n+1)} < \infty.$$

All conditions of Theorem 1 are satisfied except condition (2). In fact equation (4) has a solution $\{x_n\} = \{n+1\} \in M^+$.

Theorem 2. *In addition to condition* ii) *assume that,*

iv) $-1 < c_n \leq 0$ *for all* $n \in \mathbb{N}$;
v) $p_n \geq Mq_n \geq 0$ *for all* $n \in \mathbb{N}$.

If f *is nondecreasing and*

$$\lim_{n\to\infty} \sum_{s=n_0}^{n-1} \frac{1}{(a_s)^{\frac{1}{\alpha}}} = \infty \text{ and } \lim_{n\to\infty} \sum_{s=n_0}^{n-1} (p_s - Mq_s) = \infty \quad (5)$$

then for the equation (1), *we have* $M^+ = \emptyset$.

Proof. Suppose that equation (1) has a solution $x_n \in M^+$. Without loss of generality we may assume that $x_n > 0, \Delta x_n \geq 0, x_{n-\theta} \geq 0$ and $\Delta x_{n-\theta} \geq 0$ for all $n \geq n_1$ since the proof for the case $x_n < 0$ and $\Delta x_n \leq 0$ for all large n is similar. Let $z_n = x_n + c_n x_{n-k} \geq (1 + c_n)x_{n-k} > 0$ for $n \geq n_1$. From equation (1), we have

$$\Delta(a_n(\Delta z_n)^\alpha) \leq -(p_n - Mq_n)f(x_{n-l}) \leq 0, \ n \geq n_1.$$

So $\{a_n(\Delta z_n)^\alpha\}$ is nonincreasing and we claim that $(a_n(\Delta z_n)^\alpha) \geq 0$ for $n \geq n_1$. If $a_n(\Delta z_n)^\alpha < 0$ then $a_n(\Delta z_n)^\alpha \leq a_{n_1}(\Delta z_{n_1})^\alpha < 0$ for $n \geq n_1$. Dividing the last inequality by a_n and then summing we obtain $z_n \leq z_{n_1} + \sum_{s=n_1}^{n-1} \frac{(a_{n_1}^{\frac{1}{\alpha}} \Delta z_{n_1})}{a_s^{\frac{1}{\alpha}}}$ which implies $z_n \to -\infty$ as $n \to \infty$, a contradiction. Hence $a_n(\Delta z_n)^\alpha \geq 0$ for $n \geq n_1$. The rest of the proof is similar to that of Theorem 1 and hence the details are omitted. □

Next we examine the existence of solutions belonging to class M^-.

Theorem 3. *Assume condition* iii) *holds. Further assume that*

vi) $k \leq m \leq l$ *and* $0 < \frac{g(u)}{f(u)} \leq M$ *for all* $u \neq 0$;
vii) $\int_0^\beta \frac{du}{f^{\frac{1}{\alpha}}(u)} < \infty, \ \int_{-\beta}^0 \frac{du}{f^{\frac{1}{\alpha}}(u)} > -\infty$ *for some* $\beta > 0$;

viii) *f is nondecreasing and submultiplicative;*
ix) $c_n \geq 0$ *and nonincreasing for all* $n \in \mathbb{N}$.

If

$$\lim_{n\to\infty} \sup \sum_{s=n_0}^{n-1} \Big(\frac{1}{a_s f(1+c_s)} \sum_{t=n_0}^{s-1} (p_t - Mq_t) \Big)^{\frac{1}{\alpha}} = \infty, \; n_0 \in \mathbb{N}, \tag{6}$$

then for the equation (1)*, we have* $M^- = \emptyset$.

Proof. Suppose that equation (1) has a solution $\{x_n\} \in M^-$. Without loss of generality we may assume that there exists an integer $n_1 \geq n_0 \in \mathbb{N}$ such that $x_n > 0, \Delta x_n \leq 0, x_{n-\theta} > 0$ and $\Delta x_{n-\theta} \leq 0$ for all $n \geq n_1$ since the proof for the case $x_n < 0, \Delta x_n \geq 0$ for all large n is similar. Let $z_n = x_n + c_n x_{n-k}$ then $z_n > 0$ and $\Delta z_n \leq 0$ for all $n \geq n_1$.

From the proof of Theorem 1, we have

$$\frac{a_n (\Delta z_n)^\alpha}{f(x_{n-l})} - \frac{a_{n_1} (\Delta z_{n_1})^\alpha}{f(x_{n_1 - l})} \leq \; - \sum_{s=n_1}^{n-1} (p_s - Mq_s).$$

or

$$\frac{(\Delta z_n)^\alpha}{f(x_{n-l})} \leq - \frac{1}{a_n} \sum_{s=n_1}^{n-1} (p_s - Mq_s), \; n \geq n_1. \tag{7}$$

Since $\{x_n\}$ is nonincreasing and $k \leq l$ we see that

$$z_n \leq (1 + c_n) x_{n-l}$$

and by condition (viii) we have $f(z_n) \leq f(1+c_n) f(x_{n-l}), \; n \geq n_1$. From (7), we have

$$\frac{\Delta z_n}{f^{\frac{1}{\alpha}}(z_n)} \leq - \Big(\frac{1}{a_n f(1+c_n)} \sum_{s=n_1}^{n-1} (p_s - Mq_s) \Big)^{\frac{1}{\alpha}}, \; n \geq n_1. \tag{8}$$

Now, for $z_{n+1} \leq u \leq z_n$, we have $\frac{1}{f(u)} \geq \frac{1}{f(z_n)}$ then it follows that

$$\int_{z_{n+1}}^{z_n} \frac{du}{f^{\frac{1}{\alpha}}(u)} \geq - \frac{(\Delta z_n)}{f^{\frac{1}{\alpha}}(z_n)}. \tag{9}$$

Using (9) in (8) and then summing the resulting inequality from n_1 to $n-1$, we obtain

$$\sum_{s=n_1}^{n-1} \Big(\frac{1}{a_s f(1+c_s)} \sum_{t=n_1}^{s-1} (p_t - Mq_t) \Big)^{\frac{1}{\alpha}} \leq \int_{z_n}^{z_{n_1}} \frac{du}{f^{\frac{1}{\alpha}}(u)} \leq \int_0^{z_{n_1}} \frac{du}{f^{\frac{1}{\alpha}}(u)}$$

which in view of (6) implies that $\int_0^{z_{n_1}} \frac{du}{f^{\frac{1}{\alpha}}(u)} = \infty$, a contradiction to vii). The proof is now complete. □

Theorem 4. *In addition to condition* ix) *assume that,*

x) $l \geq m$;

xi) $p_n \geq Mq_n > 0$ *for all* $n \in \mathbb{N}, f$ *is nondecreasing and* $0 < \frac{g(u)}{f(u)} \leq M$ *for* $u \neq 0$;

If

$$\sum_{n=n_0}^{\infty} \frac{1}{a_n^{\frac{1}{\alpha}}} = \infty \tag{10}$$

then for the equation (1) *we have* $M^- = \emptyset$.

Proof. Proceeding as in the proof of Theorem 3, define $z_n = x_n + c_n x_{n-k}$ then $z_n > 0$ and $\Delta z_n \leq 0$ for $n \geq n_1$. Let $w_n = a_n(\Delta z_n)^\alpha$ then $\Delta w_n = \Delta(a_n(\Delta z_n)^\alpha) \leq -(p_n - Mq_n)f(x_{n-l}) < 0,\ n \geq n_1$. Summing the last inequality from n_1 to $n-1$, we have

$$w_n \leq w_{n_1} - \sum_{s=n_1}^{n-1} (p_s - Mq_s) f(x_{s-l}) \leq w_{n_1},$$

also

$$\Delta z_n \leq \left(\frac{w_{n_1}}{a_n}\right)^{\frac{1}{\alpha}} < 0.$$

Summing the last inequality, we obtain

$$z_{n+1} \leq z_{n_1} + (w_{n_1})^{\frac{1}{\alpha}} \sum_{s=n_1}^{n} \frac{1}{a_s^{\frac{1}{\alpha}}} \to -\infty$$

as $n \to \infty$ which contradicts the assumption $z_n > 0$ for all large n. The proof is now complete. □

The assumptions in Theorems 3 and 4 are essential as the following examples show.

Example 2. Consider the neutral difference equations

$$\Delta(2^{4n}(\Delta(x_n + \frac{1}{2}x_{n-1}))^3) + \frac{2^{n+1} + 4^n}{4} x_{n-2} - 2^n x_{n-1} = 0, \tag{11}$$

and

$$\Delta(4^n(\Delta(x_n + 2x_{n-1}))) + \frac{5(4^n) + 4(2^n)}{8} x_{n-2} - 2^n x_{n-1} = 0. \tag{12}$$

For both of these equation $\{x_n\} = \{\frac{1}{2^n}\}$ is a solution, and hence $M^- \neq \emptyset$. For the equation (11) all conditions of Theorem 3 are satisfied except condition (6). But for the equation (12) conditions of Theorem 4 are satisfied except condition (10).

Theorem 5. *Assume conditions* x) *and* xi) *hold. Further assume that*

xii) $\sum\limits_{n=n_0}^{\infty} \frac{1}{a_n^{\frac{1}{\alpha}}} = \infty$;

xiii) $\lim\limits_{n\to\infty} \sum\limits_{s=n_0}^{n-1} (p_s - Mq_s) = \infty$;

xiv) $-1 < c_n \leq 0$ *for all* $n \in \mathbb{N}$.

Then for the equation (1) *we have* $M^- = \emptyset$.

Proof. Proceeding as in the proof of Theorem 4, define $z_n = x_n + c_n x_{n-k}$ and we have $z_n \geq (1 + c_n)x_n > 0$. From the equation (1) we have

$$\Delta(a_n(\Delta z_n)^\alpha) \leq -(p_n - Mq_n)f(x_{n-l}),\ n \geq n_1.$$

Then it follows that $a_n(\Delta z_n)^\alpha$ is decreasing for $n \geq n_1$ and proceeding as in the proof of Theorem 2, in view of condition xii) we find $a_n(\Delta z_n)^\alpha > 0$ for $n \geq n_1$.

Now define $w_n = \frac{a_n(\Delta z_n)^\alpha}{f(z_n)},\ n \geq n_1$, to obtain

$$\begin{aligned}\Delta w(n) &= \frac{\Delta(a_n(\Delta z_n)^\alpha)}{(f(z_{n+1}))} - \frac{a_{n+1}(\Delta z_{n+1})^\alpha \Delta f(z_n)}{f(z_n)f(z_{n+1})} \\ &\leq -(p_n - Mq_n)\frac{f(x_{n-l})}{f(z_{n+1})}.\end{aligned} \tag{13}$$

From the definition of z_n we have in view of xiv) that $z_{n+1} \leq x_{n+1}$. Since $x_n \in M^-$, we obtain $z_{n+1} \leq x_{n-l}$. Thus $f(z_{n+1}) \leq f(x_{n-l})$ for $n \geq n_1$. Using this inequality in (13) and summing the resulting inequality we obtain

$$w_n \leq w_{n_1} - \sum_{s=n_1}^{n-1} (p_s - Mq_s)$$

which because of (xiii) implies that $w_n \to -\infty$ as $n \to \infty$. This contradiction completes the proof. □

Theorem 6. *Assume* $m = l$, *and conditions* xi), xii), xiii) *and* xiv) *hold. Then every solution of equation* (1) *is either oscillatory or weakly oscillatory.*

Proof. From Theorem 2 it clear that $M^+ = \emptyset$. Hence, to complete the proof it suffices to show that $M^- = \emptyset$. For this, let $\{x_n\}$ be a solution of equation (1) which is in the class M^-. For this solution, as earlier we assume that $x_n > 0, \Delta x_n \leq 0, x_{n-\theta} > 0$ and $\Delta x_{n-\theta} \leq 0$ for all $n \geq n_1$, and define $z_n = x_n + c_n x_{n-k}$. Then, in view of condition xiv), $z_n \geq (1 + c_n)x_n > 0$ and $\Delta z_n \leq 0$ for all $n \geq n_1$. Again, we define $w_n = a_n(\Delta z_n)^\alpha$ so that $w_n \leq 0,\ n \geq n_1$, and from the equation (1), we have

$$w_n \leq w_{n_1} - \sum_{s=n_1}^{n-1} (p_s - Mq_s)f(x_{s-l}).$$

Thus from Abel's transformation [1] it follows that

$$w_n \le w_{n_1} - f(x_{n-l}) \sum_{s=n_1}^{n-1} (p_s - Mq_s) + \sum_{s=n_1}^{n-1} \Delta f(x_{s-l}) \Big(\sum_{t=n_1}^{s} (p_t - Mq_t) \Big).$$

From the above relation, and the conditions xi) and xiii) with n sufficiently large, we find that $w_n \le w_{n_1}$, that is $\Delta z_n \le \frac{w_{n_1}^{\frac{1}{\alpha}}}{a_n^{\frac{1}{\alpha}}} < 0$. The rest of the proof of $M^- = \emptyset$ is same as that of Theorem 4. This completes the proof. $\square$

Theorem 7. *Assume conditions* iii), vi), viii) *and* ix) *hold. If* $\frac{g(u)}{f(u)} \le M$ *and condition* (6) *satisfied, then for every solution* $x_n \in M^-$ *of equation* (1), $\lim\limits_{n\to\infty} x_n = 0$.

Proof. The proof follows from the same argument as in the proof of Theorem 3, and the fact that $\lim_{n\to\infty} \int_{z_n}^{z_{n_1}} \frac{dt}{f^{\frac{1}{\alpha}}(t)} = \infty$ implies that $\lim_{n\to\infty} z_n = 0$, and $z_n \ge x_n$ for all $n \ge n_1$. This completes the proof. $\square$

Finally, we examine the asymptotic behavior of solutions in the class M^+.

Theorem 8. *Assume that*

xv) $c_n \ge 0$ *nondecreasing and bounded,*

xvi) $\alpha \ge 1, m \ge l, f$ *is nondecreasing and* $0 < \frac{g(u)}{f(u)} \le M$ *for* $u \ne 0$,

xvii) $q_n \ge 0$ *for all* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} \sup \sum_{s=n_0}^{n} (p_s - Mq_s) \Big(\sum_{t=n_0}^{s} \frac{1}{a_t} \Big) = \infty$, *for all* $n_0 \in \mathbb{N}$.

Then, every solution of equation (1) *in the class* M^+ *is unbounded.*

Proof. Let $\{x_n\}$ be a solution of equation (1) in M^+. As in Theorem 1, without loss of generality we assume that there exits an integer $n_1 \ge n_0 \in \mathbb{N}$ such that $x_n > 0, \Delta x_n \ge 0, x_{n-\theta} > 0, \Delta x_{n-\theta} \ge 0$ for all $n \ge n_1$, and set $z_n = x_n + c_n x_{n-k}$, so that $z_n > 0, \Delta z_n \ge 0$ for all $n \ge n_1$. For the function

$$w_n = -\frac{a_n (\Delta z_n)^\alpha}{f(x_{n-l})} \sum_{s=n_1}^{n} \frac{1}{a_s}, \tag{14}$$

we have

$$\Delta w_n = -\frac{\Delta(a_n(\Delta z_n)^\alpha)}{f(x_{n-l})} \sum_{s=n_1}^{n} \frac{1}{a_s} - \frac{(\Delta z_n)^\alpha}{f(x_{n-l})} + \frac{(a_{n+1}(\Delta z_{n+1})^\alpha)}{f(x_{n-l}) f(x_{n+1-l})} \Big(\sum_{s=n_1}^{n} \frac{1}{a_s} \Big) \Delta f(x_{n-l})$$

$$\geq (p_n - Mq_n)\sum_{s=n_1}^{n}\frac{1}{a_s} - \frac{(\Delta z_n)^\alpha}{f(x_{n-l})}.$$

Summing the last inequality, we obtain

$$w_n \geq \sum_{s=n_1}^{n}(p_s - Mq_s)\sum_{t=n_1}^{s}\frac{1}{a_t} - \sum_{s=n_1}^{n-1}\frac{(\Delta z_n)^\alpha}{f(x_{n-l})}. \tag{15}$$

Since $\frac{(\Delta z_n)^\alpha}{f(x_{n-l})}$ is positive for $n \geq n_1$, the following exists

$$\lim_{n\to\infty}\sum_{s=n_1}^{n-1}\frac{(\Delta z_n)^\alpha}{f(x_{n-l})} = \beta, (say).$$

We claim that $\beta = \infty$. Indeed, if $\beta < \infty$, then xvii) combined with (15) leads to $\lim_{n\to\infty} w_n = \infty$, which is a contradiction to the fact that w_n is negative for all $n \geq n_1$. Now for all $n \geq n_1$, we have $f(x_{n-l}) \geq f(x_{n_1-l}) = A$, and consequently

$$\sum_{s=n_1}^{n-1}\frac{(\Delta z_s)^\alpha}{f(x_{s-l})} \leq \frac{1}{A}\Big(\sum_{s=n_1}^{n-1}\Delta z_s\Big)^\alpha \leq \frac{z_n^\alpha}{c}.$$

Thus it follows that $\lim_{n\to\infty} z_n = \infty$. Finally $z_n = x_n + c_n x_{n-k} \leq (1+c_n)x_n$, the boundedness of $\{c_n\}$ implies that $\lim_{n\to\infty} x_n = \infty$. This completes the proof. □

Remark 1. An example of an equation with unbounded solutions belonging to class M^+ is provided by equation (4).

Remark 2. If g is nondecreasing instead of f then the conclusion of the theorems remain the same provided the condition $0 < \frac{g(u)}{f(u)} \leq M$ for $u \neq 0$ be replaced by $\frac{f(u)}{g(u)} \geq M > 0$ for $u \neq 0$.

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(Received: June 28, 2010)

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