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CURVATURE VIA THE DE SITTER'S SPACE-TIME

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Abstract. We define the central curvature and the total central curvature of a closed curve in a Lorentz $(n+1)$–space. In addition, we obtain estimates for the total central curvatures of spacelike pure polygons. The study is done by using the n-dimensional de Sitter's space-time.

1. Introduction

In the last three decades interest in Lorentzian geometry has increased, [2]. We will concentrate on a curvature of particular interest here: the total central curvature of closed curves.

In some papers, the total central curvature is related to Riemannian spaces; but this curvature has not been treated on spaces with indefinite metrics.

In [1], Thomas F. Banchoff stated that the total central curvature of a closed curve in 3-dimensional Euclidean space refers to the measure of curvedness of a space curve contained in a bounded ball. He obtained this curvature by averaging the total absolute curvature of the image curves under central projection from all points on the sphere, and he showed that the total central curvature agrees with the classical total absolute curvature of the original space.

In [5] we defined, by means of integral formulas, the total central curvature in Lorentzian spaces with dimension 2 and 3.

The aim of this paper is to generalize the total central curvature for closed curves in the Lorentzian space L^{n+1}, $n \geq 3$.

Also, we will show that if f_1 is a polygon then $tcc_2(f_1) = k$, where k is the number of its non middle vertices. In order to do that we will first give the readers the definitions of s-spherical and t-spherical image of a polygon in Lorentzian spaces and we will characterize the local support line to a polygon in the Lorentzian plane.

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Last, we will show that if f_n is a polygon with m vertices in L^{n+1}, $n \geq 2$, then $tcc_{n+1}(f_n) \leq m$.

In section two, preliminaries, we will recall the basic notions in Lorentzian geometry.

2. Preliminaries

Let x and y be two vectors in the $(n+1)-$dimensional vector space $\mathbb{R}^{n+1}$. As it is well known, [8], the Lorentzian inner product of x and y is defined by

$$\langle x, y \rangle = -x_1 y_1 + \sum_{i=2}^{n+1} x_i y_i.$$

Thus the square ds^2 of an element of arc-length is given by

$$ds^2 = -dx_1^2 + \sum_{i=2}^{n+1} dx_i^2.$$

The space $\mathbb{R}^{n+1}$ equipped with this metric is called a $(n+1)$-dimensional Lorentzian space, or Lorentz $(n+1)$-space, which has curvature zero. We write L^{n+1} instead of $\left(\mathbb{R}^{n+1},\ ds\right)$.

We say that a vector x in L^{n+1} is timelike if $\langle x, x \rangle < 0$, spacelike if $\langle x, x \rangle > 0$, and null if $\langle x, x \rangle = 0$. The null vectors are also said to be lightlike.

We consider the time orientation as follows: a timelike vector x is future-pointing if $\langle x, e_1 \rangle < 0$ and past-pointing if $\langle x, e_1 \rangle > 0$, where $e_1 = (1, 0, \ldots, 0)$.

We say that x is orthogonal to y if $\langle x, y \rangle = 0$, $x \neq y \neq 0$.

Let x be a vector in L^{n+1}, then $\|x\| = \sqrt{|\langle x, x \rangle|}$ is called the Lorentzian norm of x. We say that x is a unit vector if $\langle x, x \rangle = 1$ or $\langle x, x \rangle = -1$.

We shall give a hypersurface M in L^{n+1} by expressing its coordinates x_i as functions of n parameters in a certain interval. We consider the functions x_i to be real functions of real variables.

We say that M is a non-lightlike hypersurface if at every $p \in M$ its tangent space T_pM is equipped with a positive definite or Lorentzian metric, [2].

A parametrized curve is called timelike, spacelike or null pure curve if at every point, its tangent vector is timelike, spacelike or null, respectively.

Now, we recall the definition of a pure polygon in Lorentzian spaces (cf. [4]). If $p_1, \ldots, p_m \in L^{n+1}$, a polygon $\mathbf{P}[p_1, \ldots, p_m]$ is called a pure spacelike polygon if $\overrightarrow{p_1 p_2}, \ldots, \overrightarrow{p_{m-1} p_m}$ and $\overrightarrow{p_1 p_m}$ are spacelike vectors.

Analogous to time orientation, we define a spacelike orientation in Lorentzian plane: a spacelike vector x is a spacelike-for-future vector if $\langle x, e_2 \rangle > 0$, and spacelike-for-past vector if $\langle x, e_2 \rangle < 0$, where $e_2 = (0, 1)$.

In L^2, a vertex p_i of a pure spacelike polygon is called a middle vertex if $\overrightarrow{p_{i-1}p_i}$ and $\overrightarrow{p_ip_{i+1}}$ have the same spacelike orientation, and non middle vertex if $\overrightarrow{p_ip_{i-1}}$ and $\overrightarrow{p_ip_{i+1}}$ have the same spacelike orientation.

Classically, we say that $S_1^n = \left\{x \in L^{n+1} : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = 1\right\}$ is the $n-$dimensional de Sitter's space-time

When $n = 1$, S_1^1 is called a Lorentzian circle; in what follows, we will denote:

$$\left(S_1^1\right)_+ = \left\{(x_1, x_2) \in S_1^1 : x_2 > 0\right\}, \left(S_1^1\right)_- = \left\{(x_1, x_2) \in S_1^1 : x_2 < 0\right\}.$$

In [5], we defined the total central curvature of f_1 as follows.

Definition 1. *Let $(C_j)_{j\geq 1}$ be a sequence of connected arcs such that :*

i) $C_j \subset \left(S_1^1\right)_+$, $\forall j \geq 1$.
ii) $C_j \subset C_{j+1}$, $\forall j \geq 1$.
iii) $0 < length\left(C_j\right) < \infty$, $\forall j \geq 1$.
iv) $\lim\limits_{j\to\infty} C_j = \left(S_1^1\right)_+$.

If $f_1 : S^1 \to L^2$ is a continuous map of the circle S^1 in the Lorentzian plane such that $f_1\left(S^1\right) \cap \left(S_1^1\right)_+ = \emptyset$, then the total central curvature of f_1 is defined by

$$tcc_2\left(f_1\right) = \lim_{j\to\infty} \frac{1}{length\left(C_j\right)} \int_{\xi\in C_j} \Upsilon_\xi\left(f_1\right)\ ds_{C_j}$$

if this limit exists; in other case, we say that $tcc_2\left(f_1\right) = \infty$.

3. S-spherical and t-spherical image of a polygon

In [7], Milnor related the total curvature of a polygon in R^{n+1} to its spherical polygon. He defined the spherical image of a vector $\overrightarrow{p_{i+1} - p_i} \in \mathbb{R}^{n+1}$ as $q_i = \frac{p_{i+1}-p_i}{\|p_{i+1}-p_i\|} \in S^n$, where S^n is the unit $n-$dimensional Euclidean sphere. A spherical polygon $\mathbf{Q}$ is formed on S^n by joining each q_{i-1} to q_i by a great circle of length $0 \leq \alpha_i \leq \pi$.

If p_i, $p_{i+1} \in L^{n+1}$, the vector $\overrightarrow{p_{i+1} - p_i}$ is a timelike, spacelike or lightlike vector.

Definition 2. *Let p_i, $p_{i+1} \in L^{n+1}$.*

i) *If $\overrightarrow{p_{i+1} - p_i}$ is a timelike vector, then its t-spherical image is $q_i = \frac{p_{i+1}-p_i}{\|p_{i+1}-p_i\|} \in H_0^n$, where $H_0^n = \left\{x \in L^{n+1} : -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = -1\right\}$.*
ii) *If $\overrightarrow{p_{i+1} - p_i}$ is a spacelike vector, then its s-spherical image is $q_i = \frac{p_{i+1}-p_i}{\|p_{i+1}-p_i\|} \in S_1^n$.*

Next, we define the oriented s-spherical image in Lorentzian plane.

Definition 3. *Let p_i, $p_{i+1} \in L^2$ such that $\overrightarrow{p_{i+1} - p_i}$ is a spacelike vector.*

i) *The s–for-future spherical image of $\overrightarrow{p_{i+1} - p_i}$ is*
$$q_i^+ = \begin{cases} \frac{p_{i+1}-p_i}{\|p_{i+1}-p_i\|} & \text{if } \overrightarrow{p_{i+1} - p_i} \text{ is spacelike-for-future,} \\ \frac{p_i-p_{i+1}}{\|p_{i+1}-p_i\|} & \text{if } \overrightarrow{p_{i+1} - p_i} \text{ is spacelike-for-past.} \end{cases}$$

ii) *The s-for-past spherical image of $\overrightarrow{p_{i+1} - p_i}$ is*
$$q_i^- = \begin{cases} \frac{p_{i+1}-p_i}{\|p_{i+1}-p_i\|} & \text{if } \overrightarrow{p_{i+1} - p_i} \text{ is spacelike-for-past,} \\ \frac{p_i-p_{i+1}}{\|p_{i+1}-p_i\|} & \text{if } \overrightarrow{p_{i+1} - p_i} \text{ is spacelike-for-future.} \end{cases}$$

De Sitter's space-time is geodesically complete; however, there are points in the space which cannot be joined to each other by any geodesic. This is in contrast to spaces with a positive definite metric, where geodesic completeness guarantees that any two points of a space can be joined by at least one geodesic, (cf. [6]).

Proposition 4. *Let $\boldsymbol{P}$=$\boldsymbol{P}[p_1, \ldots, p_m]$ be a pure spacelike polygon in Lorentz 3-space. If $p_1, \ldots, p_m$ lie on the plane $x_1 = 0$, then q_i lies on the curve given by $x_2^2 + x_3^2 = 1$, $\forall i$. Hence the edge $q_{i-1}q_i$ is spacelike, that is q_{i-1} joins to q_i by a spacelike geodesic curve in S_1^2.*

Proof. By definition of s-spherical image. □

We define the s–for-future and s-for-past spherical polygon of a pure spacelike polygon $\mathbf{P}$=$\mathbf{P}[p_1, \ldots, p_m]$ in the Lorentzian plane. Without loss of generality, in what follows we assume that p_1 is a non middle vertex (cf. [4]).

Definition 5. *Let $\boldsymbol{P}$=$\boldsymbol{P}[p_1, \ldots, p_m]$ be a pure spacelike polygon in Lorentzian plane.*

i) *The s–for-future spherical polygon of $\boldsymbol{P}$, $\boldsymbol{Q}^+$, is formed on $\left(S_1^1\right)_+$ by joining each q_{i-1}^+ to q_i^+ by a connected arc with finite length.*
ii) *The s-for-past spherical polygon of $\boldsymbol{P}$, $\boldsymbol{Q}^-$, is formed on $\left(S_1^1\right)_-$ by joining each q_{i-1}^- to q_i^- by a connected arc with finite length.*

In [5], we define a local support line in Lorentzian plane.

Definition 6. *Let $f : S^1 \to L^2$ be a continuous map of the circle S^1 in the Lorentzian plane. A local support line to f at x is a line containing x and bounding a closed half-plane which contains the image of a neighborhood of x in S^1.*

We denote the number of local support to f passing through the point ξ by $\Upsilon_\xi\,(f)$.

Now we characterize the local support line to a polygon in Lorentzian plane.

Theorem 7. *Let $\boldsymbol{P}$=$\boldsymbol{P}[p_1,\ldots,p_m]$ be a pure spacelike polygon in L^2 and let $\boldsymbol{Q}$=$\boldsymbol{Q}[q_1,\ldots,q_{t_m}]$ be its s-spherical polygon. Without loss of generality, we choose the orientation such that every q_i lie on $\left(S^1_1\right)_+$.*

Let $\xi\in\left(S^1_1\right)_+$. Then:

i) *The line $\overleftrightarrow{\xi p_1}$ is a local support line to $\boldsymbol{P}$ if and only if ξ does not lie on the edge q_1q_m.*
ii) *If $i>1$ and p_i is an non middle vertex, then the line $\overleftrightarrow{\xi p_i}$ is a local support line to $\boldsymbol{P}$ if and only if ξ does not lie on the edge $q_{i-1}q_i$.*
iii) *If p_i is a middle vertex, then the line $\overleftrightarrow{\xi p_i}$ is a local support line to $\boldsymbol{P}$ if and only if ξ lies on the edge $q_{i-1}q_i$.*

Proof. Let $\mathbf{P}$=$\mathbf{P}[p_1,\ldots,p_m]$ be a pure spacelike polygon in L^2**.** A line is local line support of $\mathbf{P}$ if and only if it contains a vertex p_j.

In the other hand, the sum of non middle angles is equal to sum of middle angles of **P,** [4]**.**

By definition of middle and non middle angles, and by the definition of q_i, i)-iii) are following . Let us recall that the angle ω_i between q_i and q_{i+1} is a central angle and it is equal to length of arc joining q_i and q_{i+1}. □

4. Total central curvature of closed curves in L^{n+1}

Let us recall that $S^n_1\subset L^{n+1}$ is not a compact hypersurface and its volume is not finite either. Hence, we first will define the curvature of a continuous map f_n with respect to a connected n-region $R^n_j\subset S^n_1$ which has finite volume.

In what follows, we consider $f_n: S^1\to L^{n+1}$ a continuous map of the circle S^1 in the Lorentz $(n+1)$-space such that $\langle x,x\rangle>1\ \forall x\in f_n\left(S^1\right)$.

We first define the central projection map in L^{n+1}.

Definition 8. *Let $T_p\left(S^n_1\right)$ be the tangent space to S^n_1 at $p\in S^n_1$ and let L^n_p be the hyperplane parallel to $T_p\left(S^n_1\right)$ through the center of S^n_1. The central projection map $\pi_p: L^{n+1}-T_p\left(S^n_1\right)\to L^n_p$ is given by*

$$\pi_p\left(x\right)=\frac{1}{1-\langle x,p\rangle}\left(x-\langle x,p\rangle\ \ p\right).$$

This projection map is a one to one and onto map, (cf. [3] for more details). Note that π_p restricted to $S^2_1-T_p\left(S^2_1\right)$ gives the stereographic projection in Lorentzian 3-space.

Let us remark that L^n_p and L^n are two congruent spaces.

In what follows, we consider $n\geq 3$.

Definition 9. *Let $R_j^n \subset S_1^n$ be a connected region such that $0 < vol(R_j^n) < \infty$. The central curvature of f_n with respect to R_j^n is defined by*

$$cc_{n+1}\left(f_n, R_j^n\right) = \frac{1}{vol\left(R_j^n\right)} \int_{p\in R_j^n} cc_n\left(\pi_p \circ i_{n+1} \circ f_n,\ \left(R_j^n \cap L_p^n\right)\right)\ ds_{R_j^n},$$

where $\int_{R_j^n} ds_{R_j^n} = vol(R_j^n) =:$ volume of R_j^n; $i_{n+1} : \left\{x \in L^{n+1} : \langle x, x\rangle > 1\right\} \to L^{n+1} - T_p\left(S_1^n\right)$ is the inclusion map, and $\pi_p : L^{n+1} - T_p\left(S_1^n\right) \to L_p^n$ is the central projection map.

We now define the total central curvature of f_n.

Definition 10. *Let $(R_j^n)_{j\geq 1}$ be a sequence of connected regions such that :*

i) $R_j^n \subset S_1^n$, $\forall j \geq 1$.
ii) $R_j^n \subset R_{j+1}^n$, $\forall j \geq 1$.
iii) $0 < vol(R_j^n) < \infty$, $\forall j \geq 1$.
iv) $\forall p \in S_1^n$, $\exists j_0 \geq 1$ *such that* $(R_j^n \cap L_p^n) \neq \emptyset$, $\forall j \geq j_0$.
v) $\lim_{j\to\infty} R_j^n = S_1^n$.

The total central curvature of f_n is defined by

$$tcc_{n+1}\left(f_n\right) = \lim_{j\to\infty} cc_{n+1}\left(f_n, R_j^n\right)$$

if this limit exists; in other case, we say that $tcc_{n+1}\left(f_n\right) = \infty$.

The existence of a sequence $(R_j^n)_{j\geq 1}$ satisfying the above mentioned conditions is proven by the following Theorem.

Theorem 11. *In L^{n+1} there exists a sequence $(R_j^n)_{j\geq 1}$ of connected regions with properties i)-v) of Definition* 10.

Proof. Let $\{a_j\}_{j\geq 1}$ be a sequence of positive real numbers such that:
a) $a_1 > 1$.
b) $a_{j+1} = a_j + \frac{1}{j}$, $\forall j \geq 1$.

If $R_j^n = \left\{\left(x_1^j, \ldots, x_{n+1}^j\right) \in S_1^2 : -a_j < x_1^j < a_j\right\}$ then $(R_j^n)_{j\geq 1}$ is a sequence of connected regions with i)-v) properties. □

Lemma 12. *Let $(R_j^n)_{j\geq 1}$ be a sequence of connected regions as given in Theorem* 11. *If $(R_i^n)_{i\geq 1}$ is a subsequence of $(R_j^n)_{j\geq 1}$ with properties i)-v), then*

$$\lim_{j\to\infty} cc_{n+1}\left(f_n; R_j^n\right) = \lim_{i\to\infty} cc_{n+1}\left(f_n; R_i^n\right).$$

Proof. Since the properties i)-iv) hold for $(R_j^n)_{j\geq 1}$ and $(R_i^n)_{i\geq 1}$, and $(R_i^n)_{i\geq 1} \subset (R_j^n)_{j\geq 1}$, then:

$$\forall i \geq 1 \exists j_{1_i}, j_{2_i} \geq 1 \text{ such that } R^n_{j_{1_i}} \subset R_i^n \subset R^n_{j_{2_i}} \tag{5.1}$$

and

$$\forall j \geq 1 \exists i_{1_j}, i_{2_j} \geq 1 \text{ such that } R^n_{i_{1_j}} \subset R_j^n \subset R^n_{i_{2_j}}. \tag{5.2}$$

By (5.1) and (5.2), $\lim_{j\to\infty} cc_{n+1}(f_n; R_j^n) = \lim_{i\to\infty} cc_{n+1}(f_n; R_i^n)$. $\square$

In what follows, if $R^{n-1} \subset S_1^{n-1}$ is a non connected region such that $R^{n-1} = \cup_{i\in I} R_i^{n-1}$ is a disjoint union and R_i^{n-1} is a connected region $\forall i \in I$, we denote $\sum_{i\in I} cc_n\left(f_{n-1}, R_i^{n-1}\right)$ by $cc_n\left(f_{n-1}, R^{n-1}\right)$.

Theorem 13. *Let $(R_j^n)_{j\geq 1}$ and $(R_h^n)_{h\geq 1}$ be two sequences of connected regions as given in Theorem* 11, *then* $\lim_{j\to\infty} cc_{n+1}(f_n; R_j^n) = \lim_{h\to\infty} cc_{n+1}(f_n; R_h^n)$.

Proof. By properties i)-v), $\forall h \geq 1\ \exists j_{1_h}, j_{2_h} \geq 1$ such that:

a) $R^n_{j_{1_h}} \subset R_h^n \subset R^n_{j_{2_h}}$

b) $(R^n_{j_{1_h}})_{h\geq 1}, (R^n_{j_{2_h}})_{h\geq 1} \subset (R_j^n)_{j\geq 1}$ with properties i)-v).

By a), we have that

$$\begin{aligned}
&\frac{1}{vol(R^n_{j_{2_h}})} \int_{p\in R^n_{j_{2_h}}} cc_n(\pi_p \circ i_{n+1} \circ f_n,\ (R^n_{j_{2_h}} \cap L_p^n)) \\
&< \int_{p\in (R^n_{j_{2_h}} - R^n_h)} \frac{cc_n(\pi_p \circ i_{n+1} \circ f_n,\ ((R^n_{j_{2_h}} - R^n_h) \cap L_p^2))}{vol(R^2_{j_{2_h}} - R^2_h)} ds_{(R^n_{j_{2_h}} - R^n_h)} \\
&\quad + \int_{p\in R_h^n} \frac{cc_n(\pi_p \circ i_{n+1} \circ f_n,\ (R_h^n \cap L_p^n)_+)}{vol(R_h^n)} ds_{R_h^n} \\
&< \int_{p\in (R^n_{j_{2_h}} - R^n_{j_{1_h}})} \frac{cc_n(\pi_p \circ i_{n+1} \circ f_n,\ ((R^n_{j_{2_h}} - R^n_{j_{1_h}}) \cap L_p^n))}{vol(R^n_{j_{1_h}})} ds_{(R^n_{j_{2_h}} - R^n_{j_{1_h}})} \\
&\quad + \int_{p\in R_h^2} \frac{cc_n(\pi_p \circ i_{n+1} \circ f_n,\ (R_h^n \cap L_n^n)_+)}{vol(R_h^n)} ds_{R_h^n}.
\end{aligned}$$

By b), we have that

$$\begin{aligned}
&\lim_{j\to\infty} cc_{n+1}(f_n; R_j^n) \\
&\leq \lim_{h\to\infty} \int_{p\in (R^n_{j_{2_h}} - R^n_{j_{1_h}})} \frac{cc_n(\pi_p \circ i_{n+1} \circ f_n, ((R^n_{j_{2_h}} - R^n_{j_{1_h}}) \cap L_p^n))}{vol(R^n_{j_{1_h}})} ds_{(R^n_{j_{2_h}} - R^n_{j_{1_h}})} \\
&\qquad + \lim_{h\to\infty} cc_{n+1}(f_n; R_h^n).
\end{aligned}$$

On the other hand,

$$\lim_{h\to\infty}\int_{p\in(R^n_{j_{2_h}}-R^n_{j_{1_h}})} cc_n(\pi_p\circ i_{n+1}\circ f_n,\ ((R^n_{j_{2_h}}-R^n_{j_{1_h}})\cap L^n_p))ds_{(R^n_{j_{2_h}}-R^n_{j_{1_h}})}$$

$$= \text{constant} \Rightarrow \lim_{h\to\infty}\int_{p\in(R^n_{j_{2_h}}-R^n_{j_{1_h}})} \frac{cc_n(\pi_p\circ i_{n+1}\circ f_n,\ ((R^n_{j_{2_h}}-R^n_{j_{1_h}})\cap L^n_p))}{vol(R^n_{j_{1_h}})}ds_{(R^n_{j_{2_h}}-R^n_{j_{1_h}})}=0.$$

Then,

$$\lim_{j\to\infty} cc_{n+1}\left(f_n;R^n_j\right)\leq \lim_{h\to\infty} cc_n\left(f_n;R^n_h\right). \tag{5.3}$$

Analogously,

$$\lim_{j\to\infty} cc_{n+1}\left(f_n;R^n_j\right)\geq \lim_{h\to\infty} cc_{n+1}\left(f_n;R^n_h\right). \tag{5.4}$$

By (5.3) and (5.4), we have that

$$\lim_{j\to\infty} cc_{n+1}\left(f_n;R^n_j\right)= \lim_{h\to\infty} cc_{n+1}\left(f_n;R^n_h\right).$$

□

We now study the regions $(R^n_j\cap L^n_p)$.

Theorem 14. *Let $p\in S^n_1$ and let $(R^n_j)_{j\geq 1}$ be a sequence of connected regions as given in Theorem 11. If there exist $j_p\geq 1$ and $(R^n_{h_p}\cap L^n_p)_{h_p\geq j_p}\subset (R^n_j\cap L^n_p)_{j\geq j_p}$ such that $\lim_{h_p\to\infty}R^n_{h_p}=S^n_1$ and $\lim_{h_p\to\infty}\frac{vol(R^n_{h_p+1}\cap L^n_p)}{vol(R^n_{h_p}\cap L^n_p)}=1$, then $(R^n_{h_p}\cap L^n_p)_{h_p\geq j_p}$ is a sequence of connected regions in L^n_p with properties i)-v) of Theorem 11.*

Proof. Let p be a fixed point of S^n_1 and let $(R^n_j)_{j\geq 1}$ be a sequence of connected regions as given in Theorem 11. There exists $j_p\geq 1$ such that $p\in R^n_j$, $\forall j\geq j_p$, and there exists $(R^n_{h_p}\cap L^n_p)_{h_p\geq j_p}\subset (R^n_j\cap L^n_p)_{j\geq j_p}$ such that $\lim_{h_p\to\infty}R^n_{h_p}=S^n_1$ and $\lim_{h_p\to\infty}\frac{vol(R^n_{h_p+1}\cap L^n_p)}{vol(R^n_{h_p}\cap L^n_p)}=1$.

Hence, $\forall h_p\geq j_p$, $(S_{h_p}\cap L^2_p)_+$ is a connected region. Also:

i) Since $R^n_j\subset S^n_1$, then $(R^n_{h_p}\cap L^n_p)\subset(S^n_1\cap L^n_p)$, $\forall h_p\geq j_p$.
ii) Since $R^n_j\subset R^n_{j+1}$, then $(R^n_{h_p}\cap L^n_p)\subset(R^n_{h_p+1}\cap L^n_p)$, $\forall h_p\geq j_p$.
iii) Since $0<vol(R^n_j)<\infty$, then $0<vol(R^n_j\cap L^n_p)<\infty$, $\forall j\geq 1$. In particular, $0<vol(R^n_{h_p}\cap L^2_p)<\infty$, $\forall h_p\geq j_p$.
v) Since $\lim_{h_p\to\infty}R^n_{h_p}=S^n_1$ and $\lim_{R^n_{h_p}\to S^n_1}(R^n_{h_p}\cap L^n_p)=(S^n_1\cap L^n_p)$, then $\lim_{h_p\to\infty}(R^n_{h_p}\cap L^n_p)=(S^n_1\cap L^n_p)$.

iv) Since that for each point $p \in S_1^n$, $\exists j_p$ such that $R_j^n \cap L_p^n \neq \emptyset\ \forall j \geq j_p$ and $(R_{h_p}^n \cap L_p^n)_{h_p \geq j_p} \subset (R_j^n \cap L_p^n)_{j \geq j_p}$, then $\exists h_p \geq j_p$ such that $(R_h^n \cap L_p^n) \neq \emptyset\ \forall h \geq h_p$. By v), we obtain that for each point $q \in (S_1^n \cap L_p^n)$, $\exists h_q$ such that $(R_h^n \cap L_p^n \cap L_q^{n-1}) \neq \emptyset\ \forall h \geq h_q$.

□

We now show a main theorem in L^{n+1}.

Theorem 15. *The total central curvature of* f_n *is given by*

$$tcc_{n+1}(f_n) = \lim_{j\to\infty} \frac{1}{vol(R_j^n)} \int_{p\in R_j^n} tcc_n(\pi_p \circ i_{n+1} \circ f_n) ds_{R_j^n}.$$

Proof. Let $\gamma_p = \pi_p \circ i_{n+1} \circ f_n$ and $tcc_{n+1}(f_n) = \lim\limits_{j\to\infty} cc_{n+1}(f_n, R_{nj})$.

According to Definitions 9 and 10, we have that:

$$\lim_{j\to\infty} \frac{1}{vol(R_j^2)} \int_{p\in R_j^n} \left[tcc_n(\gamma_p) - cc_n(\gamma_p,\ R_j^n \cap L_p^n)\right] ds_{R_j^n}$$
$$= \lim_{j\to\infty} \frac{1}{vol(R_j^n)} \int_{p\in R_j^n} \left[\lim_{t_p\to\infty} cc_n(\gamma_p,\ R_{t_p}^{n-1}) - cc_n(\gamma_p,\ R_j^n \cap L_p^n)\right] ds_{R_j^n}$$
$$= \lim_{j\to\infty} \frac{1}{vol(R_j^n)} \int_{p\in R_j^n} \lim_{t_p\to\infty} \left[cc_n(\gamma_p,\ R_{t_p}^{n-1}) - cc_n(\gamma_p,\ R_j^n \cap L_p^n)\right] ds_{R_j^n}.$$

By Theorems 13 and 14, we may assume that $(R_{t_p}^{n-1})_{t_p \geq j_p} = (R_{h_p}^n \cap L_p^2)_{h_p \geq j_p}$. Then,

$$\lim_{j\to\infty} \frac{1}{vol(R_j^n)} \int_{p\in R_j^n} [tcc_n(\gamma_p) - cc_n(\gamma_p,\ R_j^n \cap L_p^n)] ds_{R_j^n}$$
$$= \lim_{j\to\infty} \frac{1}{vol(R_j^n)} \int_{p\in R_j^n} \lim_{h_p\to\infty} \left[cc_n(\gamma_p,\ R_{h_p}^n \cap L_p^n) - cc_n(\gamma_p,\ R_j^n \cap L_p^n)\right] ds_{R_j^n}.$$

Hence, by Lemma 12,

$$\lim_{j\to\infty} \frac{1}{vol(R_j^n)} \int_{p\in R_j^n} [tcc_n(\gamma_p) - cc_n(\gamma_p,\ R_j^n \cap L_p^n)] ds_{R_j^n} = 0.$$

□

5. Total central curvature of spacelike pure polygons

Now we obtain the total central curvature for a spacelike pure polygon.

Lemma 16. *Let* $\mathbf{P}[p_1, \ldots, p_m] = f_1(S^1)$ *be a spacelike pure polygon in* L^2. *Then,* $tcc_2(f_1)$ *is an positive integer number and*

$$tcc_2(f) = k$$

where k denote the number of non middle vertices of $\mathbf{P}$*,* $2 \leq k \leq m$*.*

Proof. Let $(C_j)_{j\geq 1}$ be a sequence of connected arcs according to Definition 1.

If $\xi \in C_j$, the line through ξ and $f_1(x)$ is a local support line to f_1 at x if and only if there exists i such that $f_1(x) = p_i$ vertex of $\mathbf{P}$. Let $q_i = \left(x_1^i, x_2^i\right)$ the s–for-future spherical image of $\overrightarrow{p_{i+1} - p_i}$, and let $q_{i_0} = A = (a_1, a_2)$ and $q_{i_1} = B = (b_1, b_2)$ such that $a_1 = \min_{1\leq i\leq m}\{x_1^i\}$ and $b_1 = \max_{1\leq i\leq m}\{x_1^i\}$.

There exists $j_0 \geq 1$ such that the arc $AB \subset C_j$, $\forall j \geq j_0$. Then we have:

$$\int_{\xi\in C_j} \Upsilon_\xi(f_1)\, ds_{C_j} = \left(\int_{\xi\in AB} \Upsilon_\xi(f_1)\, ds_{AB} + \int_{\xi\in(C_j-AB)} \Upsilon_\xi(f_1)\, ds_{(C_j-AB)}\right).$$

By Theorem 7 we know that $\Upsilon_\xi(f_1) = k$, then

$$tcc_2(f_1) = \lim_{j\to\infty} \frac{1}{length\,(C_j)} \int_{\xi\in C_j} \Upsilon_\xi(f_1)\, ds_{C_j} = k.$$

□

Corollary 17. *If* $\boldsymbol{P}$ *is a convex pure spacelike polygon, then* $tcc_2(f_1) = 2$*.*

Proof. Every convex pure spacelike polygon in Lorentzian plane has exactly two non middle vertices, (cf. [4]). □

Theorem 18. *Let* $\mathbf{P} = f_n\left(S^1\right)$ *be a pure spacelike polygon with* m *vertices in* L^{n+1}*,* $n \geq 2$*. Then* $tcc_{n+1}(f_n) \leq m$*.*

Proof. For each $p \in S_1^n$, $\pi_p(\mathbf{P})$ is a pure spacelike polygon in L_p^n which has at most m vertices. The inequalities follow from Theorems 14 and 15, and the relationship

$$tcc_3(f_2) = \lim_{j\to\infty} \frac{1}{area\left(R_j^2\right)} \int_{p\in R_j^2} tcc_2(\pi_p \circ i_3 \circ f_2)\, ds_{R_j^2}.$$

□

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