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SINGULAR CURVES ON K3 SURFACES

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ABSTRACT. We investigate the Clifford index of singular curves on K3 surfaces by following the lines of [10]. As a consequence, we are able to deduce from [3] that Green's conjecture holds for all integral curves on K3 surfaces.

1. INTRODUCTION

Let C be a complex integral projective curve of arithmetic genus $g \geq 2$. For any line bundle $L \in \mathrm{Pic}(C)$ and all integers p, q, let $K_{p,q}(C, L)$ denote the Koszul cohomology groups introduced in [9] as the cohomology of the complex:

$$\wedge^{p+1} H^0(L) \otimes H^0(L^{q-1}) \to \wedge^p H^0(L) \otimes H^0(L^q) \to \wedge^{p-1} H^0(L) \otimes H^0(L^{q+1}).$$

Green's conjecture states that $K_{p,1}(C, \omega_C) = 0$ if and only if $p \geq g - \mathrm{Cliff}(C) - 1$, where

$$\begin{aligned}\mathrm{Cliff}(C) = \min\{&\deg(A) - 2(h^0(A) - 1) : A \text{ is a torsion free sheaf on } C \\ &\text{with } h^0(A) \geq 2, h^1(A) \geq 2\}\end{aligned}$$

is the Clifford index of C.

Green's conjecture is known to hold for the general curve of genus g (see [14] and [15]) and has been recently verified also for every smooth curve lying on an arbitrary K3 surface (see [3], Theorem 1.2). In particular, [2] shows that Green's conjecture is satisfied for any smooth d-gonal curve verifying a suitable linear growth condition on the dimension of Brill-Noether varieties of pencils which holds for the general d-gonal curve. The arguments in [2], taking the path opened in [14], rely on suitable degenerations to irreducible nodal curves.

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This approach motivates a systematic investigation of Green's conjecture in the singular case. In our previous contribution [6] we already addressed the case of k-gonal nodal curves, here instead we consider singular curves on K3 surfaces. Following the standard terminology, by a K3 surface we mean a smooth compact complex surface X with $h^1(\mathcal{O}_X) = 0$ and $K_X \cong \mathcal{O}_X$.

Our main result is the following useful generalization of [10]:

Theorem 1. *Let X be a* K3 *surface and let $C \subset X$ be an integral curve of arithmetic genus $g \geq 2$. Then*

$$\mathrm{Cliff}(C') = \mathrm{Cliff}(C)$$

for every integral curve $C' \in |C|$.

As a consequence of Theorem 1, we obtain the following remarkable extension of [3], Theorem 1.2:

Corollary 1. *Green's conjecture holds for every integral curve C of arithmetic genus $g \geq 2$ lying on an arbitrary* K3 *surface X.*

We work over the field $\mathbb{C}$ of complex numbers.

2. The proofs

Let X be a regular smooth projective surface (i.e. $h^1(\mathcal{O}_X) = 0$). Let C be an integral curve on X and let A be a rank one torsion-free sheaf on C. Assume that A is generated by its global sections. Then we have an exact sequence

$$0 \to F(C, A) \to H^0(C, A) \otimes \mathcal{O}_X \to A \to 0,$$

where $F(C, A)$ is locally free since it has depth 2 on a smooth surface (see [11], Proposition 1.3 and Corollary 1.4).

By dualizing and setting $E(C, A) := F(C, A)^\vee$ we obtain the short exact sequence

$$0 \to H^0(C, A) \otimes \mathcal{O}_X \to E(C, A) \to \mathcal{E}xt^1_{\mathcal{O}_X}(A, \mathcal{O}_X) \to 0. \tag{1}$$

Thanks to Lemma 2 of [12] we have

$$\mathcal{E}xt^1_{\mathcal{O}_X}(A, \mathcal{O}_X) \cong \mathcal{N}_{C/X} \otimes A^\vee. \tag{2}$$

The corresponding local case is addressed in [8], Proposition 21.10 and in [7], Corollary 3.1.15.

Moreover by adjunction we obtain

$$\mathcal{N}_{C/X} \otimes A^\vee \cong \omega_C \otimes \mathcal{O}_C(-K_X) \otimes A^\vee. \tag{3}$$

From the above formulae it follows that the properties of $E(C,A)$ are the same as for the classical Lazarsfeld-Mukai bundle (see [10], §2):

$$\begin{aligned}\det E(C,A) &= \mathcal{O}_X(C),\\ c_2(E(C,A)) &= \deg A,\\ rk(E(C,A)) &= h^0(A),\\ h^1(E(C,A)) &= h^2(E(C,A)) = 0\\ h^0(E(C,A)) &= h^0(A) + h^0(\mathcal{N}_{C/X}\otimes A^\vee).\end{aligned}$$

Since $h^1(\mathcal{O}_X)=0$, it follows that if $\mathcal{N}_{C/X}\otimes A^\vee$ is globally generated away from a finite set Σ then $E(C,A)$ is globally generated away from Σ.

Proof of Theorem 1. If C is an integral curve of arithmetic genus $g\geq 2$ on the K3 surface X, then by adjunction we have $C^2 = 2g-2\geq 2$, hence X is algebraic and projective (see for instance [13], Théorème 3.4 and Corollaire 3.6). Assume that C has minimal Clifford index among all integral curves in its linear system. Let A be a rank one torsion-free sheaf computing the Clifford index of C. We claim that both A and $\omega_C\otimes A^\vee$ are globally generated. Indeed, assume for instance that $\omega_C\otimes A^\vee$ is not. Since $h^1(A)>0$, then the image of the evaluation map $H^0(C,\omega_C\otimes A^\vee)\otimes\mathcal{O}_C\to\omega_C\otimes A^\vee$ is a torsion free sheaf $B\subsetneq\omega_C\otimes A^\vee$ such that $\omega_C\otimes A^\vee/B$ has finite support. In particular, we have $h^0(B)=h^0(\omega_C\otimes A^\vee)=h^1(A)$, $\deg(B)=\deg(\omega_C\otimes A^\vee)-\deg(\omega_C\otimes A^\vee/B) < \deg(\omega_C\otimes A^\vee) = 2g-2-\deg(A)$, $h^1(B) > h^1(\omega_C\otimes A^\vee) = h^0(A)$ (notice that $(A^\vee)^\vee = A$ since C is Gorenstein). It follows that $h^0(B)\geq 2$, $h^1(B)\geq 2$, and $\deg(B)-2(h^0(B)-1) < 2g-2-\deg(A)-2(h^0(\omega_C\otimes A^\vee)-1) = \deg(A)-2(h^0(A)-1)$, contradicting the minimality of the Cliffford index of A. Hence we can freely use the auxiliary results collected above. Furthermore we can assume $\mathrm{Cliff}(C) < [(g-1)/2]$. Indeed, integral curves on a smooth surface have planar singularities, which are smoothable, hence the above inequality follows (see [1], Theorem 9 or [5], Proposition 1.5). Now we can proceed exactly as in the proof of the main theorem of [10] and deduce our statement. □

Proof of Corollary 1. As already pointed out in [4], a simple analysis of the proof of [9], Theorem (3.b.7), shows that under our assumptions on C and on X we have $K_{p,q}(X,C)\cong K_{p,q}(C,K_C)$ for every p and q. Hence Corollary 1 follows from [3], Theorem 1.3, and Theorem 1 above. □

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