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HEREDITARILY IRREDUCIBLE MAPPINGS OF CARTESIAN PRODUCT OF CONTINUA

IVAN LONČAR

ABSTRACT. In Section 2 we shall prove that if Cartesian product $X \times Y$ of nondegenerate continua admits a hereditarily irreducible mapping $f : X \times Y \to Z$, then $\mathrm{w}(X \times Y) = \mathrm{w}(Z)$.

The main section of the paper, Section 3, contains theorems concerning the Whitney maps on continua. In particular, it is proved that the product $\Pi\{X_s : s \in S\}$ of nondegenerate continua admits a Whitney map for $C(\Pi\{X_s : s \in S\})$ if and only if it is metrizable.

1. PRELIMINARIES

All spaces in this paper are compact Hausdorff and all mappings are continuous. The weight of a space X is the least cardinal of a basis for X and is denoted by $\mathrm{w}(X)$.

Let X be a space. We define its hyperspaces as the following sets:

$$\begin{aligned} 2^X &= \{F \subseteq X : F \text{ is closed and nonempty}\}, \\ C(X) &= \{F \in 2^X : F \text{ is connected}\}, \\ C^2(X) &= C(C(X)), \\ X(n) &= \{F \in 2^X : F \text{ has at most } n \text{ points}\},\ n \in \mathbb{N}. \end{aligned}$$

For any finitely many subsets $S_1, \ldots, S_n$, let

$$\langle S_1, dots, S_n \rangle = \left\{ F \in 2^X : F \subset \bigcup_{i=1}^{n} S_i, \text{ and } F \cap S_i \neq \emptyset, \text{ for each } i \right\}.$$

The topology on 2^X is the Vietoris topology, i.e., the topology with a base $\{< U_1, \ldots, U_n >: U_i$ is an open subset of X for each i and each $n < \infty$ $\}$, and $C(X)$ is a subspace of 2^X.

We shall use the notion of a network of a topological space.

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A family $\mathcal{N} = \{M_s : s \in S\}$ of a subsets of a topological space X is a *network* for X if for every point $x \in X$ and any neighborhood U of x there exists an $s \in S$ such that $x \in M_s \subset U$ [2, p. 170]. The *network weight* of a space X is defined as the smallest cardinal number of the form card$(\mathcal{N})$, where $\mathcal{N}$ is a network for X; this cardinal number is denoted by nw(X).

Theorem 1.1. [2, p. 171, Theorem 3.1.19]. *For every compact space X we have* $\mathrm{nw}(X) = \mathrm{w}(X)$.

2. Hereditarily irreducible mappings on the product $X \times Y$

A mapping $f : X \to Y$ is said to be *hereditarily irreducible* [6, p. 204, (1.212.3)] provided that for any given subcontinuum Z of X, no proper subcontinuum of Z maps onto $f(Z)$.

A continuum X is said to be *colocally connected* provided that for each point $x \in X$ and each open set $U \ni x$ there exists an open set V containing x such that $V \subset U$ and $X \setminus V$ is connected.

Lemma 2.1. *Product $X \times Y$ of nondegenerate continua is a colocally connected continuum.*

Proof. Let U be a neighborhood of a point $(x, y) \in X \times Y$. There exists a neighborhood $V = U_x \times U_y$ such that $V \subset U$. Let us prove that $E = X \times Y \setminus V$ is connected. We may assume that $U_x \neq X$ and $U_y \neq Y$. Let (x_1, y_1), (x_2, y_2) be a pair of different points in E. We shall prove that there exists a continuum contained in E and containing the points (x_1, y_1),(x_2, y_2). This implies that E is connected. Define the following continua:

$$\begin{array}{ll} E_{x_1} = \{(x_1, y) : y \in Y\}, & E_{x_2} = \{(x_2, y) : y \in Y\}, \\ E_{y_1} = \{(x, y_1) : x \in X\}, & E_{y_2} = \{(x, y_2) : x \in X\}, \\ E_1 = E_{x_1} \cup E_{y_1}, & E_2 = E_{x_2} \cup E_{y_2}. \end{array}$$

In order to complete the proof we shall consider the following cases.

Case 1. *The continuum E_1 or E_2 is a subset of E.* Suppose that E_1 is a subset of E. If $E_2 \subset E$, then $E_1 \cup E_2$ is a continuum contained in E and containing the points (x_1, y_1),(x_2, y_2). If $E_2 \nsubseteq E$, then $E_{x_2} \subset E$ or $E_{y_2} \subset E$. Now, $E_1 \cup E_{x_2}$ or $E \cup E_{y_2} \subset E$ is a continuum contained in E and containing the points (x_1, y_1),(x_2, y_2).

Case 2. $E_1 \nsubseteq E$ and $E_2 \nsubseteq E$. It easy to see that now $E_{x_1} \subset E$ or $E_{y_1} \subset E$. Similarly, $E_{x_2} \subset E$ or $E_{y_2} \subset E$. It is enough to consider the case when $E_{x_1} \subset E$. If $E_{x_2} \subset E$ and $E_{x_1} = E_{x_2}$, then E_{x_1} is a continuum which contain the points (x_1, y_1),(x_2, y_2) and is contained in E. If $E_{x_1} \neq E_{x_2}$, then there exists a point $y_3 \in Y \setminus U_y$ such that $E_{y_3} = \{(x, y_3) : x \in X\}$ is a continuum contained in E. Now $E_{x_1} \cup E_{x_2} \cup E_{y_3}$ is a continuum contained in E and containing the points (x_1, y_1),(x_2, y_2). Finally, if $E_{x_1} \subset E$ and

$E_{y_2} \subset E$, then $E_{x_1} \cap E_{y_2} \neq \emptyset$. It follows that $E_{x_1} \cup E_{y_2}$ is the required continuum, i.e., $E_{x_1} \cup E_{y_2} \subset E$ and (x_1, y_1),$(x_2, y_2) \in E_{x_1} \cup E_{y_2}$. Finally, we infer that $E = X \times Y \setminus U$ is connected. Hence, $X \times Y$ is colocally connected. □

Now we shall prove the following result.

Proposition 2.2. *Let $W = X \times Y$ be a product of two nondegenerate continua. If there exists a hereditarily irreducible mapping $f : W \to Z$, then $C(f) : C(W) \to C(Z)$ is one-to-one on $C(W) \setminus (W)(1)$. Moreover, $C(f)$ is a homeomorphism of $C(W) \setminus (W)(1)$ onto $C(f)(C(W) \setminus (W)(1))$.*

Proof. Suppose that $C(f)$ is not one-to-one. Then there exist a continuum F in Z and two continua C, D in W such that $f(C) = f(D) = F$. It is impossible that $C \subset D$ or $D \subset C$ since f is hereditarily irreducible. Otherwise, if $C \cap D \neq \emptyset$, then for a continuum $Z = C \cup D$ we have that C and D are subcontinua of Z and $f(Z) = f(C) = f(D) = F$ which is impossible since f is hereditarily irreducible. We infer that $C \cap D = \emptyset$. Let (x, y) and (z, u) be the different points of D. There exists an open set $U = U_x \times U_y$ such that $C \cap U = \emptyset$ and $(z, u) \notin U$. By Lemma 2.1 $E = W \setminus U$ is a continuum. Moreover, $C \subset E, D \neq D \cap E \neq \emptyset$. Now $f(E \cup D) = f(E)$ which is impossible since f is hereditarily irreducible. Furthermore, $C(f)^{-1}(Z(1)) = W(1)$ since from the hereditarily irreducibility of f it follows that no non-degenerate subcontinuum of W maps under f onto a point.We infer that $C(f)^{-1}[Z \setminus Z(1)] = C(W) \setminus W(1)$. It follows that the restriction $P = C(f)|(C(W) \setminus W(1))$ is one-to-one and closed [2, p. 95, Proposition 2.1.4]. Hence, P is a homeomorphism of $C(W) \setminus W(1)$ onto $P(C(W) \setminus W(1)) \subset Z$. □

Proposition 2.2 enables to prove the following main result of this section.

Theorem 2.3. *Let $W = X \times Y$ be a product of two nondegenerate continua. If there exists a hereditarily irreducible mapping $f : W \to Z$, then $\mathrm{w}(W) = \mathrm{w}(Z)$.*

Proof. It is obvious that $\mathrm{w}(Z) \leq \mathrm{w}(W)$ [2, p. 171, Theorem 3.1.22]. Let us prove that $\mathrm{w}(Z) \geq \mathrm{w}(W)$. Firstly, from the fact that $C(f)|C(W) \setminus W(1) : C(W) \setminus W(1) \to C(f)(C(W) \setminus W(1))$ is a homeomorphism (Proposition 2.2) we have

$$\mathrm{w}(C(W) \setminus W(1)) \leq \mathrm{w}(Z) \tag{2.1}$$

since $\mathrm{w}(C(W) \setminus W(1)) = \mathrm{w}(C(f)(C(W) \setminus W(1)) \leq \mathrm{w}(2^Z) = \mathrm{w}(Z)$ [2, p. 306, Problem 3.12.26 (a)]. Now we shall prove that

$$\mathrm{w}(W) = \mathrm{w}(C(W) \setminus W(1)). \tag{2.2}$$

Let $\mathcal{B} = \{B_\alpha : \alpha \in A\}$ be a base of $C(W)\diagdown W(1)$ such that $\operatorname{card}(A) = \mathrm{w}(C(W)\diagdown W(1))$. For each $B_\alpha \in \mathcal{B}$ let $C_\alpha = \{w \in W : w \in B,\, B \in B_\alpha\}$, i.e., the union of all continua B contained in B_α.

The family $\{C_\alpha : \alpha \in A\}$ is a network of W. Let w be a point of W and let U be an open subsets of W such that $w \in U$. There exists an open set V such that $w \in V \subset \operatorname{Cl} V \subset U$. Let K be a component of $\operatorname{Cl} V$ containing w. By the Boundary Bumping Theorem [7, p. 73, Theorem 5.4] K is non-degenerate and, consequently, $K \in C(W)\diagdown W(1)$. Now, $\langle U\rangle \cap (C(W)\diagdown W(1))$ is a neighborhood of K in $C(W)\diagdown W(1)$. It follows that there exists a $B_\alpha \in \mathcal{B}$ such that $K \in B_\alpha \subset \langle U\rangle \cap (C(W)\diagdown W(1))$. It is clear that $C_\alpha \subset U$ and $w \in C_\alpha$ since $w \in K$. Hence, the family $\{C_\alpha : \alpha \in A\}$ is a network of W. Hence, $\mathrm{nw}(W) = \mathrm{w}(C(W)\diagdown W(1))$ and $\mathrm{w}(W) = \mathrm{w}(C(W)\diagdown W(1))$ (Theorem 1.1). The equation (2.2) is proved. Finally, from the relations (2.1) and (2.2) it follows $\mathrm{w}(W) \leq \mathrm{w}(Z)$. The proof is complete since we have $\mathrm{w}(W) \leq \mathrm{w}(Z)$ and $\mathrm{w}(Z) \leq \mathrm{w}(W)$. $\square$

We obtain an interesting result if Z is metrizable, i.e., $\mathrm{w}(Z) = \aleph_0$.

Theorem 2.4. *The product $X \times Y$ of two nondegenerate continua is metrizable if and only if there exists a hereditarily irreducible mapping $f : X \times Y \to Z$ onto a metrizable continuum Z.*

Proof. If $X \times Y$ is metrizable, then the identity $\mathrm{i} : X \times Y \to X \times Y$ is hereditarily irreducible. Conversely, if there exists a hereditarily irreducible mapping $f : X \times Y \to Z$ onto a metrizable continuum Z, then by Theorem 2.3 it follows that $\mathrm{w}(X \times Y) = \mathrm{w}(Z) = \aleph_0$. Hence, $X \times Y$ is metrizable. $\square$

In the case of hereditarily irreducible mappings on $\Pi\{X_s : s \in S\}$ the following result is obtained.

Theorem 2.5. *Let $\Pi\{X_s : s \in S\}$ be a product of nondegenerate continua such that $\operatorname{card}(S) \geq 2$. If there exists a hereditarily irreducible mapping $f : \Pi\{X_s : s \in S\} \to Z$, then $\mathrm{w}(\Pi\{X_s : s \in S\}) = \mathrm{w}(Z)$.*

Proof. It is clear that $\Pi\{X_s : s \in S\}$ and $X_t \times \Pi\{X_s : s \in S, s \neq t\}$ are homeomorphic [2, Proposition 2.3.7, p, 109]. From Theorem 2.3 it follows that $\mathrm{w}(X_t \times \Pi\{X_s : s \in S, s \neq t\}) = \mathrm{w}(Z)$. Hence, $\mathrm{w}(\Pi\{X_s : s \in S\}) = \mathrm{w}(Z)$. $\square$

Corollary 2.6. *Let $\Pi\{X_s : s \in S\}$ be a product of nondegenerate continua such that $\operatorname{card}(S) \geq 2$. If there exists a hereditarily irreducible mapping $f : \Pi\{X_s : s \in S\} \to Z$ onto metrizable continuum Z, then $\Pi\{X_s : s \in S\}$ is metrizable.*

3. Whitney maps

Let Λ be a subspace of 2^X. By a *Whitney map* for Λ [6, p. 24, (0.50)] we will mean any mapping $W : \Lambda \to [0, +\infty)$ satisfying

a) if $A, B \in \Lambda$ such that $A \subset B$ and $A \neq B$, then $W(A) < W(B)$ and
b) $W(\{x\}) = 0$ for each $x \in X$ such that $\{x\} \in \Lambda$.

If X is a metric continuum, then there exists a Whitney map for 2^X and $C(X)$ ([6, pp. 24-26], [3, p. 106]). On the other hand, if X is non-metrizable, then it admits no Whitney map for 2^X [1]. It is known that there exist non-metrizable continua which admit and ones which do not admit a Whitney map for $C(X)$ [1].

Let X, Y be compact spaces. A mapping $f : X \to Y$ is *light* (*zero-dimensional*) if all fibers $f^{-1}(y)$ are hereditarily disconnected (zero-dimensional or empty) [2, p. 450], i.e., if $f^{-1}(y)$ does not contain any connected subsets of cardinality larger that one ($\dim f^{-1}(y) \leq 0$). Every zero-dimensional mapping is light, and in the realm of mappings with compact fibers the two classes of mappings coincide.

The following theorem explains the role of light mappings in the study of Whitney maps for continua.

Theorem 3.1. *A continuum X admits a Whitney map for $C(X)$ if and only if there exists a light mapping $f : C(X) \to Y$ onto a metric continuum Y.*

Proof. **a)** Suppose that X admits a Whitney map for $C(X)$. By [5, Theorem 1.8] there exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric continua X_a such that X is homeomorphic to $\lim \mathbf{X}$. Now we have a σ-directed inverse system $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ of metric continua such that $C(X)$ is homeomorphic to $\lim C(\mathbf{X})$. From [4, Corollary 3.2.] it follows that the projections $C(p_b) : C(\lim \mathbf{X}) \to C(X_b)$ are light for every $b \in B$, where B is cofinal subset of A. Hence, each $C(p_b)$ is a required light mapping onto a metric continuum $Y = C(p_b)(\lim C(\mathbf{X}))$.

b) Suppose now that there exist a light mapping $f : C(X) \to Y$ onto a metric continuum Y. Consider, as in a), a σ-directed inverse system $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ of metric continua such that $C(X)$ is homeomorphic to $\lim C(\mathbf{X})$. There is a subset B cofinal in A such that there exists a mapping $f_b : C(p_b)(\lim C(X)) \to Y$ such that $f = f_b C(p_b)$ since $C(\mathbf{X})$ is σ-directed and Y is metric. Let us prove that $C(p_b)$ is light. Suppose that it is not light. Then there exist a point $z \in C(p_b)(\lim C(X))$ such that $C(p_b)^{-1}(z)$ contains a continuum Z. It follows that $f^{-1}(f_b(z))$ contains Z since $C(p_b)^{-1}(z) \subset f^{-1}(f_b(z))$. This is impossible since f is light. Hence, $C(p_b)$ is light. By [4, Corollary 3.2.] we infer that X admits a Whitney map for $C(X)$. The proof of Theorem is completed. $\square$

Proposition 3.2. [6, p. 204, (1.212.3)]. *If $f : X \to Y$ is a mapping between continua, then $C(f) : C(X) \to C(Y)$ is light if and only if f is hereditarily irreducible.*

Proposition 3.2 and Theorem 3.1 imply the following result which shows the role of hereditarily irreducible mappings in the study of Whitney maps for continua.

Corollary 3.3. *A continuum X admits a Whitney map for $C(X)$ if and only if there exists a hereditarily irreducible mapping $f : X \to Y$ onto a metric continuum Y.*

In the study of a Whitney map for $C(\Pi\{X_s : s \in S\})$ the following result is useful.

Theorem 3.4. *If a space X contains a transfinite increasing (decreasing) sequence of subcontinua, then X admits no Whitney map for $C(X)$.*

Proof. Suppose that $\omega : C(X) \to \mathbb{R}$ is a Whitney map and that X contains the transfinite increasing sequence $C_0 \subset C_1 \subset \cdots \subset C_\xi \subset \ldots,\ \xi < \omega_1$ (decreasing sequence $C_0 \supset C_1 \supset \cdots \supset C_\xi \supset \ldots, \xi < \omega_1$) of subcontinua of X (where ω_1 is the first uncountable ordinal). Then $\omega(C_0) < \omega(C_1) < \cdots < \omega(C_\xi) < \ldots,\ \xi < \omega_1$ is an increasing transfinite sequence of real numbers ($\omega(C_0) > \omega(C_1) > \cdots > \omega(C_\xi) > \ldots, \xi < \omega_1$ is decreasing transfinite sequence of real numbers). This is impossible since $\mathrm{w}(\mathbb{R}) = \aleph_0$. $\square$

Using Theorem 3.4 we obtain the following theorem.

Theorem 3.5. *If a product $\Pi\{X_s : s \in S\}$ of nondegenerate continua admits a Whitney map for $C(\Pi\{X_s : s \in S\})$, then $\mathrm{card}(S) \leq \aleph_0$.*

Proof. Suppose, on the contrary, that $\mathrm{card}(S) \geq \aleph_1$. We may assume that S contains all countable ordinal numbers. For each countable ordinal number α we define $C_\alpha = \Pi\{X_\beta : \beta \leq \alpha\}$. It is clear that $C_0 \subset C_1 \subset \cdots \subset C_\xi \subset \ldots$, $\xi < \omega_1$ (where ω_1 is the first uncountable ordinal). This contradicts Theorem 3.4. Hence, $\mathrm{card}(S) \leq \aleph_0$. $\square$

Theorem 3.6. *Let $X \times Y$ be a product of nondegenerate continua. If $X \times Y$ admits a Whitney map for $C(X \times Y)$, then both X and Y are metrizable.*

Proof. Apply Theorem 2.3 and the facts that if a continuum admits a Whitney map for $C(X \times Y)$ then there exist a hereditarily irreducible mapping onto a metrizable continuum (Corollary 3.3). $\square$

We close this section with the following results.

Theorem 3.7. *The product $\Pi\{X_s : s \in S\}$ of nondegenerate continua admits a Whitney map for $C(\Pi\{X_s : s \in S\})$ if and only if it is metrizable.*

Proof. If $\Pi\{X_s : s \in S\}$ is a metric continuum, then there exists a Whitney map for 2^X and $C(X)$ ([6, pp. 24-26], [3, p. 106]). Conversely, if $\Pi\{X_s : s \in S\}$ admits a Whitney map, then Theorem 3.5 implies that $\text{card}(S) \leq \aleph_0$. Thus, we may assume that S is the set $\mathbb{N}$ of all positive integers. For each $i \in \mathbb{N}$ we consider the product $X_i \times \Pi\{X_j : j \neq i\}$ which is homeomorphic to $\Pi\{X_k : k \in \mathbb{N}\}$. Thus, $X_i \times \Pi\{X_j : j \neq i\}$ admits a Whitney map. By Theorem 3.6 we infer that X_i is metrizable. Hence, each X_i is metrizable and, consequently, $\Pi\{X_k : k \in \mathbb{N}\}$ is metrizable. □

It is known [2, p. 171, Corollary 3.1.20] that if a compact space X is the countable union of its subspaces $X_n, n \in \mathbb{N}$, such that $\mathrm{w}(X_n) \leq \aleph_0$, then $\mathrm{w}(X) \leq \aleph_0$. Using this fact and the above proved theorems we obtain the following theorem.

Theorem 3.8. *If a continuum X is the countable union of its subcontinua which are products, then X admits a Whitney map for $C(X)$ if and only if it is metrizable.*

Proof. Let $X = \cup\{X_i \times Y_i : i \in \mathbb{N}\}$. If X admits a Whitney map for $C(X)$, then each $X_i \times Y_i$ admits a Whitney map for $C(X_i \times Y_i)$. By Theorem 3.6 it follows that each $X_i \times Y_i$ is metrizable. Hence, X is metrizable [2, p. 171, Corollary 3.1.20]. □

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Faculty of Organizations and
Informatics Varaždin
Croatia
E–mail: ivan.loncar1@vz.htnet.hr