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SCREEN TRANSVERSAL LIGHTLIKE SUBMANIFOLDS OF INDEFINITE KENMOTSU MANIFOLDS

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Abstract. In this paper, we introduce screen transversal lightlike submanifolds of indefinite Kenmotsu manifolds. We obtain a characterization of screen transversal anti-invariant lightlike submanifolds as well as a condition for induced connection to be a metric connection and provide an example of ST-anti-invariant lightlike submanifold of R^9_2. Also, we obtain necessary and sufficient conditions for radical screen transversal lightlike submanifolds to be totally geodesic.

Introduction

In the theory of submanifolds of semi-Riemannian manifolds it is interesting to study the geometry of lightlike submanifolds due to the fact that the intersection of the normal vector bundle and the tangent bundle is non-trivial making it more interesting and remarkably different from the study of non-degenerate submanifolds. The geometry of lightlike submanifolds of indefinite Kaehler manifolds was studied by Duggal and Bejancu [4]. Further, Duggal and Sahin studied screen CR-lightlike and generalized CR-lightlike [5, 6] submanifolds of indefinite Kaehler manifolds. Later, B. Sahin initiated the study of transversal lightlike and screen transversal lightlike submanifolds of an indefinite Kaehler manifold [1, 2] which are different from previously introduced submanifolds. Recently, we have studied lightlike submanifolds, slant lightlike submanifolds, screen slant lightlike submanifolds, generalised CR-lightlike submanifolds of indefinite Kenmotsu manifolds [8,9,10,11] and obtained many interesting results. However, a general notion of screen transversal lightlike submanifolds of indefinite Kenmotsu manifolds has not been introduced as yet.

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In Section 1, we have collected the formulae and information which are useful in subsequent sections. In Section 2, we define screen transversal, screen transversal anti-invariant and radical transversal lightlike submanifolds. In Section 3, we obtained a characterization of screen transversal anti-invariant lightlike submanifolds as well as a condition for induced connection to be a metric connection and provided an example of ST-anti-invariant lightlike submanifold of R^9_2. In Section 4, we have studied radical screen transversal lightlike submanifolds.

1. Preliminaries

An odd-dimensional semi-Riemannian manifold $\overline{M}$ is said to be an indefinite almost contact metric manifold if there exist structure tensors $\{\phi, V, \eta, \overline{g}\}$, where ϕ is a (1,1) tensor field, V a vector field, η a 1-form and $\overline{g}$ is the semi-Riemannian metric on $\overline{M}$ satisfying

$$\begin{cases} \phi^2 X = -X + \eta(X)V, \quad \eta \circ \phi = 0, \quad \phi V = 0, \quad \eta(V) = 1 \\ \overline{g}(\phi X, \phi Y) = \overline{g}(X, Y) - \eta(X)\eta(Y), \quad \overline{g}(X, V) = \eta(X) \end{cases} \tag{1.1}$$

for any $X, Y \in T\overline{M}$, where $T\overline{M}$ denotes the Lie algebra of vector fields on $\overline{M}$.

An indefinite almost contact metric manifold $\overline{M}$ is called an indefinite Kenmotsu manifold if [3],

$$(\overline{\nabla}_X \phi)Y = -\overline{g}(\phi X, Y)V + \eta(Y)\phi X, \text{ and } \overline{\nabla}_X V = -X + \eta(X)V \tag{1.2}$$

for any $X, Y \in T\overline{M}$, where $\overline{\nabla}$ denote the Levi-Civita connection on $\overline{M}$.

A submanifold M^m immersed in a semi-Riemannian manifold $\{\overline{M}^{m+k}, \overline{g}\}$ is called a lightlike submanifold if it admits a degenerate metric g induced from $\overline{g}$ whose radical distribution $\mathrm{Rad}(TM)$ is of rank r, where $1 \leq r \leq m$.

Now, $\mathrm{Rad}(TM) = TM \cap TM^{\perp}$, where

$$TM^{\perp} = \bigcup_{x \in M} \{u \in T_x\overline{M} : \overline{g}(u, v) = 0, \forall v \in T_x M\}. \tag{1.3}$$

Let $S(TM)$ be a *screen distribution* which is a semi-Riemannian complementary distribution of $\mathrm{Rad}(TM)$ in TM, that is, $TM = \mathrm{Rad}(TM) \perp S(TM)$. We consider a *screen transversal vector bundle* $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $\mathrm{Rad}(TM)$ in $TM^{\perp}$. Since, for any local basis $\{\xi_i\}$ of $\mathrm{Rad}(TM)$, there exists a local frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\overline{g}(\xi_i, N_j) = \delta_{ij}$ and $\overline{g}(N_i, N_j) = 0$, it follows that there exists a *lightlike transversal vector bundle* $ltr(TM)$ locally spanned by $\{N_i\}$ (cf.[4], pg. 144). Let $\mathrm{tr}(TM)$ be the complementary (but not orthogonal) vector bundle to

TM in $T\overline{M}|_M$. Then

$$\begin{cases} tr(TM) = ltr(TM) \perp S(TM^\perp) \\ T\overline{M}|_M = S(TM) \perp [\mathrm{Rad}(TM) \bigoplus ltr(TM)] \perp S(TM^\perp). \end{cases} \tag{1.4}$$

A submanifold $(M, g, S(TM), S(TM^\perp))$ of $\overline{M}$ is said to be

(i) r-lightlike if $r < min\{m, k\}$;

(ii) Coisotropic if $r = k < m$, $S(TM^\perp) = \{0\}$;

(iii) Isotropic if $r = m < k$, $S(TM) = \{0\}$;

(iv) Totally lightlike if $r = m = k$, $S(TM) = \{0\} = S(TM^\perp)$.

Let $\overline{\nabla}$, ∇ and ∇^t denote the linear connections on $\overline{M}$, M and vector bundle tr(TM), respectively. Then the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM), \tag{1.5}$$

$$\overline{\nabla}_X U = -A_U X + \nabla^t_X U, \quad \forall U \in \Gamma(tr(TM)), \tag{1.6}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^t_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively and A_U is the shape operator of M with respect to U. Moreover, according to the decomposition (1.4), h^l, h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^\perp))$-valued *lightlike second fundamental form* and *screen second fundamental form* of M, respectively, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad \forall X, Y \in \Gamma(TM), \tag{1.7}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X(N) + D^s(X, N), \quad N \in \Gamma(ltr(TM)), \tag{1.8}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X(W) + D^l(X, W), \quad W \in \Gamma(S(TM^\perp)), \tag{1.9}$$

where $D^l(X, W)$, $D^s(X, N)$ are the projections of ∇^t on $\Gamma(ltr(TM))$ and $\Gamma(S(TM^\perp))$, respectively and ∇^l, ∇^s are linear connections on $\Gamma(ltr(TM))$ and $\Gamma(S(TM^\perp))$, respectively. We call ∇^l, ∇^s the lightlike and screen transversal connections on M, and A_N, A_W are shape operators on M with respect to N and W, respectively. Using (1.5) and (1.7)$\sim$(1.9), we obtain

$$\overline{g}(h^s(X, Y), W) + \overline{g}(Y, D^l(X, W)) = g(A_W X, Y), \tag{1.10}$$

$$\overline{g}(D^s(X, N), W) = \overline{g}(N, A_W X). \tag{1.11}$$

Let $\overline{P}$ denote the projection of TM on $S(TM)$ and let ∇^*, ∇^{*t} denote the linear connections on $S(TM)$ and Rad(TM), respectively. Then from the decomposition of tangent bundle of lightlike submanifold, we have

$$\nabla_X \overline{P} Y = \nabla^*_X \overline{P} Y + h^*(X, \overline{P} Y), \tag{1.12}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{1.13}$$

for X, $Y \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}TM)$, where h^*, A^* are the second fundamental form and shape operator of distributions $S(TM)$ and Rad(TM),

respectively.
From (1.12) and (1.13), we get

$$\overline{g}(h^l(X,\overline{P}Y),\xi) = g(A^*_\xi X,\overline{P}Y), \tag{1.14}$$

$$\overline{g}(h^*(X,\overline{P}Y),N) = g(A_N X,\overline{P}Y), \tag{1.15}$$

$$\overline{g}(h^l(X,\xi),\xi) = 0, \quad A^*_\xi \xi = 0. \tag{1.16}$$

In general, the induced connection ∇ on M is not a metric connection. Since $\overline{\nabla}$ is a metric connection, from (1.7), we obtain

$$(\nabla_X g)(Y,Z) = \overline{g}(h^l(X,Y),Z) + \overline{g}(h^l(X,Z),Y). \tag{1.17}$$

However, it is important to note that ∇^*, ∇^{*t} are metric connections on $S(TM)$ and $\mathrm{Rad}(TM)$, respectively.

A plane Π in $T_x\overline{M}$ of a Kenmotsu manifold $\overline{M}$ is called a ϕ-section if it is spanned by a unit vector X orthogonal to V and ϕX, where X is a non-null vector field on $\overline{M}$. The sectional curvature $K(\Pi)$ with respect to Π determined by X is called a ϕ-sectional curvature. If $\overline{M}$ has a ϕ-sectional curvature c which does not depend on the ϕ-section at each point, then c is constant in $\overline{M}$. Then, $\overline{M}$ is called an indefinite Kenmotsu space form and is denoted by $\overline{M}(c)$. The curvature tensor $\overline{R}$ of $\overline{M}(c)$ is given by [3, 7]

$$\begin{aligned}\overline{R}(X,Y)Z = \frac{c-3}{4}\{\overline{g}(Y,Z)X - \overline{g}(X,Z)Y\} + \frac{c+1}{4}\{\eta(X)\eta(Z)Y \\ - \eta(Y)\eta(Z)X + \overline{g}(X,Z)\eta(Y)V - \overline{g}(Y,Z)\eta(X)V + \overline{g}(\phi Y,Z)\phi X \\ + \overline{g}(\phi Z,X)\phi Y - 2\overline{g}(\phi X,Y)\phi Z\end{aligned} \tag{1.18}$$

for any X, Y and Z vector fields on $\overline{M}$.

2. Screen transversal lightlike submanifolds

In this section, we introduce the screen transversal (ST), the radical screen transversal and the screen transversal anti-invariant lightlike submanifolds of indefinite Kenmotsu manifolds.

Lemma 2.1. *Let M be an r-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ with structure vector field tangent to M. Suppose that $\phi\mathrm{Rad}TM$ is a vector subbundle of $S(TM^\perp)$. Then, $\phi ltrTM$ is also vector subbundle of the screen transversal bundle $S(TM^\perp)$. Moreover, $\phi ltrTM \cap \phi\mathrm{Rad}TM = \{0\}$.*

Proof. Let us assume that $ltrTM$ is invariant with respect to ϕ, then by the definition of a lightlike submanifold, there exist vector fields $\xi \in \Gamma(\mathrm{Rad}TM)$ and $N \in \Gamma(ltr(TM))$ such that $\overline{g}(\xi,N) = 1$. Also from (1.1), we get

$$\overline{g}(\phi\xi,\phi N) = \overline{g}(\xi,N) - \eta(N)\eta(\xi) = \overline{g}(\xi,N) = 1.$$

However, if $\phi N \in \Gamma(ltr(TM))$ then by the hypothesis we get $\overline{g}(\phi\xi, \phi N) = 0$. Hence, we obtain a contradiction which implies that ϕN does not belong to $ltr(TM)$.

Now, suppose that $\phi N \in \Gamma S(TM)$. Then, in a similar way, we have

$$0 = \overline{g}(\phi\xi, \phi N) = \overline{g}(\xi, N) - \eta(N)\eta(\xi) = \overline{g}(\xi, N) = 1$$

which is again a contradiction. Thus ϕN does not belong to $S(TM)$.

We can also obtain that ϕN does not belong to $\mathrm{Rad}TM$. Then, from the decomposition of a lightlike submanifold, we conclude that $\phi N \in S(TM^{\perp})$.

Now, suppose that there exists a vector field $X \in \Gamma(\phi ltrTM \cap \phi \mathrm{Rad}TM)$. Then, we have $\overline{g}(X, \phi N) = 0$, since $X \in \Gamma(\phi ltrTM)$. However, for an r-lightlike submanifold there exists some vector fields $\phi X \in \Gamma(\mathrm{Rad}TM)$ for $X \in \Gamma\phi(\mathrm{Rad}TM)$ such that $\overline{g}(\phi X, N) \neq 0$. Since ϕ is skew symmetric, we get

$$0 \neq \overline{g}(\phi X, N) = -\overline{g}(X, \phi N) = 0$$

which is a contradiction. □

Definition 2.1. *Let M be an r-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ with structure vector field tangent to M. Then M is called screen transversal lightlike (ST-lightlike) submanifold of $\overline{M}$ if there exists a screen transversal vector bundle $S(TM^{\perp})$ such that $\phi\mathrm{Rad}TM \subset S(TM^{\perp})$.*

Definition 2.2. *Let M be ST-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ with structure vector field tangent to M. Then*

(a) *M is called a radical ST-lightlike submanifold of $\overline{M}$ if $S(TM)$ is invariant with respect to ϕ.*
(b) *M is called a ST-anti-invariant lightlike submanifold of $\overline{M}$ if $S(TM)$ is screen transversal with respect to ϕ i.e. $\phi S(TM) \subset S(TM^{\perp})$.*

From Lemma 2.1 and Definition 2.1, it follows that $\phi ltrTM \subset S(TM^{\perp})$. Also it is obvious that there are no co-isotropic and totally lightlike ST-lightlike submanifolds of indefinite Kenmotsu manifolds. It is important to point out that $\phi\mathrm{Rad}TM$ and $\phi ltrTM$ are not orthogonal otherwise $S(TM^{\perp})$ would be degenerate.

For ST-anti-invariant lightlike submanifold M of an indefinite Kenmotsu manifold $\overline{M}$ with structure vector field tangent to M, we have

$$S(TM^{\perp}) = \phi(\mathrm{Rad}TM) \oplus \phi(ltrTM) \perp \phi(D') \perp D_0 \qquad (2.1)$$

where $S(TM) = D' \perp \{V\}$ and D_0 is complementary distribution orthogonal to $\phi(\mathrm{Rad}TM) \oplus \phi(ltrTM) \perp \phi(D')$ in $S(TM^{\perp})$.

Proposition 2.1. *Let M be ST-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ with structure vector field tangent to M. Then the distribution D_0 is invariant with respect to ϕ.*

Proof. For $X \in \Gamma(D_0)$, $\xi \in \Gamma(\text{Rad}TM)$, $N \in \Gamma(ltrTM)$, we have

$$\overline{g}(\phi X, \xi) = -\overline{g}(X, \phi\xi) = 0 \text{ and } \overline{g}(\phi X, N) = -\overline{g}(X, \phi N) = 0$$

which implies that $\phi(D_0) \cap \text{Rad}TM = \{0\}$ and $\phi(D_0) \cap ltr(TM) = \{0\}$. From (1.1), we get

$$\overline{g}(\phi X, \phi\xi) = \overline{g}(X, \xi) - \eta(X)\eta(\xi) = \overline{g}(X, \xi) = 0$$

and

$$\overline{g}(\phi X, \phi N) = \overline{g}(X, N) - \eta(X)\eta(N) = \overline{g}(X, N) = 0$$

which shows that $\phi(D_0) \cap \phi(\text{Rad}TM) = \{0\}$ and $\phi(D_0) \cap \phi(ltr(TM)) = \{0\}$. Moreover, since $\phi(S(TM))$ and D_0 are orthogonal, we obtain

$$\overline{g}(\phi X, Z) = -\overline{g}(X, \phi Z) = 0$$

and

$$\overline{g}(\phi X, \phi Z) = \overline{g}(X, Z) - \eta(X)\eta(Z) = \overline{g}(X, Z) = 0$$

for $Z \in \Gamma(S(TM))$, $\phi Z \in \Gamma(\phi(S(TM)))$, which shows that

$$\phi(D_0) \cap S(TM) = \{0\} \text{ and } \phi(D_0) \cap \phi(S(TM)) = \{0\}.$$

Thus, we find that

$$\phi(D_0) \cap TM = \{0\}, \phi(D_0) \cap ltr(TM) = \{0\}$$

and

$$\phi(D_0) \cap \{\phi(S(TM)) \perp \phi(ltr(TM)) \oplus \phi(\text{Rad}TM)\} = \{0\}$$

which shows that D_0 is invariant. □

3. ST-anti-invariant lightlike submanifolds

In this section, we study ST-anti invariant lightlike submanifolds of an indefinite Kenmotsu manifold.

Hereafter, $(R_q^{2m+1}, \phi_0, V, \eta, \overline{g})$ will denote the manifold R_q^{2m+1} with its usual Kenmotsu structure given by

$$\begin{cases} \eta = dz, \qquad V = \partial z, \\ \overline{g} = \eta \bigotimes \eta + e^{2z}(-\sum_{i=1}^{q/2}(dx^i \bigotimes dx^i + dy^i \bigotimes dy^i) \\ \qquad + \sum_{i=q+1}^{m}(dx^i \bigotimes dx^i + dy^i \bigotimes dy^i)), \\ \phi_0(\sum_{i=1}^{m}(X_i\partial x^i + Y_i\partial y^i) + Z\partial z) = \sum_{i=1}^{m}(Y_i\partial x^i - X_i\partial y^i) \end{cases}$$

where (x^i, y^i, z) are the Cartesian coordinates. We have:

Example 3.1. Let $\overline{M} = (R_2^9, \overline{g})$ be a semi-Euclidean space, where $\overline{g}$ is of signature (-, +, +, +, -, +, +, +, +) with respect to the canonical basis

$$\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}.$$

Consider a submanifold M of R^9_2, defined by

$$\begin{cases} x_1 = \sin u_1 \cosh u_2, & x_2 = \cos u_1 \cosh u_2, \\ x_3 = \sin u_1 \sinh u_2, & x_4 = \cos u_1 \sinh u_2, \\ x_5 = u_1, \quad x_6 = 0, & x_7 = \cos u_3, \quad x_8 = \sin u_3 \\ z = t. & \end{cases}$$

Then a local frame of TM is given by

$$\begin{cases} Z_1 = e^{-z}(\cos u_1 \cosh u_2 \partial x_1 - \sin u_1 \cosh u_2 \partial x_2 + \cos u_1 \sinh u_2 \partial x_3 \\ \qquad - \sin u_1 \sinh u_2 \partial x_4 + \partial x_5), \\ Z_2 = e^{-z}(\sin u_1 \sinh u_2 \partial x_1 + \cos u_1 \sinh u_2 \partial x_2 + \sin u_1 \cosh u_2 \partial x_3 \\ \qquad + \cos u_1 \cosh u_2 \partial x_4), \\ Z_3 = e^{-z}(- \sin u_3 \partial x_7 + \cos u_3 \partial x_8), \qquad Z_4 = V = \partial z. \end{cases}$$

Thus, M is a 1-lightlike submanifold with $\mathrm{Rad}TM = span\{Z_1\}$, and screen distribution $S(TM) = span\{Z_2, Z_3\}$. It is easy to see that $S(TM)$ is not invariant with respect to ϕ. Since $\{\phi Z_2, \phi Z_3\}$ is non-degenerate it follows that $\phi(S(TM)) \subset S(TM^\perp)$.

The lightlike transversal bundle $ltr(TM)$ is spanned by

$$N = \frac{e^{-z}}{2}(- \cos u_1 \cosh u_2 \partial x_1 + \sin u_1 \cosh u_2 \partial x_2 - \cos u_1 \sinh u_2 \partial x_3 \\ + \sin u_1 \sinh u_2 \partial x_4 + \partial x_5)$$

and the screen transversal bundle is

$$S(TM^\perp) = span\{W_1 = \phi N, W_2 = \phi Z_1, W_3 = \phi Z_3, W_4 = \phi Z_2\}$$

where

$$\begin{cases} W_1 = \frac{e^{-z}}{2}(- \sin u_1 \cosh u_2 \partial x_1 - \cos u_1 \cosh u_2 \partial x_2 - \sin u_1 \sinh u_2 \partial x_3 \\ \qquad - \cos u_1 \sinh u_2 \partial x_4 + \partial x_6), \\ W_2 = e^{-z}(\sin u_1 \cosh u_2 \partial x_1 + \cos u_1 \cosh u_2 \partial x_2 + \sin u_1 \sinh u_2 \partial x_3 \\ \qquad + \cos u_1 \sinh u_2 \partial x_4 + \partial x_6), \\ W_3 = e^{-z}(- \cos u_3 \partial x_7 - \sin u_3 \partial x_8), \\ W_4 = e^{-z}(- \cos u_1 \sinh u_2 \partial x_1 + \sin u_1 \sinh u_2 \partial x_2 - \cos u_1 \cosh u_2 \partial x_3 \\ \qquad + \sin u_1 \cosh u_2 \partial x_4). \end{cases}$$

Then it is easy to see that M is a ST-anti-invariant lightlike submanifold.

Now, we give a characterization for ST-anti-invariant lightlike submanifolds of indefinite Kenmotsu space forms.

Theorem 3.1. *Let M be a lightlike submanifold of an indefinite Kenmotsu space form $\overline{M}(c)$ with structure vector field tangent to M. Suppose that $c \neq -1$ and $\phi(\mathrm{Rad}TM) \subset S(TM^\perp)$. Then M is ST-anti-invariant lightlike submanifold if and only if*

$$\overline{g}(\overline{R}(X,Y)\xi, \phi N) = 0 \tag{3.1}$$

for $X, Y \in \Gamma(S(TM))$*,* $\xi \in \Gamma(\mathrm{Rad}TM)$ *and* $N \in \Gamma(ltr(TM))$*.*

Proof. From Lemma 2.1 we have $\phi(ltr(TM)) \subset S(TM^{\perp})$, since $\phi(\mathrm{Rad}TM) \subset S(TM^{\perp})$. From (1.1), we have

$$\overline{g}(\phi X, N) = -\overline{g}(X, \phi N) = 0$$

for $X \in \Gamma(S(TM))$ and $N \in \Gamma(ltr(TM))$. Hence $\phi(S(TM)) \cap \mathrm{Rad}TM = \{0\}$.

Moreover, we find that

$$\overline{g}(\phi X, \phi\xi) = 0 \text{ and } \overline{g}(\phi X, \phi N) = 0$$

for $X \in \Gamma(S(TM))$ and $N \in \Gamma(ltr(TM))$. Hence, we get

$$\phi(S(TM)) \cap \mathrm{Rad}TM = \{0\} \text{ and } \phi(S(TM)) \cap \phi(ltr(TM)) = \{0\}.$$

Similarly, we can obtain that $\phi(S(TM)) \cap ltr(TM) = \{0\}$.

On the other hand, since $\phi\xi \in \Gamma(S(TM^{\perp}))$, from (1.18), we get

$$\overline{g}(\overline{R}(X, Y)\xi, \phi N) = \frac{(c+1)}{2}\overline{g}(X, \phi Y)\overline{g}(\xi, N).$$

Since $c \neq -1$ and $\overline{g}(\xi, N) \neq 0$, for $\xi \in \Gamma(\mathrm{Rad}TM)$, $N \in \Gamma(ltr(TM))$. Thus, $\overline{g}(\overline{R}(X, Y)\xi, \phi N) = 0$ if and only if $\phi(S(TM)) \perp S(TM)$.

Therefore, $\phi(S(TM)) \subset S(TM^{\perp})$ as $\phi(S(TM)) \cap ltr(TM) = \{0\}$. □

Let F_1, F_2, F_3 and F_4 be the projection morphisms on $\phi(\mathrm{Rad}TM)$, $\phi(S(TM)), \phi(ltr(TM))$ and D_0 respectively. Then, for $W \in \Gamma(S(TM^{\perp}))$, we have

$$W = F_1W + F_2W + F_3W + F_4W. \tag{3.2}$$

On the other hand, for $W \in \Gamma(S(TM^{\perp}))$ we write

$$\phi W = BW + CW \tag{3.3}$$

where BW and CW are tangential and transversal parts of ϕW. Then applying ϕ to (3.2), we get

$$\phi W = \phi F_1W + \phi F_2W + \phi F_3W + \phi F_4W. \tag{3.4}$$

Separating tangential and transversal parts in (3.4), we find

$$BW = \phi F_1W + \phi F_2W, \quad CV = \phi F_3W + \phi F_4W. \tag{3.5}$$

We put
$\phi F_1 = B_1, \phi F_2 = B_2, \phi F_3 = C_1$ and $\phi F_4 = C_2$, we can write (3.4) as follows:

$$\phi W = B_1W + B_2W + C_1W + C_2W, \tag{3.6}$$

where $B_1V \in \Gamma(RadTM), B_2V \in \Gamma(S(TM)), C_1V \in \Gamma(ltrTM)$ and $C_2V \in \Gamma(D_0)$.

Theorem 3.2. *Let M be a ST-anti-invariant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Then the induced connection is a metric connection if and only if $\nabla^s_X\phi\xi$ has no components in $\phi(S(TM))$ for $X \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}TM)$.*

Proof. From (1.2), we have

$$\begin{aligned}\overline{\nabla}_X\phi\xi = (\overline{\nabla}_X\phi)\xi + \phi(\overline{\nabla}_X\xi) &= -\overline{g}(\phi X, \xi)V + \eta(\xi)\phi X + \phi(\overline{\nabla}_X\xi)\\ \Rightarrow \qquad \overline{\nabla}_X\phi\xi &= \phi(\overline{\nabla}_X\xi)\\ \Rightarrow \qquad \overline{\nabla}_X\xi &= -\phi\overline{\nabla}_X\phi\xi\end{aligned}$$

for $X \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{RadTM})$.

Using (1.7), (1.9) and (3.6), we get

$$\begin{aligned}\nabla_X\xi + h^l(X,\xi) + h^s(X,\xi) = \phi A_{\phi\xi}X &- B_1\nabla^s_X\phi\xi\\ &- B_2\nabla^s_X\phi\xi - C_1\nabla^s_X\phi\xi - C_2\nabla^s_X\phi\xi - \phi D^l(X,\phi\xi).\end{aligned}$$

Taking the tangential parts of above equation, we get

$$\nabla_X\xi = -B_1\nabla^s_X\phi\xi - B_2\nabla^s_X\phi\xi.$$

Thus assertion follows from (cf. [4], Theorem 2.4, p. 161). □

4. Radical ST lightlike submanifolds

In this section, we study radical ST-lightlike submanifolds.

We have:

Theorem 4.1. *Let M be a radical ST-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Then, screen distribution $S(TM)$ is integrable if and only if*

$$\overline{g}(h^s(X,\phi Y),\phi N) = \overline{g}(h^s(\phi Y, X),\phi N) \tag{4.1}$$

for $X, Y \in \Gamma(S(TM))$ and $N \in \Gamma(ltr(TM))$.

Proof. From (1.1) and (1.2), we have

$$\overline{g}([X,Y],N) = \overline{g}(\overline{\nabla}_X\phi Y,\phi N) - \overline{g}(\overline{\nabla}_Y\phi X,\phi N)$$

for $X, Y \in \Gamma(S(TM))$ and $N \in \Gamma(ltr(TM))$. Then, using (1.5) and (1.7), we get (4.1). □

Theorem 4.2. *Let M be a radical ST-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Then, radical distribution is integrable if and only if*

$$\overline{g}(h^s(\xi_1,\phi X),\phi\xi_2) = \overline{g}(h^s(\xi_2,\phi X),\phi\xi_1)$$

for $X \in \Gamma(S(TM))$ and $\xi_1, \xi_2 \in \Gamma(\mathrm{Rad}TM)$.

Proof. From (1.1), (1.2), (1.13) and (1.16), we have

$$\overline{g}([\xi_1, \xi_2], X) = \overline{g}(\overline{\nabla}_{\xi_1} \phi\xi_2, \phi X) - \overline{g}(\overline{\nabla}_{\xi_2} \phi\xi_1, \phi X).$$

Then, using (1.5), we get $g([\xi_1, \xi_2], X) = -g(A_{\phi\xi_2}\xi_1, \phi X) + g(A_{\phi\xi_1}\xi_2, \phi X)$. Thus, from (1.10), we obtain

$$g([\xi_1, \xi_2], \phi X) = \overline{g}(h^s(\xi_1, \phi X), \phi\xi_2) - \overline{g}(h^s(\xi_2, \phi X), \phi\xi_1)$$

which proves the assertion. □

Theorem 4.3. *Let M be a radical ST-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Then S(TM) defines a totally geodesic foliation on M if and only if $h^s(X, \phi Y)$ has no components in $\phi(\mathrm{Rad}TM)$ for $X, Y \in \Gamma(S(TM))$.*

Proof. Using (1.1), (1.2), (1.5), (1.7) and (1.10), we obtain

$$\overline{g}(\nabla_X Y, N) = \overline{g}(h^s(X, \phi Y), \phi N)$$

which proves the assertion. □

Similarly, we have:

Theorem 4.4. *Let M be a radical ST-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Then,* $\mathrm{Rad}TM$ *defines a totally geodesic foliation on M if and only if $h^s(\xi_1, \phi X)$ has no components in $\phi(ltr(TM))$ for $\xi_1 \in \Gamma(RadTM)$ and $X \in \Gamma(S(TM))$.*

Now, we have:

Theorem 4.5. *Let M be a radical ST-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Then, the induced connection is a metric connection if and only if $h^s(X, \phi Y)$ has no components in $\phi(ltr(TM))$ for $X, Y \in \Gamma(S(TM))$.*

Proof. From (1.2), we have $\overline{\nabla}_X \xi = -\phi\overline{\nabla}_X \phi\xi$ for $X \in \Gamma(TM)$ and $\xi \in \Gamma(\mathrm{Rad}TM)$. Hence, using (1.7) and (1.9), we get

$$\nabla_X \xi + h^l(X, \xi) + h^s(X, \xi) = \phi A_{\phi\xi} X - \phi\nabla^s_X \phi\xi - \phi D^l(X, \phi\xi).$$

Taking inner product in above with $Y \in \Gamma(S(TM))$, we obtain

$$g(\nabla_X \xi, Y) = -g(A_{\phi\xi} X, \phi Y).$$

Hence, using (1.10), we get

$$g(\nabla_X \xi, Y) = -\overline{g}(h^s(X, \phi Y), \phi\xi).$$

Thus, the proof is complete. □

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