SARAJEVO JOURNAL OF MATHEMATICS
Vol.7 (19) (2011), 123–133

# A STRICT FIXED POINT PROBLEM FOR $\delta$-ASYMPTOTICALLY REGULAR MULTIFUNCTIONS AND WELL-POSEDNESS

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ABSTRACT. In 2005, Lj. Ćirić has established a fixed point theorem for asymptotically regular selfmappings of complete metric spaces. The purpose of this paper is to extend this theorem to the case of $\delta$-asymptotically regular multifunctions on an orbitally complete metric space $X$ which satisfy a variant of Ćirić's contractive condition. The well-posedness of the strict fixed point problem of these multifunctions is studied. We provide also a general result when the metric space $X$ is compact. Our results are natural extensions to some recent results of Lj. B. Ćirić and some old results obtained by Sharma and Yuel and Guay and Singh.

## 1. INTRODUCTION

Many authors have extended the Banach fixed point theorem (see [2]) by introducing more general contractive conditions, which imply the existence of a fixed point. Almost all of these conditions imply the asymptotic regularity of the mappings under consideration. We recall that the notion of asymptotic regularity for mappings was introduced by Browder and Petryshyn (see [3]).

**Definition 1.1.** *A selfmapping $T$ on a metric space $(X, d)$ is said to be* asymptotically regular at a point $x$ in $X$, if

$$d(T^n x, T^n Tx) \to 0 \quad as \quad n \to \infty,$$

*where $T^n x$ denotes the $n$-th iterate of $T$ at $x$.*

So the investigation of the asymptotically regular maps plays an important role in fixed point theory.

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2000 *Mathematics Subject Classification.* 54H25, 47H10.
*Key words and phrases.* Strict fixed point of a multifunction, well-posedness of strict fixed point problem, $\delta$-asymptotically regular multifunctions, $T$-orbitally complete metric spaces, compact metric spaces.

Sharma and Yuel [16] and Guay and Singh [12] were among the first who used concept of asymptotic regularity to prove fixed point theorems for a wider class of mappings than a class of mappings introduced and studied by Ćirić in [5]. In (2005), Ćirić (see [6]) has generalized the results of [16] and [12] by the following result.

**Theorem 1.1.** *Let* $R^+$ *be the set of nonnegative reals and let* $F_i : R^+ \to R^+$ *be functions such that* $F_i(0) = 0$ *and* $F_i$ *is continuous at* $0$ $(i = 1, 2)$.

*Let* $(X, d)$ *be a complete metric space and* $T$ *be a selfmapping on* $X$ *satisfying the following condition:*

$$d(Tx, Ty) \leq a_1 F_1[\min\{d(x, Tx), d(y, Ty)\}] + a_2 F_2[d(x, Tx), d(y, Ty)]$$
$$+a_3 d(x, y) + a_4[d(x, Tx) + d(y, Ty)] + a_5[d(x, Ty) + d(y, Tx)] \tag{1.1}$$

*for all* $x, y$ *in* $X$, *where* $a_i = a_i(x, y)$ $(i = 1, 2, 3, 4, 5)$ *are nonnegative functions for which there exist a constant* $K > 0$ *and* $0 < \lambda_1, \lambda_2 < 1$ *such that:*

$$a_1(x, y), a_2(x, y) \leq K, \tag{1.2}$$

$$a_4(x, y) + a_5(x, y) \leq \lambda_1, \tag{1.3}$$

$$a_3(x, y) + 2a_5(x, y) \leq \lambda_2, \tag{1.4}$$

*for all* $x, y$ *in* $X$.

*If* $T$ *is asymptotically regular at some* $x_0$ *in* $X$, *then* $T$ *has a unique fixed point in* $X$ *and at this point* $T$ *is continuous.*

The purpose of this paper is to extend Theorem 1.1 to the case of multifunctions. To state our main result, we need to introduce some preliminaries. These preliminaries are gathered in the second section. In the third section, we present our main result (see Theorem 3.1) in which we investigate existence and uniqueness of strict fixed points for a multifunction $T : X \to B(X)$, where $B(X)$ is the set of all nonempty bounded sets of a metric space $(X, d)$ satisfying the contractive condition (3.1) when $X$ is $T$-orbitally complete (see Definition 2.4) and $T$ is $\delta$-asymptotically regular (see Definition 3.1). Two other related general results (see Theorem 3.2 and Theorem 3.3) are also established in Section 3. In the fourth section we establish the well-posedness of the strict fixed point problem for these multifunctions.

## 2. Preliminaries

Throughout this paper, $\mathbb{N}$ will be the set of non negative integers. Let $(X, d)$ be a metric space and $B(X)$ the set of all nonempty bounded sets of $X$. As in [8], [9] and [10], we define the functions $\delta(A, B)$ and $D(A, B)$ by

$$\delta(A, B) := \sup\{d(a, b) : a \in A,\ b \in B\},$$

$$D(A,B) := \inf\{d(a,b) : a \in A,\, b \in B\}.$$

If $A$ consists of single point "$a$" we write $\delta(A,B) = \delta(a,B)$.
If $B$ consists of single point "$b$", we write $\delta(A,B) = \delta(A,b)$. It follows immediately from the definition of $\delta(A,B)$ that

$$\delta(A,B) = \delta(B,A), \quad \forall A, B \in B(X),$$

and

$$\delta(A,B) \leq \delta(A,C) + \delta(C,B), \quad \forall A, B, C \in B(X).$$

**Definition 2.1.** *A sequence $\{A_n\}$ of nonempty subsets of $X$ is said to converge to a subset $A$ of $X$ if:*
(i) *Each point $a \in A$ is the limit of a convergent sequence $\{a_n\}$, where $a_n \in A_n$, for all $n \in \mathbb{N}$.*
(ii) *For arbitrary $\epsilon > 0$ there exists an integer $m > 0$ such that $A_n \subset A(\epsilon)$ for all integer $n \geq m$ , where*

$$A(\epsilon) := \{x \in X : \exists a \in A : d(x,a) < \epsilon\}.$$

*The set $A$ is said to be the limit of the sequence $\{A_n\}$.*

**Lemma 2.1.** (Fisher ([8])). *If $\{A_n\}$ and $\{B_n\}$ are two sequences in $B(X)$ converging to the sets $A$ and $B$ respectively in $B(X)$, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A,B)$.*

**Lemma 2.2.** (Fisher and Sessa ([10])). *Let $\{A_n\}$ be a sequence in $B(X)$ and $y \in X$ such that $\lim_{n\to\infty} \delta(A_n, y) = 0$. Then the sequence $\{A_n\}$ converges to $\{y\}$ in $B(X)$.*

**Definition 2.2.** *Let $T : X \to B(X)$ be a multifunction.*
a) *A point $x \in X$ is a fixed point of $F$ if $x \in Tx$.*
b) *A point $x \in X$ is a strict fixed point of $T$ if $\{x\} = Tx$.*

In 1974, Ćirić (see [4]) has first introduced orbitally complete metric spaces.

**Definition 2.3.** *Let $f : (X,d) \to (X,d)$. If for any $x \in X$, every Cauchy sequence of the orbit $O(f,x) := \{x, fx, f^2x, \ldots\}$ is convergent in $X$, then the metric space is said to be $f$-orbitally complete.*

**Remark 2.1.** Every complete metric space is $f$-orbitally complete for any $f$. An orbitally complete space may not be a complete metric space (see [17]).

Let $T : X \to B(X)$ and $x_0 \in X$. An orbit of $T$ at point $x_0$, is a sequence $\{x_n\}$ given by

$$O(T, x_0) := \{x_n : x_{n+1} \in T(x_n),\, n = 0, 1, 2, \ldots\}.$$

**Definition 2.4.** *Let* $(X, d)$ *be a metric space. Let* $T : X \to B(X)$ *be a multifunction.* $(X, d)$ *is said to be* $T$*-orbitally complete, if for all* $x \in X$*, every Cauchy subsequence of the orbit* $O(T, x)$ *converges to a point in* $X$*.*

The notion of well-posednes of a fixed point problem has evoked much interest to several mathematicians (see for example [15], [7], [11], [13], [14] and [1]).

**Definition 2.5.** *Let* $(X, d)$ *be a metric space and* $T : (X, d) \to (X, d)$ *be a mapping. The fixed point problem of* $T$ *is said to be well posed if:*
(i) $T$ *has a unique fixed point* $z$ *in* $X$*,*
(ii) *for any sequence* $\{x_n\}$ *of points in* $X$ *such that* $\lim_{n\to\infty} d(Tx_n, x_n) = 0$*, we have* $\lim_{n\to\infty} d(x_n, z) = 0$.

We extend Definition 2.5 for multifunctions.

**Definition 2.6.** *Let* $(X, d)$ *be a metric space and* $T : X \to B(X)$ *be a multifunction. The strict fixed point problem of* $T$ *is said to be well-posed if:*
(i) $T$ *has a unique strict fixed point* $z$ *in* $X$*,*
(ii) *for any sequence* $\{x_n\}$ *of points in* $X$ *such that* $\lim_{n\to\infty} \delta(Tx_n, x_n) = 0$*, we have* $\lim_{n\to\infty} d(x_n, z) = 0$.

## 3. Main result

To present our main result, we need to introduce the following class of functions.

Let $\mathbb{R}^+$ be the set of nonnegative reals and let $\mathcal{F}$ be the set of functions $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following properties:

(F 1) $F(t, 0) = 0 = F(0, t)$ for all $t \in \mathbb{R}^+$.

(F 2) $F$ is continuous at the point $(0, 0)$.

(F 3) For all $t \in \mathbb{R}^+$, the functions $F(., t)$ and $F(t, .)$ are continuous at 0.

Examples of such functions are given below.

**Example 3.1.** $F(s, t) = F_1(\min\{s, t\}) + F_2(s\,t)$, where $F_i : \mathbb{R}^+ \to \mathbb{R}^+$ are functions such that $F_i(0) = 0$ and $F_i$ is continuous at 0 for $(i = 1, 2)$.

**Example 3.2.** $F(s, t) = \alpha[\min\{s, t\} + s\,t]$, where $\alpha \in [0, \infty)$.

**Example 3.3.** $F(s, t) = F(\min\{s, t\} + s\,t)$, where $F : \mathbb{R}^+ \to \mathbb{R}^+$ is a function such that $F(0) = 0$ and $F$ is continuous at 0.

**Example 3.4.** $F(s, t) = \frac{\min\{s,t\}+s\,t}{1+s+t}$, for all $s, t \in \mathbb{R}^+$.

**Example 3.5.** $F(s, t) = \frac{\min\{s,t\}^p+s^q\,t^r}{1+(s+t)^m}$, for all $s, t \in \mathbb{R}^+$, where $p, q, r, m \in (0, \infty)$.

We generalize Definition 1.1 by the following definition.

**Definition 3.1.** *A selfmapping $T$ on a metric space $(X,d)$ is said to be* $\delta$-asymptotically regular at a point $x$ in $X$, if

$$\lim_{n\to\infty} \delta(x_n, Tx_n) = 0,$$

*where $\{x_n\}$ denotes the sequence of points such that $x_{n+1} \in Tx_n$ for each non-negative integer $n$ with $x_0 := x$.*

The main result of this paper runs as follows.

**Theorem 3.1.** *Let $(X,d)$ be a metric space and $T : X \to B(X)$ a multifunction such that*

$$\begin{aligned}\delta(Tx,Ty) &\le a_0F(\delta(x,Tx),\delta(y,Ty)) \\ &\quad + a_1d(x,y) + a_2[\delta(x,Tx)+\delta(y,Ty)] + a_3[D(x,Ty)+D(y,Tx)] \quad (3.1)\end{aligned}$$

*for all $x, y$ in $X$, where $F \in \mathcal{F}$ and $a_i = a_i(x,y)$ $(i = 0,1,2,3)$ are nonnegative functions for which there exist three constants $K > 0$ and $\lambda_1, \lambda_2 \in (0,1)$, such that the following inequalities:*

$$a_0(x,y) \le K, \quad (3.2)$$

$$a_2(x,y) + a_3(x,y) \le \lambda_1, \quad (3.3)$$

$$a_1(x,y) + 2a_3(x,y) \le \lambda_2 \quad (3.4)$$

*are satisfied for all $x, y$ in $X$.*

*If $(X,d)$ is $T$-orbitally complete and if $T$ is $\delta$-asymptotically regular at some $x_0$ in $X$, then the multifunction $T$ has an unique fixed $u$ point in $X$ which is strict fixed point for $T$.*

*Moreover, $T$ is $\delta$-continuous at the unique strict fixed point $u$ in the following sense:*

*For each sequence $\{u_n\}$ converging to $u$, we have $\lim_{n\to\infty}\delta(u,Tu_n) = 0$.*

*Proof.* We first show that $\{x_n\}$ is a Cauchy sequence, where $\{x_n\}$ denotes the sequence of points such that $x_{n+1} \in Tx_n$ for each non-negative integer $n$ with $x_0 := x$. Denote

$$d_n = \delta(x_n, Tx_n).$$

Using the triangle inequality, from (1.1) we have

$$\begin{aligned}d(x_n,x_m) &\le d_n + \delta(Tx_n,Tx_m) + d_m \\ &\le d_n + d_m + a_0F(d_n,d_m) + a_1d(x_n,x_m) \\ &\quad + a_2(d_n+d_m) + a_3[D(x_n,Tx_m)+D(x_m,Tx_n)],\end{aligned}$$

where $a_i = a_i(x_n,x_m)$. Using again the triangle inequality, we get

$$d(x_n,x_m) \le (a_1+2a_3)d(x_n,x_m) + (1+a_2+a_3)(d_n+d_m) + a_0F(d_n,d_m).$$

Hence, because of (3.2), (3.3) and (3.4), we obtain

$$d(x_n, x_m) \leq \lambda_2 d(x_n, x_m) + (1+\lambda_1)(d_n + d_m) + KF(d_n, d_m),$$

from which we get

$$d(x_n, x_m) \leq \frac{1+\lambda_1}{1-\lambda_2}(d_n + d_m) + \frac{K}{1-\lambda_2} F(d_n, d_m).$$

Since $T$ is $\delta$-asymptotically regular and $F$ is continuous at $(0,0)$, taking the limit as $m$ and $n$ tend to infinity we obtain

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0,$$

which implies that $\{x_n\}$ is a Cauchy sequence.

Since $X$ is $T$-orbitally complete, there is some $u$ in $X$ such that

$$\lim x_n = u.$$

For each non-negative integer $n$, we have

$$\delta(u, Tx_n) \leq d(u, x_n) + \delta(x_n, Tx_n),$$

which implies that

$$\lim_{n\to\infty} \delta(u, Tx_n) = 0. \tag{3.5}$$

Now we show that $u$ is a unique strict fixed point of $T$. Suppose $\delta(u, Tu) > 0$. From (1.1) we have

$$\begin{aligned}
\delta(u, Tu) &\leq \delta(u, Tx_n) + \delta(Tx_n, Tu) \\
&\leq \delta(u, Tx_n) + a_0 F(d_n, \delta(u, Tu)) + a_1 d(x_n, u) \\
&+ a_2[d_n + \delta(u, Tu)] + a_3[D(x_n, Tu) + D(u, Tx_n)],
\end{aligned}$$

where $a_i = a_i(x_n, u)$. Using the triangle inequality we get

$$\begin{aligned}
\delta(u, Tu) \leq\ & (1+a_3)\delta(u, Tx_n) + a_0 F(d_n, \delta(u, Tu)) \\
&+ (a_1 + a_3) d(x_n, u) + a_2 d_n + (a_2 + a_3)\delta(u, Tu).
\end{aligned}$$

Therefore, from (3.2), (3.3) and (3.4), we have

$$\begin{aligned}
\delta(u, Tu) \leq\ & (1+\lambda_1)\delta(u, Tx_n) + KF(d_n, \delta(u, Tu)) \\
&+ \lambda_2 d(x_n, u) + \lambda_1 d_n + \lambda_1 \delta(u, Tu).
\end{aligned}$$

Since $F$ is continuous at the point $(0, \delta(u, Tu))$ and since $F(0, t) = 0$, then by taking the limit we get

$$\delta(u, Tu) \leq \lambda_1 \delta(u, Tu) < \delta(u, Tu),$$

a contradiction. Therefore, $\delta(u, Tu) = 0$; hence $Tu = \{u\}$. That is $u$ is a strict fixed point for $T$.

To prove the uniqueness of $u$, let us suppose that $u$ and $v$ are two strict fixed points of $T$. From (3.1), with $a_i = a_i(u, v)$,

$$\begin{aligned} d(u,v) = \delta(Tu,Tv) &\le a_0F(0,0) + a_1d(u,v) \\ &\quad + a_2 \cdot 0 + 2a_3d(u,v) \\ &= (a_1 + 2a_3)d(u,v). \end{aligned}$$

Hence, because of (3.4),

$$(1 - \lambda_2)d(u,v) \le 0,$$

which implies $v = u$.

To prove that $T$ is $\delta$-continuous at $u$, suppose that $x_n \to u$. Then from (3.1),

$$\begin{aligned} \delta(Tx_n,u) = \delta(Tx_n,Tu) &\le a_0 \cdot F(\delta(x_n,Tx_n),0) + a_1d(x_n,u) \\ &\quad + a_2\delta(x_n,Tx_n) + a_3[d(x_n,u) + \delta(Tx_n,u)] \\ &\le (a_1 + a_2 + a_3)d(x_n,u) + (a_2 + a_3)\delta(u,Tx_n), \end{aligned}$$

where $a_i = a_i(x_n, u)$. Hence, using (3.3) and (3.4),

$$\delta(u,Tx_n) \le (\lambda_1 + \lambda_2)d(x_n,u) + \lambda_1\delta(u,Tx_n),$$

from which we obtain

$$\delta(u,Tx_n) \le \frac{\lambda_1 + \lambda_2}{1 - \lambda_1}d(x_n,u).$$

Letting $n$ go to infinity, we obtain

$$\lim_{n\to\infty} \delta(u,Tx_n) = 0,$$

which implies that $T$ is $\delta$-continuous at the point $u$. This completes the proof. □

**Remark 3.1.** Theorem 3.1 generalizes Theorem 1.1 obtained by Lj. B. Ćirić in [6].

The following corollary provides an extension of a result of Sharma and Yuel [16].

**Corollary 3.1.** *Let $(X, d)$ be a metric space and $T : X \to B(X)$ be a multifunction such that*

$$\delta(Tx,Ty) \le \alpha\frac{\min\{\delta(x,Tx),\delta(y,Ty)\} + \delta(x,Tx) \cdot \delta(y,Ty)}{1 + d(x,y)} + \beta d(x,y),$$

*for all $x, y$ in $X$, where $\alpha, \beta \in [0, 1)$.*

*If $(X, d)$ is $T$-orbitally complete and if $T$ is $\delta$-asymptotically regular at some $x_0$ in $X$, then the multifunction $T$ has a unique fixed u point in $X$ which is strict fixed point for $T$. Moreover, $T$ is $\delta$-continuous at the point $u$.*

This corollary follows from Theorem 3.1, by the following considerations:

$$a_0(x,y) = \frac{1}{1+d(x,y)}, \quad F(s,t) = \alpha(\min\{s,t\} + st),$$
$$a_1(x,y) = \beta, \quad a_2(x,y) = a_3(x,y) = 0.$$

Clearly (3.2) (3.3) and (3.4) are realized by taking $K = 1$ and $\lambda_1 = \lambda_2 = \beta < 1$.

The following corollary extends a result of Guay and Singh [12]

**Corollary 3.2.** *Let $(X,d)$ be a metric space and $T : X \to B(X)$ a multifunction such that*

$$\delta(Tx,Ty) \leq pd(x,y) + q[\delta(x,Tx) + \delta(y,Ty)] + r[D(x,Tx) + D(y,Ty)],$$

*for all $x,y$ in $X$, where $p,q$ and $r$ are fixed non-negative real numbers such that $q + r < 1$ and $p + 2r < 1$.*

*If $(X,d)$ is $T$-orbitally complete and if $T$ is $\delta$-asymptotically regular at some $x_0$ in $X$, then the multifunction $T$ has a unique strict fixed point $u$ in $X$ at which $T$ is $\delta$-continuous.*

This corollary follows from Theorem 3.1 by setting $a_0 = 0$, $F = 0$; $a_1 = p$, $a_2 = q$, $a_3 = r$; $\lambda_1 = q + r$ and $\lambda_2 = p + 2r$.

**Remark 3.2.** An example of Yuel and Sharma [16] shows that assumption of asymptotically regularity in the above theorems can not be dropped.

The following theorem unifies and generalizes two theorems obtained by Sharma and Yuel in [16] and by Lj. Ćirić in [6].

**Theorem 3.2.** *Let $(X,d)$ be a metric space, not necessarily orbitally complete, and let $T$ be as in Theorem 3.1. If an orbit of $T$ at some $x_0$ has a subsequence converging to a point $u$ in $X$, then $u$ is the unique strict fixed point of $T$, and the sequence defined by $x_{n+1} \in Tx_n$ also converges to $u$ and $T$ is $\delta$-continuous at $u$.*

*Proof.* Suppose that $T$ is $\delta$-asymptotically regular at some point $x$ in $X$. Let $\{x_n\}$ be an orbit of $T$ at $x$ such that

$$\lim_{n\to\infty} \delta(x_n, Tx_n) = 0$$

As it is shown in the proof of the Theorem 3.1, the sequence $\{x_n\}$ is a Cauchy sequence. Since it contains a subsequence converging to $u$, then we have

$$\lim_{n\to\infty} d(u,x_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} \delta(u,Tx_n) = 0.$$

The rest of the result follows by the same method of proof as in the Theorem 3.1. So, we omit the details. $\square$

As a consequence, we have the following result for compact metric spaces.

**Theorem 3.3.** *Let* $(X, d)$ *be a compact metric space and let* $T$ *be a multifunction as in Theorem* 3.1. *Then* $T$ *has a unique stricte fixed point in* $X$ *at which* $T$ *is* $\delta$*-continuous.*

## 4. Well-posedness

We end this paper by a result establishing the well-posedness of the strict fixed point problem for a multifunction $T$ on a metric space satisfying the conditions of Theorem 3.1. More precisely, we have.

**Theorem 4.1.** *Let* $(X, d)$ *be a metric space and* $T : X \to B(X)$ *a multifunction such that*

$$\delta(Tx, Ty) \leq a_0 F(\delta(x, Tx), \delta(y, Ty)) \\ + a_1 d(x, y) + a_2[\delta(x, Tx) + \delta(y, Ty)] + a_3[D(x, Ty) + D(y, Tx)] \quad (4.1)$$

*for all* $x, y$ *in* $X$*, where* $F \in \mathcal{F}$ *and* $a_i = a_i(x, y)$ $(i = 0, 1, 2, 3)$ *are nonnegative functions for which there exist three constants* $K > 0$ *and* $\lambda_1, \lambda_2 \in (0, 1)$*, such that the following inequalities:*

$$a_0(x, y) \leq K, \quad (4.2)$$

$$a_2(x, y) + a_3(x, y) \leq \lambda_1, \quad (4.3)$$

$$a_1(x, y) + 2a_3(x, y) \leq \lambda_2 \quad (4.4)$$

*are satisfied for all* $x, y$ *in* $X$.

*If* $(X, d)$ *is* $T$*-orbitally complete and if* $T$ *is* $\delta$*-asymptotically regular at some* $x_0$ *in* $X$*, then the strict fixed point problem for* $T$ *is well posed.*

*Moreover,* $T$ *is* $\delta$*-continuous at its unique strict fixed point.*

*Proof.* By Theorem 2.1, $T$ has a unique strict fixed point $u$ in $X$. Let $\{u_n\}$ be a sequence in $X$ such that

$$\lim_{n\to\infty} \delta(u_n, Tu_n) = 0.$$

We have to show that $\lim_{n\to\infty} d(u_n, u) = 0$. To this end, we use the inequality (4.1). Then (by using the triangle inequality) we have

$$\begin{aligned} \delta(u_n, u) &\leq \delta(u_n, Tu_n) + \delta(Tu_n, Tu) \\ &\leq \delta(u_n, Tu_n) + a_0 F(\delta(u_n, Tu_n), 0) + a_1 d(u_n, u) \\ &\qquad + a_2 \delta(u_n, Tu_n) + a_3[2d(u_n, u) + \delta(u_n, Tu_n)] \\ &\leq (a_1 + 2a_3) d(u_n, u) + (1 + a_2 + a_3)\delta(u_n, Tu_n), \end{aligned}$$

where $a_i = a_i(u_n, u)$. Hence, using (4.3) and (4.4),

$$\delta(u_n, u) \leq \lambda_2 d(u_n, u) + (1 + \lambda_1)\delta(u_n, Tu_n),$$

from which we obtain

$$\delta(u_n, u) \le \frac{1+\lambda_1}{1-\lambda_2}\delta(u_n, Tu_n).$$

Letting $n$ go to infinity, we obtain

$$\lim_{n\to\infty} \delta(u, u_n) = 0,$$

which implies that the strict fixed point problem for $T$ is well posed.

The remainder follows from Theorem 2.1. This completes the proof. □

**Acknovledgement.** The author thanks very much the referee for his (her) many valuable comments and useful suggestions.

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(Received: December 5, 2009)
(Revised: April 21, 2010)

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