SARAJEVO JOURNAL OF MATHEMATICS
Vol.6 (19) (2010), 169–188

ON WEIGHTED EXTENSIONS OF BAJRAKTAREVIĆ MEANS

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ABSTRACT. For real functions f, g, α, β defined in an interval I we introduce a mean $B^{[f,g]}_{\alpha,\beta}$. which extends the Bajraktarević mean $B^{[f,g]}$ in I. The problem of symmetry of $B^{[f,g]}_{\alpha,\beta}$, leading to a functional with two unknown functions, is solved. We show that, under some conditions, every Bajraktarević mean $B^{[f,g]}$ in $(0,\infty)$ can be embedded in a two-parameter family of means $\left\{B^{[f,g]}_{a,b} : a, b > 0\right\}$. As a special case a new family of means $\left\{B^{[p,q]}_{t} : t > 0\right\}$, which can be treated as the weighted Gini means, is constructed. As an application, the pairs of the these means which leave the geometric mean invariant are indicated, the effective limits of the sequence of iterates of the relevant mean-type mappings are given, as well as some functional equations are solved.

1. Introduction

A function $M : I \times I \to \mathbb{R}$ is called a *mean* in an interval $I \subset \mathbb{R}$ if

$$\min(x, y) \le M(x, y) \le \max(x, y), \qquad x, y \in I.$$

A mean M is called *strict* if these inequalities are strict for all $x, y \in I$, $x \ne y$; and *symmetric* if it is a symmetric function, i.e. if $M(x, y) = M(y, x)$ for all $x, y \in I$. A function $M : I \times I \to \mathbb{R}$ is called *increasing* if it is increasing with respect to each of the variables. It is obvious that an increasing function M is a mean iff it is reflexive, i.e. if $M(x, x) = x$ for all $x \in I$. A mean $M : (0,\infty)^2 \to (0,\infty)$ is called *homogeneous* if

$$M(tx, ty) = tM(x, y), \qquad t, x, y > 0.$$

If the continuous functions $f : I \to \mathbb{R}$ and $g : I \to (0,\infty)$ are such that $\frac{f}{g}$ is strictly monotonic, then they generate the mean $B^{[f,g]} : I^2 \to I$ defined

2000 *Mathematics Subject Classification.* Primary: 26E30, 39B22.
Key words and phrases. Beckenbach-Mitrinović-Gini mean, weighted mean, functional equation, mean-type mappings, iteration.

by

$$B^{[f,g]}(x,y) := \left(\frac{f}{g}\right)^{-1}\left(\frac{f(x)+f(y)}{g(x)+g(y)}\right), \qquad x,y \in I,$$

which is symmetric in I (in general, not increasing). It is called *Bajraktarević mean* (briefly, *B-mean)* (cf. Bajractarević [1], Bullen, Mitrinović Vasić [4], p. 263; Bullen [3], p. 310-316, where it is written in an equivalent fashion). For a constant g the mean $B^{[f,g]}$ becomes a *quasi-arithmetic* mean $A^{[f]} : I^2 \to I$ defined by

$$A^{[f]}(x,y) := f^{-1}\left(\frac{f(x)+f(y)}{2}\right).$$

Every B-mean $M^{[f,g]}$ can be imbedded in a family $\left\{M_w^{[f,g]} : w \in [0,1]\right\}$ of means defined by

$$M_w^{[f,g]}(x,y) := \left(\frac{f}{g}\right)^{-1}\left(\frac{(1-w)f(x)+wf(y)}{(1-w)g(x)+wg(y)}\right), \qquad x,y \in I.$$

Clearly, $M_{1/2}^{[f,g]} = B^{[f,g]}$, and $M_w^{[f,g]}$ is symmetric iff $w = \frac{1}{2}$. The means $M_w^{[f,g]}$ are treated as weighted extensions of $B^{[f,g]}$ and the number $w \in (0,1)$ is referred to as its *weight.*

Let $\alpha, \beta : I \to I$ be a bijective and continuous mappings of an interval I. In Section 4 of the present paper we show in particular that, if the function

$$\psi_{\alpha,\beta}^{[f,g]}(x) := \frac{f(\alpha(x))+f(\beta(x))}{g(\alpha(x))+g(\beta(x))}$$

is strictly monotonic, then (Theorem 3) the function $B_{\alpha,\beta}^{[f,g]} : I^2 \to I$ defined by

$$B_{\alpha,\beta}^{[f,g]}(x,y) := \left[\psi_{\alpha,\beta}^{[f,g]}\right]^{-1}\left(\frac{f(\alpha(x))+f(\beta(y))}{g(\alpha(x))+g(\beta(y))}\right), \qquad x,y \in I,$$

is a strict and continuous mean and $B_{\alpha,\beta}^{[f,g]} = B^{[f,g]}$ for $\alpha = \beta = \mathrm{id}|_I$. The problem of the symmetry $B_{\alpha,\beta}^{[f,g]}$ is decided. It leads to the functional equation

$$\frac{F[\gamma(x)]+F[\gamma^{-1}(y)]}{G[\gamma(x)]+G[\gamma^{-1}(y)]} = \frac{F(x)+F(y)}{G(x)+G(y)}, \qquad x,y \in I,$$

where $\gamma = \alpha \circ \beta^{-1}$ and the functions F, G are unknown.

Applying this result we show that if $I = (0,\infty)$, then any Bajraktarević mean, can be, in an natural way, imbedded in a two-parameter family of means $\left\{B_{a,b}^{[f,g]} : a, b > 0\right\}$ defined by

$$B_{a,b}^{[f,g]}(x,y) := \left[\psi_{a,b}^{[f,g]}\right]^{-1}\left(\frac{f(ax)+f(by)}{g(ax)+g(by)}\right), \qquad x, y > 0.$$

We prove that $B_{a,b}^{[f,g]}$ is symmetric iff $a = b$.

For the power functions $f(x) = x^p$, $g(x) = x^q$ $(x > 0)$, where $p, q \in \mathbb{R}$, $p \neq q$ are arbitrarily fixed, in Section 5 we introduce a one-parameter family of means $\left\{B_t^{[p,q]} : t > 0\right\}$, different than the respective $\left\{M_w^{[p,q]} : w \in [0,1]\right\}$, which may be regarded as the weighted Gini means (cf. [3], p. 232). In Section 6 we indicate the mean type mappings of the forms $\left(B_t^{[p,q]}, B_t^{[r,s]}\right)$ and $\left(M_w^{[p,q]}, M_{1-w}^{[r,s]}\right)$ which leave the geometric mean invariant. We apply these properties to find effectively the limits of the sequence of iterates, as well as, to solve some functional equations.

2. Auxiliary results

We begin this section with the following

Remark 1. If the functions $f, F : I \to \mathbb{R}$ and $g, G : I \to (0, \infty)$ satisfy the equation

$$\frac{F(x)+F(y)}{G(x)+G(y)} = \frac{f(x)+f(y)}{g(x)+g(y)}, \qquad x, y \in I, \tag{1}$$

then

$$\left(\frac{f(x)}{g(x)} - \frac{f(y)}{g(y)}\right)\left(\frac{G(x)}{g(x)} - \frac{G(y)}{g(y)}\right) = 0, \qquad x, y \in I. \tag{2}$$

Proof. Taking $y = x$ in (1) we get

$$F(x) = \frac{f(x)}{g(x)}G(x), \qquad x \in I. \tag{3}$$

By (1) we hence obtain

$$\frac{\frac{f(x)}{g(x)}G(x) + \frac{f(y)}{g(y)}G(y)}{G(x)+G(y)} = \frac{f(x)+f(y)}{g(x)+g(y)}, \qquad x, y \in I,$$

which is equivalent to (2). $\square$

Lemma 1. *Let $I \subset \mathbb{R}$ be an interval. Suppose that the functions $f, F : I \to \mathbb{R}$, $g, G : I \to (0, \infty)$ are continuous and $\frac{f}{g}$ is one-to-one. Then equation* (1) *is satisfied if, and only if, there is a real constant a such that*

$$F(x) = af(x), \qquad G(x) = ag(x), \qquad x \in I.$$

Proof. Let us fix a $y \in I$ and put $a := \frac{G(y)}{g(y)}$. Since $\frac{f}{g}$ is one-to-one, by formula (2) of Remark 1, we get $\frac{G(x)}{g(x)} = a$ for all $x \in I$, $x \neq y$, whence, by the continuity of the functions G and g,

$$G(x) = ag(x), \qquad x \in I.$$

Since the relation (3) holds true under the assumption of the lemma, we infer that

$$F(x) = af(x), \qquad x \in I.$$

□

Theorem 1. *Suppose that the functions $f, F, H : I \to \mathbb{R}$, $g, G, K : I \to (0, \infty)$ are continuous and $\frac{f}{g}$ is one-to-one in an interval $I \subset \mathbb{R}$. Then the equation*

$$\frac{F(x) + H(y)}{G(x) + K(y)} = \frac{f(x) + f(y)}{g(x) + g(y)}, \qquad x, y \in I, \tag{4}$$

is satisfied if, and only if, there is a real constant $a > 0$ such that

$$H(x) = af(x) - F(x), \qquad K(x) = ag(x) - G(x), \qquad x \in I, \tag{5}$$

and the function

$$I^2 \ni (x, y) \to \frac{F(x) - F(y) + af(y)}{G(x) - G(y) + ag(y)} \quad \textit{is symmetric.} \tag{6}$$

Proof. To prove the "only if" part of the theorem suppose that the functions f, F, H, g, G, K satisfy equation (4). Setting $y = x$ in (4) we get

$$H(x) = -F(x) + \frac{f(x)}{g(x)}[G(x) + K(x)], \qquad x \in I.$$

Hence, making use of (4), we have

$$\frac{F(x) - F(y) + \frac{f(y)}{g(y)}[G(y) + K(y)]}{G(x) + K(y)} = \frac{f(x) + f(y)}{g(x) + g(y)}, \qquad x, y \in I. \tag{7}$$

Since the right-hand side of this equation is symmetric with respect to x and y, we infer that, for all $x, y \in I$,

$$\frac{F(x) - F(y) + \frac{f(y)}{g(y)}[G(y) + K(y)]}{G(x) + K(y)} = \frac{F(y) - F(x) + \frac{f(x)}{g(x)}[G(x) + K(x)]}{G(y) + K(x)},$$

whence, for all $x, y \in I$,

$$F(x) - F(y)$$
$$= \frac{\frac{f(x)}{g(x)}[G(x) + K(x)][G(x) + K(y)] - \frac{f(y)}{g(y)}[G(y) + K(y)][G(y) + K(x)]}{G(y) + K(x) + G(x) + K(y)}.$$

Replacing the difference $F(x) - F(y)$ in the denominator of the left-hand side of (7) by the expression of the right-hand side of the above relation, after simple calculation, we obtain

$$\frac{\frac{f(x)}{g(x)}[G(x)+K(x)] + \frac{f(y)}{g(y)}[G(y)+K(y)]}{[G(x)+K(x)] + [G(y)+K(y)]} = \frac{f(x)+f(y)}{g(x)+g(y)}, \qquad x, y \in I.$$

Applying Lemma 1 with $F(x) := \frac{f(x)}{g(x)}[G(x)+K(x)]$ and G replaced by $G+K$ we infer that there is an $a \in \mathbb{R}$ such that

$$G(x) + K(x) = ag(x), \qquad x \in I.$$

The number a is positive as, by the assumption, the values of the functions G, K and g are positive. Hence, setting $y = x$ in (4), we obtain

$$F(x) + H(x) = af(x), \qquad x \in I.$$

Thus $H = -F + af$, $K = -G + ag$ and equation (4) takes the form

$$\frac{F(x) - F(y) + af(y)}{G(x) - G(y) + ag(y)} = \frac{f(x)+f(y)}{g(x)+g(y)}, \qquad x, y \in I.$$

Since the right-hand side of this equation is a symmetric function, so is the function (6). This completes the proof of the "only if" part of the result.

Now suppose that the conditions (5) and (6) hold true. From (5), for a positive real a, we have

$$\frac{F(x) + H(y)}{G(x) + K(y)} = \frac{F(x) - F(y) + af(y)}{G(x) - G(y) + ag(y)}, \qquad x, y \in I.$$

From (6) we have

$$\frac{F(x) - F(y) + af(y)}{G(x) - G(y) + ag(y)} = \frac{F(y) - F(x) + af(x)}{G(y) - G(x) + ag(x)}, \qquad x, y \in I,$$

which implies that, for all $x, y \in I$,

$$\begin{aligned}&[F(x) - F(y)][g(x) + g(y)] - [G(x) - G(y)][f(x) + f(y)] \\ &= af(x)g(y) - af(y)g(x).\end{aligned}$$

Since

$$af(x)g(y) - af(y)g(x) = ag(y)[f(x) + f(y)] - af(y)[g(x) + g(y)]$$

we hence get, for all $x, y \in I$,

$$[F(x) - F(y) + af(y)][g(x) + g(y)] - [G(x) - G(y) + ag(y)][f(x) + f(y)] = 0,$$

whence

$$\frac{F(x) - F(y) + af(y)}{G(x) - G(y) + ag(y)} = \frac{f(x)+f(y)}{g(x)+g(y)}, \qquad x, y \in I,$$

which shows that

$$\frac{F(x)+H(y)}{G(x)+K(y)} = \frac{f(x)+f(y)}{g(x)+g(y)}, \qquad x,y \in I.$$

This completes the proof. □

3. A functional equation

Theorem 2. *Let $I \subset \mathbb{R}$ be an interval and $\gamma : I \to I$ a homeomorphism of I. Suppose that the continuous functions $F, G : I \to \mathbb{R}$, $G(x) \neq 0$ for all $x \in I$, satisfy the functional equation*

$$\frac{F[\gamma(x)] + F[\gamma^{-1}(y)]}{G[\gamma(x)] + G[\gamma^{-1}(y)]} = \frac{F(x)+F(y)}{G(x)+G(y)}, \qquad x,y \in I, \tag{8}$$

where γ^{-1} is the inverse function of γ.

Then

(1) *if G is constant then eq.* (8) *is satisfied if, and only if, for some $c \in \mathbb{R}$,*

$$F[\gamma(x)] - F(x) = c, \qquad x \in I;$$

(2) *if G is not constant then eq.* (8) *is satisfied if, and only if, there are $p, q \in \mathbb{R}$, $q \neq 0$, such that*

either $F = pG$
or $F = pG + q$ and $G \circ \gamma = G$.

Proof. The proof of the first part is obvious.

To prove the second part suppose that $F, G : I \to \mathbb{R}$ satisfy equation (8) and G is not constant. Applying Theorem 1 with F, H, G, K, f, g replaced, respectively, by the functions $F \circ \gamma$, $F \circ \gamma^{-1}$, $G \circ \gamma$, $K \circ \gamma^{-1}$, F, G, we infer that there is $a > 0$ such that

$$F(\gamma(x)) + F(\gamma^{-1}(x)) = aF(x), \qquad G(\gamma(x)) + G(\gamma^{-1}(x)) = aG(x), \quad x \in I,$$

and, consequently,

$$\frac{F(\gamma(x)) - F(\gamma^{-1}(y)) + aF(y)}{G(\gamma(x)) - G(\gamma^{-1}(y)) + aG(y)} = \frac{F(x)+F(y)}{G(x)+G(y)} \qquad x,y \in I,$$

whence, for all $x, y \in I$,

$$\begin{aligned}
&F(\gamma(x))G(x) + F(\gamma(x))G(y) - G(x)F(\gamma(y)) - F(\gamma(y))G(y) \\
&= G(\gamma(x))F(x) + G(\gamma(x))F(y) - F(x)G(\gamma(y)) - F(y)G(\gamma(y)) \\
&+ aF(x)G(y) - aG(x)F(y).
\end{aligned}$$

Replacing y by $\bar{y}$ we hence get, for all $x, \bar{y} \in I$,

$$\begin{aligned}&F(\gamma(x))G(x) + F(\gamma(x))G(\bar{y}) - G(x)F(\gamma(\bar{y})) - F(\gamma(\bar{y}))G(y)\\&= G(\gamma(x))F(x) + G(\gamma(x))F(\bar{y}) - F(x)G(\gamma(\bar{y})) - F(\bar{y})G(\gamma(\bar{y}))\\&+ aF(x)G(\bar{y}) - aG(x)F(\bar{y}).\end{aligned}$$

Subtracting these two equations by sides we obtain, for all $x, y, \bar{y} \in I$,

$$\begin{aligned}&F(\gamma(x))[G(y) - G(\bar{y})] - G(\gamma(x))[F(y) - F(\bar{y})]\\&= F(x)\{[aG(y) - G(\gamma(y))] - [aG(\bar{y}) - G(\gamma(\bar{y}))]\}\\&- G(x)\{[aF(y) - F(\gamma(y))] - [aF(\bar{y}) - F)\gamma(\bar{y}))]\}\\&+ \{[F(\gamma(y))G(y) - G(\gamma(y))F(y)] - [F(\gamma(\bar{y}))G(\bar{y})] - G(\gamma(\bar{y}))F(\bar{y})]\}.\end{aligned}$$

Choosing arbitrarily some pairs $(y_k, \bar{y}_k) \in I^2$, $k = 1, 2, 3$ and putting

$$A_k := G(y_k) - G(\bar{y}_k), \quad B_k := F(y_k) - F(\bar{y}_k),$$
$$C_k := [aG(y_k) - G(\gamma(y_k))] - [aG(\bar{y}_k) - G(\gamma(\bar{y}_k))],$$
$$D_k := [aF(y_k) - F(\gamma(y_k)] - [aF(\bar{y}_k) - F(\gamma(\bar{y}_k))],$$
$$E_k := [F(\gamma(y_k))G(y_k) - G(\gamma(y_k))F(y_k)] - [F(\gamma(y_k))G(y_k) - G(\gamma(y_k)F(y_k)]$$

in the above formula we obtain the system of functional equations

$$A_kF(\gamma(x)) - B_kG(\gamma(x)) = C_kF(x) - D_kG(x) + E_k, \quad x \in I,\ k = 1, 2, 3. \tag{9}$$

Since G is not constant, there are $y_1, \bar{y}_1 \in I$ such that $A_1 \neq 0$. For $k = 1$, from (9), we get

$$F(x+1) = \frac{B_1}{A_1}G(\gamma(x)) + \frac{C_1}{A_1}F(x) - \frac{D_1}{A_1}G(x) + \frac{E_1}{A_1}, \quad x \in \mathbb{R}.$$

Hence, making use of (9) we obtain, for $k = 2, 3$,

$$\begin{aligned}&A_k\left[\frac{B_1}{A_1}G(\gamma(x)) + \frac{C_1}{A_1}F(x) - \frac{D_1}{A_1}G(x) + \frac{E_1}{A_1}\right] - B_kG(\gamma(x))\\&= C_kF(x) - D_kG(x) + E_k,\end{aligned}$$

whence, for $k = 2, 3$ and all $x \in I$,

$$\begin{aligned}&(A_kB_1 - A_1B_k)\,G(\gamma(x))\\&= (A_1C_k - A_kC_1)\,F(x) - (A_1D_k - A_kD_1)\,G(x) + (A_1E_k - A_kE_1)\,.\end{aligned} \tag{10}$$

If for all pairs $(y_1, \bar{y}_1)$, $(y_2, \bar{y}_2) \in I^2$ we have $A_2B_1 - A_1B_2 = 0$, that is, if

$$[G(y_2) - G(\bar{y}_2)][F(y_1) - G(\bar{y}_1)] = [G(y_1) - G(\bar{y}_1)][F(y_2) - G(\bar{y}_2)],$$

then, for all $x, y, u, v \in I$ such that $G(x) \neq G(y)$, $G(u) \neq G(v)$,

$$\frac{F(x) - F(y)}{G(x) - G(y)} = \frac{F(u) - F(v)}{G(u) - G(v)}.$$

Hence, by the the continuity of F and G, there are some real constant p, q such that $F(x) = pG(x) + q$ for all $x \in \mathbb{R}$.

Consider the opposite case when for some pairs $(y_1, \bar{y}_1)$ and $(y_2, \bar{y}_2)$ we have $A_2B_1 - A_1B_2 \neq 0$. Then from (10) with $k = 2$ we get, for all $x \in I$,

$$G(x+1) = \frac{A_1C_2 - A_2C_1}{A_2B_1 - A_1B_2}F(x) - \frac{A_1D_2 - A_2D_1}{A_2B_1 - A_1B_2}G(x) + \frac{A_1E_2 - A_2E_1}{A_2B_1 - A_1B_2}.$$

This relation and (10) with $k = 3$ imply that

$$(A_3B_1 - A_1B_3)\left[\frac{A_1C_2 - A_2C_1}{A_2B_1 - A_1B_2}F(x) - \frac{A_1D_2 - A_2D_1}{A_2B_1 - A_1B_2}G(x) + \frac{A_1E_2 - A_2E_1}{A_2B_1 - A_1B_2}\right]$$
$$= (A_1C_3 - A_3C_1)\, F(x) - (A_1D_3 - A_3D_1)\, G(x) + (A_1E_3 - A_3E_1)\,,$$

whence, for all $x \in I$,

$$lF(x) + mG(x) + n = 0, \qquad x \in \mathbb{R}, \tag{11}$$

where

$$l := \frac{A_1C_2 - A_2C_1}{A_2B_1 - A_1B_2} - \frac{A_1C_3 - A_3C_1}{A_2B_1 - A_1B_2},$$
$$m := \frac{A_1D_2 - A_2D_1}{A_2B_1 - A_1B_2} - \frac{A_1D_3 - A_3D_1}{A_2B_1 - A_1B_2},$$
$$n := \frac{A_1E_3 - A_3E_1}{A_2B_1 - A_1B_2} - \frac{A_1E_2 - A_2E_1}{A_2B_1 - A_1B_2}.$$

If for some pairs $(y_k, \bar{y}_k)$, $k = 1, 2, 3$, we have $l \neq 0$ then, from (11), $F(x) = pG(x) + q$ for all $x \in \mathbb{R}$.

Now suppose that $l = 0$. Since G is not constant, it follows from (11) that $m = 0$ and $n = 0$ for all the choices of $(y_k, \bar{y}_k)$, $k = 1, 2, 3$ such that $A_2B_1 - A_1B_2$. In this case, from the definitions of the numbers l, m and n we easily infer that

$$\det\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \det\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ D_1 & D_2 & D_3 \end{bmatrix} = \det\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ E_1 & E_2 & E_3 \end{bmatrix} = 0.$$

Since $A_2B_1 - A_1B_2 \neq 0$, the vectors (A_1, A_2, A_3) and (B_1, B_2, B_3) are linearly independent. It follows that there exist $\alpha, \beta, \gamma, \delta, \kappa, \rho \in \mathbb{R}$ such that

$$C_k = \alpha A_k + \beta B_k, \quad D_k = \gamma A_k + \delta B_k, \quad E_k = \kappa A_k + \rho B_k, \qquad k = 1, 2, 3.$$

Hence and from the definitions of C_k and D_k we get, that for $k = 1, 2, 3$,

$$(a - \alpha)A_k - \beta B_k = G(\gamma(y_k)) - G(\gamma(\bar{y}_k)), \tag{12}$$

$$(a - \delta)A_k - \gamma B_k = F(\gamma(y_k)) - F(\gamma(\bar{y}_k)), \tag{13}$$

whence, using the definition of E_k, by simple calculation, we also get

$$A_k[\kappa + \gamma G(y_k) - F(\gamma(\bar{y}_k))] = B_k[(a - \delta)G(y_k) - \rho], \qquad k = 1, 2, 3,$$

which can be written in the form

$$\frac{\kappa+\gamma G(y_k)-F(\gamma(\bar{y}_k))}{(a-\delta)G(y_k)-\rho}=\frac{B_k}{A_k}, \qquad k=1,2,3.$$

Putting here $\bar{y}_3 := x$ and $y_3 := y$, we hence get

$$\frac{\kappa+\gamma G(y)-F(\gamma(x))}{(a-\delta)G(y)-\rho}=\frac{F(x)-F(y)}{G(x)-G(y)}$$

for all $x, y \in I$, $x \neq y$. From the symmetry of the right-hand side we obtain

$$\frac{\kappa+\gamma G(y)-F(\gamma(x))}{(a-\delta)G(y)-\rho}=\frac{\kappa+\gamma G(x)-F(\gamma(y))}{(a-\delta)G(x)-\rho},$$

whence

$$\begin{aligned}&[\kappa(a-\delta+\gamma\rho]G(x)-[(a-\delta)G(x)-\rho]F(\gamma(x))\\&=[\kappa(a-\delta+\gamma\rho]G(y)-[(a-\delta)G(y)-\rho]F(\gamma(y))\end{aligned}$$

which implies the existence of a constant c_0 such that

$$[\kappa(a-\delta+\gamma\rho)]G(x)-[(a-\delta)G(x)-\rho]F(\gamma(x))=c_0,$$

and, consequently,

$$F(\gamma(x))=\frac{[\kappa(a-\delta+\gamma\rho]G(x)-c_0}{(a-\delta)G(x)-\rho}$$

for all the admissible $x \in I$. Assume that $a-\delta \neq 0$. Then we have

$$F(\gamma(x))=\frac{bG(x)+c}{G(x)+d} \tag{14}$$

for some constant b, c, d. Similarly, by the symmetry of the considered functional equation we conclude that, if $a-\alpha \neq 0$ then, for some constant B, C, D,

$$G(\gamma(x))=\frac{BF(x)+C}{F(x)+D}. \tag{15}$$

From (13) and (14) we obtain

$$\left[\frac{bd-c}{[G(x)+d][G(y)+d]}-(a-\delta)\right][G(x)-G(y)]=\gamma[F(x)-F(y)].$$

Similarly, from (12) and (15) we obtain

$$(a-\alpha)[G(x)-G(y)]=\left[\frac{BD-C}{[F(x)+D][F(y)+D]}+\beta\right][F(x)-F(y)].$$

Dividing the last two equations by sides gives the equation

$$\left[\frac{bd-c}{[G(x)+d][G(y)+d]}-(a-\delta)\right]\left[\frac{BD-C}{[F(x)+D][F(y)+D]}+\beta\right]=\gamma(a-\alpha). \tag{16}$$

Setting here $y = y_0 \in I$ we infer that, for some constant A_0, B_0, C_0, D_0,

$$F(x) = \frac{A_0 G(x) + B_0}{C_0 G(x) + D_0}.$$

Replacing $F(x)$ and $F(y)$ in (16) by the suitable values given by this formula, we easily conclude that G is a constant function. This contradicts our assumption. Since in the case when $a - \delta = 0$ or $a - \alpha = 0$ the respective argument substantially simplifies, we omit it.

Thus we have shown that in all the possible cases there are $p, q \in \mathbb{R}$ such that $F = pG + q$. Suppose that $q \neq 0$. From (8) we have

$$\frac{pG(\gamma(x)) + G(\gamma^{-1}(y)) + 2q}{G(\gamma(x)) + G(\gamma^{-1}(y))} = \frac{pG(x) + pG(y) + 2q}{G(x) + G(y)}, \qquad x, y \in I,$$

whence

$$G(\gamma(x)) - G(x) = G(y) - G(\gamma^{-1}(y)), \qquad x, y \in I.$$

Thus, for a constant $c \in \mathbb{R}$,

$$G(\gamma(x)) = G(x) + c, \qquad x \in I,$$

whence, by induction,

$$G(\gamma^k(x)) = G(x) + ck, \qquad x \in I,\ k \in \mathbb{Z},$$

where $\mathbb{Z}$ stands for the set of all integers. If $c \neq 0$, taking into account the continuity of G, we get $G(I) = \mathbb{R}$. This is a contradiction, because, by assumption, the function G does not vanish in $\mathbb{R}$. Thus $c = 0$ and, consequently,

$$G(\gamma(x)) = G(x), \qquad x \in I.$$

The proof is completed. □

Corollary 1. *Let γ be a homeomorphic map of an interval $I \subset \mathbb{R}$. Suppose that the continuous functions $F, G : I \to \mathbb{R}$, $G \neq 0$ in $I \subset \mathbb{R}$, satisfy the functional equation*

$$\frac{F(\gamma(x)) + F(\gamma^{-1}(y))}{G(\gamma(x)) + G(\gamma^{-1}(y))} = \frac{F(x) + F(y)}{G(x) + G(y)}, \qquad x, y \in I.$$

If G is not constant and $\frac{F}{G}$ is monotonic then $\frac{F}{G}$ is a constant function.

Proof. Suppose that in the second part of the above theorem the number q is different than 0. Then $F = pG + q$ and $G(\gamma(x)) = G(x)$ for all $x \in I$. Hence we get

$$\frac{F(\gamma(x))}{G(\gamma(x))} = p + \frac{q}{G(\gamma(x))} = p + \frac{q}{G(x)} = \frac{pG(x) + q}{G(x)} = \frac{F(x)}{G(x)}, \quad x \in I,$$

i.e. the function $\frac{F}{G}(\gamma(x)) = \frac{F}{G}(x)$ for all $x \in I$. This contradicts to the assumption that $\frac{F}{G}$ is monotonic. Thus $q = 0$ and the proof is completed. □

4. A two parameter family of means related to a B-mean

As we have observed, if g is a non-zero constant, then $B^{[f,g]}$ reduces to a quasi-arithmetic mean. In this section we extend these means.

Theorem 3. *Let $f : (0,\infty) \to \mathbb{R}$ and $g : (0,\infty) \to (0,\infty)$ be continuous and non-constant. Suppose that, for some bijective and continuous functions $\alpha, \beta : I \to I$, the function $\psi_{\alpha,\beta}^{[f,g]} : I \to \mathbb{R}$,*

$$\psi_{\alpha,\beta}^{[f,g]}(x) := \frac{f(\alpha(x)) + f(\beta(x))}{g(\alpha(x)) + g(\beta(x))} \tag{17}$$

is strictly monotonic. Then

(1) *the function $B_{\alpha,\beta}^{[f,g]} : I^2 \to I$ given by*

$$B_{\alpha,\beta}^{[f,g]}(x,y) := \left[\psi_{\alpha,\beta}^{[f,g]}\right]^{-1}\left(\frac{f(\alpha(x)) + f(\beta(y))}{g(\alpha(x)) + g(\beta(y))}\right), \qquad x, y \in I, \tag{18}$$

is a strict and continuous mean and, for $\alpha = \beta = \mathrm{id}|_I$,

$$B_{\alpha,\beta}^{[f,g]} = B^{[f,g]};$$

(2) *the following conditions are equivalent:*
   (i) *$B_{\alpha,\beta}^{[f,g]}$ is a symmetric mean,*
   (ii) *$B_{\alpha,\beta}^{[f,g]} = B_{\beta,\alpha}^{[f,g]}$,*
   (iii) *either g is constant and $f \circ \beta - f \circ \alpha = c$ for some $c \in \mathbb{R}$, or $\beta = \alpha$.*

Proof. Suppose that $\psi_{\alpha,\beta}^{[f,g]}$ is strictly increasing and take $x, y \in (0,\infty)$, $x < y$. Then

$$\psi_{\alpha,\beta}^{[f,g]}(x) = \frac{f(\alpha(x)) + f(\beta(x))}{g(\alpha(x)) + g(\beta(x))} < \frac{f(\alpha(y)) + f(\beta(y))}{g(\alpha(y)) + g(\beta(y))} = \psi_{a,b}(y)$$

whence, as $\left[\psi_{\alpha,\beta}^{[f,g]}\right]^{-1}$ is strictly increasing,

$$\min(x,y) = x < \left[\psi_{\alpha,\beta}^{[f,g]}\right]^{-1}\left(\frac{f(ax) - f(by)}{g(ax) - g(by)}\right) < y = \max(x,y).$$

In the case when $\psi_{\alpha,\beta}^{[f,g]}$ is strictly decreasing the respective argument is similar. This proves that $B_{\alpha,\beta}^{[f,g]}$ is a strict mean. The continuity of $B_{\alpha,\beta}^{[f,g]}$ is obvious.

2) The equivalence $(i) \iff (ii)$ is obvious. To prove that $(i) \implies (iii)$, suppose that $B_{\alpha,\beta}^{[f,g]}$ is symmetric, that is that

$$B_{\alpha,\beta}^{[f,g]}(x,y) = B_{\alpha,\beta}^{[f,g]}(y,x), \qquad x, y \in I.$$

Hence, by the definition of $\psi_{\alpha,\beta}^{[f,g]}$,

$$\frac{f(\alpha(x))+f(\beta(y))}{g(\alpha(x))+g(\beta(y))}=\frac{f(\alpha(y))+f(\beta(x))}{g(\alpha(y))+g(\beta(x))},\qquad x,y\in I.$$

With $\gamma:=\alpha\circ\beta^{-1}$ this is equivalent to

$$\frac{f[\gamma(x)]+f[\gamma^{-1}(y)]}{g[\gamma(x)]+g[\gamma^{-1}(y)]}=\frac{f(x)+f(y)}{g(x)+g(y)},\qquad x,y\in I.$$

Now we can apply Theorem 2. If g is constant then this equation is satisfied iff, for some $c\in\mathbb{R}$,

$$f[\gamma(x)]-f(x)=c,\qquad x\in I,$$

that is iff, for some $c\in\mathbb{R}$,

$$f[\beta(x)]-f[\alpha(x)]=c,\qquad x\in I.$$

If g is not constant then this equation satisfies iff there are $p,q\in\mathbb{R}$, $q\neq 0$ such that either $f=pg$ or $f=pg+q$ and $g\circ\gamma=g$. The case $f=pg$ cannot happen because then the function $\psi_{\alpha,\beta}^{[f,g]}$ would be constant. Assume that $f=pg+q$ and $g\circ\gamma=g$.

Then $g\circ\beta=g\circ\alpha$ and, for all $x\in I$, we have

$$\psi_{\alpha,\beta}^{[f,g]}(x)=\frac{(pg[\alpha(x)]+q)+(g[\alpha(x)]+q)}{g[\alpha(x)]+g[\alpha(x)]}=p+\frac{q}{g[\alpha(x)]}.$$

Since $\psi_{\alpha,\beta}^{[f,g]}$ and α are one-to-one it follows that so is g. Now the equality $g\circ\beta=g\circ\alpha$ implies that $\alpha=\beta$.

As the implication $(i)\Longleftarrow(iii)$ is easy to verify, the proof is complete. $\square$

Remark 2. In the case when the function f is unknown and $I=\mathbb{R}$, the functional equation $f[\gamma(x)]-f(x)=c$ due to Abel (cf. M. Kuczma [5]).

Applying Theorem 3 and Theorem 2 we prove

Theorem 4. *Let $f:I\to\mathbb{R}$, $g:I\to(0,\infty)$ be continuous and let $a,b>0$ be fixed. Suppose that the function $\psi_{a,b}^{[f,g]}:(0,\infty)\to\mathbb{R}$,*

$$\psi_{a,b}^{[f,g]}(x):=\frac{f(ax)+f(bx)}{g(ax)+g(bx)},\qquad x\in(0,\infty),$$

is strictly monotonic. Then

(1) *the function $B_{a,b}^{[f,g]}:(0,\infty)^2\to(0,\infty)$ given by*

$$B_{a,b}^{[f,g]}(x,y):=\left[\psi_{a,b}^{[f,g]}\right]^{-1}\left(\frac{f(ax)+f(bx)}{g(ax)+g(bx)}\right),\qquad x,y\in(0,\infty),$$

is a strict and continuous mean and, for $a = b = 1$,

$$B_{1,1}^{[f,g]} = B^{[f,g]};$$

(2) *the following conditions are equivalent:*
 (i) $B_{a,b}^{[f,g]}$ *is a symmetric mean,*
 (ii) $B_{a,b}^{[f,g]} = B_{b,a}^{[f,g]}$
 (iii) $b = a$.

Proof. Define $\alpha, \beta : (0, \infty) \to (0, \infty)$ by $\alpha(x) := ax$, $\beta(x) := bx$ for $x > 0$. Since $B_{a,b}^{[f,g]} = B_{\alpha,\beta}^{[f,g]}$, the first result follows from part 1 of Theorem 3.

Now assume that $B_{a,b}^{[f,g]}$ is symmetric, that is that

$$B_{a,b}^{[f,g]}(x, y) = B_{a,b}^{[f,g]}(y, x), \qquad x, y > 0.$$

By the definition of $B_{a,b}^{[f,g]}$ this equation holds if, and only if,

$$\frac{f(ax) + f(by)}{g(ax) + g(by)} = \frac{f(ay) + f(bx)}{g(ay) + g(bx)}, \qquad x, y > 0.$$

Replacing x by $\frac{x}{a}$, y by $\frac{y}{b}$ and setting $c := \frac{b}{a}$ we obtain

$$\frac{f(x) + f(y)}{g(x) + g(y)} = \frac{f(cx) + f(c^{-1}y)}{g(cx) + g(c^{-1}y)}, \qquad x, y > 0.$$

Setting

$$F(u) := f(c^u), \qquad G(u) := g(c^u), \qquad u \in \mathbb{R},$$

we can write this equation in the form

$$\frac{F(u) + F(v)}{G(u) + G(v)} = \frac{F(u + 1) + F(v - 1)}{G(u + 1) + G(v - 1)}, \qquad u, v \in \mathbb{R}.$$

Since F and G are continuous, G is non-constant, and $\frac{F}{G}(u) = \frac{f}{g}(c^u)$ where $c \neq 1$, the function $\frac{F}{G}$ is strictly monotonic. On the other hand, in view of Corollary 1, the function $\frac{F}{G}$ must be constant. This contradiction proves that $a = b$ and completes the proof of part 2. □

In the case when the generator g of $B_{a,b}^{[f,g]}$ is constant we have the following

Theorem 5. *Let* $f : (0, \infty) \to \mathbb{R}$ *be continuous and strictly monotonic. Then*

(1) *for* $a, b > 0$ *the function* $\psi_{a,b}^{[f]} : (0, \infty) \to \mathbb{R}$,

$$\psi_{a,b}^{[f]}(x) := f(ax) + f(bx)$$

is strictly monotonic; the function $B_{a,b}^{[f]} : (0,\infty)^2 \to (0,\infty)$ *given by*

$$B_{a,b}^{[f]}(x,y) := \left[\psi_{a,b}^{[f]}\right]^{-1} (f(ax) + f(by)), \qquad x, y > 0,$$

is a strict and continuous mean and, for $a = b = 1$,

$$B_{1,1}^{[f]} = A^{[f]};$$

(2) *the following conditions are equivalent:*
  (i) $B_{a,b}^{[f]}$ *is symmetric,*
  (ii) $B_{a,b}^{[f]} = B_{b,a}^{[f]}$,
  (iii) $a = b$;

(3) $B_{a,a}^{[f]}$ *is* φ*-conjugate of the mean* $A^{[f]}$ *with* $\varphi(x) = ax$, $(x > 0)$, *that is*

$$B_{a,a}^{[,f]}(x,y) = \frac{1}{a} A^{[f]}(ax, ay), \qquad x, y > 0;$$

(4) *for every* $(x,y) \in (0,\infty)^2$, *the function*

$$(0,\infty)^2 \ni (a,b) \to B_{a,b}^{[f]}(x,y)$$

is continuous.

Remark 3. Let $f : (0,\infty) \to \mathbb{R}$ and $g : I \to (0,\infty)$ be continuous and nonconstant. If the function $\psi_{a,b}^{[f,g]}$ is strictly monotonic for all $a, b > 0$ then, according to the first part of the theorem, the B-mean $B^{[f,g]} : (0,\infty)^2 \to (0,\infty)$ exists, can be imbedded in the two-parameter family of the means $\left\{B_{a,b}^{[f,g]} : a, b > 0\right\}$ such that $B_{1,1}^{[f,g]} = B^{[f,g]}$.

To show that the assumption of the strict monotonicity of the function $\psi_{a,b}^{[f,g]}$ for all $a, b > 0$ is essential, consider the following

Example 1. Let $f(x) := x(2 + \sin x)$ and $g(x) := 2 + \sin x$ for $x \in \mathbb{R}$. Since $\frac{f}{g}(x) = x$ for $x \in (0,\infty)$, the function $\frac{f}{g}$ is strictly increasing, which implies the existence of the mean $B^{[f,g]}$. It is to verify that for some $a, b > 0$ the function $\psi_{a,b}^{[f,g]}$ is not strictly monotonic.

Remark 4. It is easy to show that, under the assumptions of Theorem 3 (or Theorem 4), the mean $B_{a,b}^{[g,f]}$ is well defined and

$$B_{a,b}^{[g,f]} = B_{a,b}^{[f,g]}.$$

Moreover $B_{a,a}^{[f,g]}$ is φ-conjugate of $B^{[g,f]}$ with $\varphi(x) = ax$, $(x > 0)$, that is

$$B_{a,a}^{[g,f]}(x,y) = \frac{1}{a} B_{a,a}^{[f,g]}(ax, ay), \qquad x, y > 0;$$

Example 2. For $f(x) = \exp(x)$, $g(x) = x$ and $a > 0$ we have $\psi_{a,a}^{[f,g]}(x) = \exp(ax)$ and

$$D_{a,a}^{[f,g]}(x,y) = \frac{1}{a}\log\left(\frac{e^{ax}+e^{ay}}{x+y}\right).$$

If $a \neq b$, then $\psi_{a,b}^{[f,g]}(x) = \frac{e^{ax}+e^{bx}}{ax+bx}$, and we do not know the effective formula for $\left(\psi_{a,b}^{[f,g]}\right)^{-1}$ as well as for $B_{a,b}^{[f,g]}$.

In this connection consider the following

Remark 5. Suppose that $f : (0,\infty) \to \mathbb{R}$ and $g : (0,\infty) \to (0,\infty)$ satisfy the assumptions of Theorem 3. It is seen from the definition of $\psi_{a,b}^{[f,g]}$ that, in finding the effective formula for $B_{a,b}^{[f,g]}$, the relation $\psi_{a,b}^{[f,g]} = \phi(a,b)\psi$, for some functions $\phi : (0,\infty)^2 \to \mathbb{R}$ and $\psi : (0,\infty) \to \mathbb{R}$, can be helpful.

Motivated by this remark we prove

Proposition 1. *Let $f : (0,\infty) \to \mathbb{R}$ and $g : (0,\infty) \to (0,\infty)$ satisfy the assumptions of Theorem 3. Suppose that, for some functions $\phi : (0,\infty)^2 \to \mathbb{R}$ and $\psi : (0,\infty) \to \mathbb{R}$,*

$$\psi_{a,b}^{[f,g]}(x) = \phi(a,b)\psi(x), \qquad a,b,x > 0. \tag{19}$$

Then there are $p,q \in \mathbb{R}\setminus\{0\}$ such that

$$f(x) = f(1)x^p, \qquad g(x) = g(1)x^q, \qquad \psi(x) = \psi(1)x^{p-q}, \qquad x > 0.$$

Proof. Setting $x = 1$ in (19) and taking into account (17), we get

$$\phi(a,b)\psi(1) = \psi_{a,b}^{[f,g]}(1) = \frac{f(a)+f(b)}{g(a)+g(b)}, \qquad a \neq b.$$

Hence, by (19),

$$\frac{f(ax)+f(bx)}{g(ax)+g(bx)} = \frac{f(a)+f(b)}{g(a)+g(b)}\frac{\psi(x)}{\psi(1)}, \qquad a,b,x > 0,\ a \neq b. \tag{20}$$

Letting $b \to a$ we obtain

$$\frac{f(ax)}{g(ax)} = \frac{f(a)}{g(a)}\frac{\psi(x)}{\psi(1)}, \qquad a,x > 0. \tag{21}$$

Setting $a = 1$ gives

$$\frac{\psi(x)}{\psi(1)} = \frac{f(1)}{g(1)}\frac{f(x)}{g(x)}, \qquad x > 0,$$

whence, by (21),

$$\frac{g(1)}{f(1)}\frac{f(ax)}{g(ax)} = \left(\frac{g(1)}{f(1)}\frac{f(a)}{g(a)}\right)\left(\frac{g(1)}{f(1)}\frac{f(x)}{g(x)}\right), \qquad a, x > 0,$$

which proves that the continuous function $\frac{g(1)}{f(1)}\frac{f}{g}$ is multiplicative. Thus there is an $r \in \mathbb{R}$ such that

$$\frac{f(x)}{g(x)} = \frac{f(1)}{g(1)}x^r, \qquad x > 0.$$

Of course $r \neq 0$ (in the opposite case the function $\frac{f}{g}$ would not be strictly monotonic). It follows that

$$f(x) = cx^r g(x), \quad x > 0; \quad \text{where} \quad c := \frac{f(1)}{g(1)}, \tag{22}$$

and

$$\psi(x) = \psi(1)x^r, \qquad x > 0. \tag{23}$$

From (20) and (23) we obtain

$$[g(a) + g(b)][f(ax) + f(bx)] = x^r[f(a) + f(b)][g(ax) + g(bx)], \quad a, b, x > 0,$$

which reduces to the equation

$$(b^r - a^r)\,[g(a)g(bx) - g(b)g(ax)] = 0, \qquad a, b, x > 0.$$

Since $r \neq 0$ we hence get

$$\frac{g(ax)}{g(a)} = \frac{g(bx)}{g(b)}, \qquad a, b, x > 0.$$

As the left-hand side does not depend on b we infer that

$$\frac{g(ax)}{g(a)} = c(x), \qquad a, x > 0,$$

for some function $c : (0, \infty) \to (0, \infty)$. Setting here $a = 1$ we get $c(x) = \frac{g(x)}{g(1)}$ for all $x > 0$, and consequently,

$$\frac{g(ax)}{g(1)} = \frac{g(a)}{g(1)}\frac{g(x)}{g(1)}, \qquad a, x > 0,$$

which shows that the function $\frac{g}{g(1)}$ is multiplicative. By the continuity of g, there exists a $q \in \mathbb{R}$ such that

$$g(x) = g(1)x^q, \qquad x > 0.$$

Put $p := q + r$. Hence, making use of (22), we obtain

$$f(x) = f(1)x^{q+r} = f(1)x^p, \qquad x > 0.$$

$\square$

5. A weighted extension of Gini means

For power functions $f(x) = x^p$, $g(x) = x^q$ $(x > 0)$, where $p \neq q$, and for the constant $a, b > 0$, for convenience, we put $\psi_{a,b}^{[p.q]}$ instead of $\psi_{a,b}^{[f.g]}$ and $B_{a,b}^{[p.q]}$ instead of $B_{a,b}^{[f.g]}$. Note that

$$\psi_{a,b}^{[p.q]}(x) = \frac{a^p + b^p}{a^q + b^q} x^{p-q}, \qquad x > 0,$$

is strictly monotonic. From Theorems 3 and 4 we obtain

$$B_{a,b}^{[p,q]}(x,y) = \left(\frac{a^q + b^q}{a^p + b^p} \frac{a^p x^p + b^p y^p}{a^q x^q + b^q y^q} \right)^{1/(p-q)}, \qquad x, y > 0.$$

In particular, for $p = 1$ and $q = 0$, we get $\psi_{a,b}(x) = (a+b)x$ and

$$B_{a,b}^{[1,0]}(x,y) = \frac{a}{a+b} x + \frac{b}{a+b} y, \qquad a, b, x, y > 0.$$

Remark 6. Note that the mean $B_{a,b}^{[p,q]}$ coincides with $B_{a,b}^{[f.g]}$ where the functions f, g are described in Proposition 1, and we have

$$B_{a,b}^{[p,q]}(x,y) = B_{\frac{a}{b},1}^{[p,q]}(x,y), \qquad a, b, x, y > 0.$$

Thus $B_{a,b}^{[p,q]}$depends only on the parameter $t = \frac{a}{b}$.

Applying Theorems 3 and 4 (if $q = 0$) and making some obvious calculations in the cases when $p = q \neq 0$ and $p = q = 0$, we obtain the following

Theorem 6. *For every $p, q \in \mathbb{R}$ and $t > 0$ the function $B_t^{[p,q]} : (0,\infty)^2 \to (0,\infty)$ defined by the following formulas*

$$B_t^{[p,q]}(x,y) := \left(\frac{t^q + 1}{t^p + 1} \frac{t^p x^p + y^p}{t^q x^q + y^q} \right)^{\frac{1}{p-q}}, \qquad p \neq q; \quad t > 0,$$

$$B_t^{[p,q]}(x,y) := t^{-\frac{t^p}{t^p+1}} (tx)^{\frac{(tx)^p}{(tx)^p + y^p}} y^{\frac{y^p}{(tx)^p + y^p}}, \qquad p = q \neq 0; \quad t > 0,$$

$$B_t^{[p,q]}(x,y) := \sqrt{xy} \qquad p = q = 0; \quad t > 0,$$

is a strict, increasing and homogeneous mean.

Moreover,

(1) *for all $p, q \in \mathbb{R}$ and $t > 0$,*

$$B_t^{[p,q]} \textit{ is symmetric iff } t = 1; \qquad B_t^{[p,q]} = B_t^{[q,p]},$$

and

$$B_{1/t}^{[p,q]}(x,y) = B_t^{[p,q]}(y,x); \qquad x, y > 0;$$

(2) *for every $x, y > 0$, the function*

$$(0,\infty)^3 \ni (p,q,t) \to B_t^{[p,q]}(x,y)$$

is continuous;

(3) *if $p^2 + q^2 > 0$ then, for all $x, y > 0$,*

$$\lim_{t\to 0+} B_t^{[p,q]}(x,y) = \lim_{t\to\infty} B_t^{[p,q]}(y,x) = \begin{cases} y, & p \geq 0,\ q \geq 0 \\ x^{\frac{-q}{p-q}} y^{\frac{p}{p-q}}, & p > 0,\ q < 0 \\ x^{\frac{-p}{-p+q}} y^{\frac{q}{-p+q}}, & p < 0,\ q > 0 \\ x, & p \leq 0,\ q \leq 0 \end{cases}.$$

Remark 7. For $p, q \in \mathbb{R}$, $p \neq q$, and $w \in (0,1)$ the function $M_{[w]}^{p,q]} : (0,\infty)^2 \to (0,\infty)$,

$$M_{[w]}^{p,q]}(x,y) := \left(\frac{(1-w)x^p + wy^p}{(1-w)x^q + wy^y}\right)^{1/(p-q)}$$

is a mean and $M_{[1/2]}^{p,q]} = B^{[p,q]}$. Letting $q \to p$ in this formula, one could define $M_{[w]}^{p,p]}$ by the formula

$$M_{[w]}^{p,p]}(x,y) := x^{\frac{(1-w)x^p}{(1-w)x^p+wy^p}} y^{\frac{wx^p}{(1-w)x^p+wy^p}} \text{ for } p \neq 0, \text{ and } M_{[w]}^{[0,0]}(x,y) := \sqrt{xy}.$$

Note that the families $\{B_t^{[p,q]} : t > 0\}$ and $\{M_w^{[p,q]} : w \in (0,1)\}$ are different. To see this note that

$$\frac{t^q+1}{t^p+1}\frac{t^p x^p + y^p}{t^q x^q + y^q} = \frac{(1-w)x^p + wy^p}{(1-w)x^q + wy^y}, \qquad x, y > 0,$$

for some $t > 0$ and $w \in (0,1)$ iff $t = 1$ and $w = 1/2$. In particular

$$\{B_t^{[p,q]} : t > 0\} \cap \{M_w^{[p,q]} : w \in (0,1)\} = \left\{B^{[p,q]} : p \neq q\right\},$$

and a counterpart of Theorem 5.3 is no longer true for the means $M_{[w]}^{p,p]}$. The family $\{B_t^{[p,q]} : p, q \in \mathbb{R},\ t > 0\}$ can be treated as the weighted extension of Gini means.

Remark 8. Note that $B_t^{[0,0]}$, for all $t > 0$, coincides with the symmetric geometric mean $G(x,y) := \sqrt{xy}$ and does not depend on t.

6. An application to iteration theory

Proposition 2. *For every $p, q \in \mathbb{R}$, $t > 0$ and $w \in (0,1)$, the geometric mean $G : (0,\infty)^2 \to (0,\infty)$, $G(x,y) := \sqrt{xy}$, is invariant with respect to the mean-type mappings $\left(B_t^{[p,q]}, B_t^{[-p,-q]}\right) : (0,\infty)^2 \to (0,\infty)^2$ and $\left(M_{[w]}^{[p,q]}, M_{[1-w]}^{[-p,-q]}\right) : (0,\infty)^2 \to (0,\infty)^2$, i.e.*

$$G \circ \left(B_t^{[p,q]}, B_t^{[-p,-q]}\right) = G, \qquad G \circ \left(M_{[w]}^{[p,q]}, M_{[1-w]}^{[-p,-q]}\right) = G.$$

Moreover, for all $(x,y) \in (0,\infty)^2$,

$$\lim_{n\to\infty} \left(B_t^{[p,q]}, B_t^{[-p,-q]}\right)^n (x,y) = (\sqrt{xy}, \sqrt{xy}) = \lim_{n\to\infty} \left(B_t^{[p,q]}, B_t^{[-p,-q]}\right)^n (x,y),$$

where $\left(B_t^{[p,q]}, B_t^{[-p,-q]}\right)^n$ denotes the nth iterate of $\left(B_t^{[p,q]}, B_t^{[-p,-q]}\right)$.

Proof. The invariance of the geometric mean is easy to verify. Now the result follows from Theorem 1 in [6] which asserts the existence and uniqueness of the invariant mean and the pointwise convergence of sequence of iterations of every continuous mean-type mapping (M,N) if M or N is strict (cf. also J.M. Borwein, P.B. Borwein [2], p.244, Theorem 8.2, where the comparability of means M and N is also assumed). □

Proposition 3. *Let $p, q \in \mathbb{R}$ and $t > 0$ be fixed, and let $F : (0,\infty)^2 \to \mathbb{R}$ be continuous on the diagonal $\{(x,x) : x > 0\}$. The function F satisfies the functional equation*

$$F\left(B_t^{[p,q]}(x,y), B_t^{[-p,-q]}(x,y)\right) = F(x,y), \qquad x,y > 0, \tag{24}$$

if, and only if, there is a single variable function $f : (0,\infty) \to \mathbb{R}$ such that

$$F(x,y) = f(xy), \qquad x,y > 0. \tag{25}$$

Proof. Assume that $F : (0,\infty)^2 \to \mathbb{R}$ satisfies equation (24). Hence, by induction,

$$F(x,y) = F\left[\left(B_t^{[p,q]}(x,y), B_t^{[-p,-q]}(x,y)\right)^n\right], \qquad x,y > 0,$$

for all positive integers $n \in \mathbb{N}$. Applying Proposition 2 and taking into account the continuity of F on the diagonal, we get, for all $x, y > 0$,

$$F(x,y) = \lim_{n\to\infty} F\left[\left(B_t^{[p,q]}(x,y), B_t^{[-p,-q]}(x,y)\right)^n\right] = F\left(\sqrt{xy}, \sqrt{xy}\right) = f(xy),$$

where $f(t) := F\left(\sqrt{t}, \sqrt{t}\right)$, $t > 0$.

Since it is easy to verify that every function F of form (25) satisfies the equation (24), the proof is complete. □

Remark 9. A counterpart of Proposition 3 for the mean-type mappings $\left(M_{[w]}^{[p,q]}, M_{[1-w]}^{[-p,-q]}\right)$ is also true.

References

[1] M. Bajractarević, *Sur une équation fonctionelle aux valeurs moyennes*, Glasnik Mat.-Fiz. Astronom. Društvo Mat. Fiz. Hrvatske, Ser. II 13 (1958), 243-248.

[2] J. M. Borwein and P. B. Borwein, *Pi and the AGM, A study in Analytic Number Theory and Computational Complexity*, John Wiley and Sons, New York, 1987.

[3] P. S. Bullen, *Handbook of Means and Their Inequalities,* Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht-Boston-London, 2003.

[4] P. S. Bullen, D. S. Mitrinović and P. M. Vasić, *Means and their inequalities*, D. Reidel Publishing Company, Dordrecht - Boston - Lancaster - Tokyo, 1988.

[5] M. Kuczma, *Functional eequations in a Single Variable*, Monografie Matematyczne 46, PWN-Polish Scientific Publishers, Warszawa 1968.

[6] J. Matkowski, *Iterations of mean-type mappings and invariant means*, Ann. Math. Siles., 13 (1999), 211–226.

(Received: August 19, 2009)
(Revised: October 22, 2009)

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