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# A NEW REGULARIZATION METHOD FOR A CLASS OF ILL-POSED CAUCHY PROBLEMS

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Abstract. In this paper, the Cauchy problem for the elliptic equation is investigated. We use a quasireversibility method to solve it and present convergence estimates under different assumptions for the exact solution. Some numerical tests illustrate that the proposed method is feasible and effective.

## 1. Introduction

Let $H$ be a Hilbert space with the inner product $< , >$ and the norm $\|.\|$. Let $A : D(A) \subset H \to H$ be a positive-definite, self-adjoint operator with compact inverse on $H$. Assume that A admits an orthonormal eigenbasis $\{\phi_p\}_{p\geq 1}$in $H$, associated with the eigenvalues such that

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \lim_{p\to\infty} \lambda_p = \infty.$$

Let $\epsilon$ be a given positive number. We consider the problem of finding the function $u : [0, 1] \to H$ from the system

$$\begin{cases} u_{tt} = Au + k^2 u, \ 0 < t < 1 \\ u_t(0) = 0 \\ \|u(0) - \varphi\| \leq \epsilon \end{cases} \tag{1}$$

where $k$ is a nonnegative real number and $\varphi$ is a given vector in $H$. The case $k = 0$, the problem (1) becomes

$$\begin{cases} u_{tt} = Au, \\ u_t(0) = 0 \\ \|u(0) - \varphi\| \leq \epsilon. \end{cases} \tag{2}$$

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The problems (1) and (2) are well known to be severely ill-posed and regularization methods for them are required. There are many regularization methods for the problem (2); we refer the reader to [2, 4, 3, 12, 5, 6, 8, 11, 15] and the references therein.

The problem (2) is an abstract version of a Cauchy problem which generalizes Cauchy problems for second-order elliptic partial differential equations in which the geometry and the coefficients enable the use of separation of variables. An example of (2) is the Cauchy problem for the Laplace equation in a rectangle. The operator is taken to be $\frac{-\partial^2}{\partial x^2}$ with the domain $D(A) = H_0^1(0,\pi) \in H = L^2(0,\pi)$. Then we can write formula (2) in the form

$$\begin{cases} u_{tt} + u_{xx} = 0, (x,t) \in (0,\pi) \times (0,1) \\ u(0,t) = u(\pi,t) = 0, t \in (0,1) \\ u_t(x,0) = 0, (x,t) \in (0,\pi) \times (0,1) \\ u(x,0) = \varphi(x), 0 < x < \pi. \end{cases} \tag{3}$$

The problem (3) as paper [3] pointed out, appears in many applications, such as in implicit marching schemes for the heat equation, in Debye-Huckel theory, in the linearization of the Poisson-Boltzmann equation [9, 16] and so on. In recent years, the Cauchy problems associated with the modified Helmholtz equation have been studied by using different numerical methods, such as the Landweber method with boundary element method (BEM) [13], the conjugate gradient method [5] and so on.

Very recently, (2) was considered by Dinh Nho Hao and co-workers [7]. They applied the method of non-local boundary value problems to regularize the above problem as follows

$$\begin{cases} u_{tt} = Au, \\ u_t(0) = 0 \\ u(0) + \alpha u(aT) = \varphi \end{cases} \tag{4}$$

with $a \geq 1$ being given and $\alpha > 0$ as the regularization parameters. Notice that this method has been used by several authors, such as Abdulkeromov [1], Vabishchevich and co-workers [18, 19], and Melnikova and co-workers [14], etc. Further, Vabishchevich and Denisenko [18] suggested (2) for regularizing (1). However, these authors did not investigate the error estimates in detail and prove their methods yield order-optimal ones.

To the author's knowledge, although (2) has been thoroughly investigated, there are rarely results for treating the problem (1) until now.

In this paper, we use a quasi-reversibility method to solve the Cauchy problem (1), which is an extension of (2). The basic idea of the quasi-reversibility method was first proposed by Lattes and Lions in their book

[10] and then by Weber in his paper [20], who used similar methods to solve the inverse heat conduction problem. Recently, the quasi-reversibility method was also widely used to solve the backward heat conduction problem [17].

This paper is organized as follows. In Section 2, we present the formulation of the Cauchy problem for the modified Helmholtz equation and propose a quasi-reversibility regularization method . In Section 3, the convergence estimates are given under two different a priori assumptions for the exact solution. Finally, the numerical implementation is given in Section 4.

## 2. Mathematical problem and regularization.

By the method of separation of variables, the solution of problem (1) is as follows

$$u(t) = \sum_{p=1}^{\infty} \left[ \cosh(\sqrt{\lambda_p + k^2} t) < u(0), \phi_p > \right] \phi_p. \tag{5}$$

Since $t > 0$, we know from (5) that, when $\lambda_p$ becomes large, $e^{\sqrt{\lambda_p+k^2}t}$ increases rather quickly. Thus, the term $e^{\sqrt{\lambda_p+k^2}t}$ is the cause of the instability. So, we hope to recover the stability of problem (5) by replacing $e^{\sqrt{\lambda_p+k^2}t}$ with a better term. In fact, we consider the following regularized problem

$$\begin{cases} v^\epsilon_{tt} = Av^\epsilon + k^2 v^\epsilon + \beta A v^\epsilon_{tt}, \ 0 \le t \le 1 \\ v^\epsilon_t(0) = 0 \\ v^\epsilon(0) = \varphi \end{cases} \tag{6}$$

where $\beta$ is a regularization parameter depending on $\epsilon$. The solution of (6) is written as follows:

$$v^\epsilon(t) = \sum_{p=1}^{\infty} \left[ \cosh\left( \sqrt{\frac{\lambda_p + k^2}{1+\beta\lambda_p}} t \right) < \varphi, \phi_p > \right] \phi_p. \tag{7}$$

Let $u^\epsilon$ be the function defined by

$$u^\epsilon(t) = \sum_{p=1}^{\infty} \left[ \cosh\left( \sqrt{\frac{\lambda_p + k^2}{1+\beta\lambda_p}} t \right) < u(0), \phi_p > \right] \phi_p. \tag{8}$$

The following theorem proves that the solution of problem (6) given by (7) depends continuously on the given Cauchy data $\varphi$.

**Theorem 1.** *Let $\varphi \in H$. Then the solution $v^\epsilon \in C([0,1]; H)$ depends continuously on $\varphi$ for any positive $\epsilon$.*

*Proof.* For every $\varphi$ in $H$ we have the expansion $\varphi = \sum_{p=1}^{\infty} < \varphi, \phi_p > \phi_p$ and thus

$$\|v^\epsilon(t)\|^2 = \sum_{p=1}^{\infty} \left[ \cosh\left( \sqrt{\frac{\lambda_p + k^2}{1+\beta\lambda_p}} t \right) < \varphi, \phi_p > \right]^2 .$$

Using the inequality

$$\sqrt{\frac{\lambda_p + k^2}{1+\beta\lambda_p}} \le \frac{1}{\sqrt{\beta}} + k$$

we get

$$\|v^\epsilon(t)\|^2 \le \frac{e^{\frac{2}{\sqrt{\beta}}+2k} + 1}{2} \|\varphi\|^2.$$

The proof is completed. □

In the following section, we present the error estimates between the exact solution $u$ given by (5) and the regularized approximation solution $v^\epsilon$ given by (7).

## 3. Error estimates

In this section, we give the convergence estimates of the quasi-reversibility regularization method for the case of $0 < t \le 1$. In the following Theorems 2 and 3, we give the convergence estimates of the quasi-reversibility method for the cases of $0 < t < 1$ and $t = 1$ based on two different a priori assumptions for the exact solution.

**Theorem 2.** *Let $u(t)$ be given by (5) and $v^\epsilon$ be given by (7). If there is a positive constant $E$ such that $\|u(1)\| \le E$ then with $\beta = \left(\frac{2}{a\ln(\frac{1}{\epsilon})}\right)^2$, $0 < a < 2$, we have for every $t \in [0, 1)$*

$$\|u(t) - v^\epsilon(t)\|^2 \le \sqrt{\frac{e^{2k}}{2}\epsilon^{2-a} + \frac{\epsilon^2}{2}} + \frac{1}{3a^2(1-t)^3\ln^2(\frac{1}{\epsilon})} E. \tag{9}$$

*Proof.* First, note that condition $\|u(0) - \varphi\| \le \epsilon$ gives

$$\begin{aligned} I_\epsilon(t) = \|u^\epsilon(t) - v^\epsilon(t)\|^2 &= \sum_{p=1}^{\infty} \cosh^2\left( \sqrt{\frac{\lambda_p + k^2}{1+\beta\lambda_p}} t \right) | < \varphi - u(0), \phi_p > |^2 \\ &\le \sum_{p=1}^{\infty} \frac{e^{\frac{2}{\sqrt{\beta}}+2k} + 1}{2} | < \varphi - u(0), \phi_p > |^2 \\ &\le \frac{e^{\frac{2}{\sqrt{\beta}}+2k} + 1}{2} \epsilon^2 = \frac{e^{2k}}{2}\epsilon^{2-a} + \frac{\epsilon^2}{2}. \end{aligned} \tag{10}$$

Combining (5) and (8), we derive that

$$< u(t) – u^\epsilon(t), \phi_p >= \left[\cosh(\sqrt{\lambda_p + k^2}t) - \cosh\left(\sqrt{\frac{\lambda_p + k^2}{1+\beta\lambda_p}}t\right)\right] < u(0), \phi_p > .$$

From (8) we have

$$< u(0), \phi_p >= \frac{< u(1), \phi_p >}{\cosh(\sqrt{\lambda_p + k^2}t)}.$$

gives

$$< u(t) - u^\epsilon, \phi_p >= A(\epsilon, t, p, k) < u(1), \phi_p >$$

where

$$A(\epsilon, t, p, k) = \frac{\cosh(\sqrt{\lambda_p + k^2}t) - \cosh\left(\sqrt{\frac{\lambda_p+k^2}{1+\beta\lambda_p}}t\right)}{\cosh(\sqrt{\lambda_p + k^2}t)}.$$

Denote

$$C = \sqrt{\lambda_p + k^2}, D = \sqrt{\frac{\lambda_p + k^2}{1+\beta\lambda_p}}.$$

Using $\sqrt{1+\beta\lambda_p} \leq 1 + \frac{1}{2}\beta\lambda_p$, we get

$$C - D = \sqrt{\lambda_p + k^2} - \sqrt{\frac{\lambda_p + k^2}{1+\beta\lambda_p}} \leq \frac{1}{2}\beta\lambda_p\sqrt{\lambda_p + k^2}.$$

This implies that

$$|A(\epsilon, t, p, k)| = \frac{e^{Ct} - e^{Dt} - \frac{e^{Ct}-e^{Dt}}{e^{Ct+Dt}}}{e^C + e^{-C}} \leq \frac{e^{Ct} - e^{Dt}}{e^C} = e^{-C(1-t)}(1 - e^{-(C-D)t}).$$

Applying the inequality $1 - e^{-(C-D)t} \leq (C - D)t \leq C - D$, we obtain

$$\begin{aligned} |A(\epsilon, t, p, k)| &\leq e^{-C(1-t)}(C - D) \leq \frac{1}{2}\beta\lambda_p\sqrt{\lambda_p + k^2}e^{-C(1-t)} \\ &= \frac{1}{2}\beta(C^2 - k^2)Ce^{-C(1-t)} \leq \frac{1}{2}\beta C^3 e^{-C(1-t)}. \end{aligned} \tag{11}$$

For $x > 0$, it is easy to prove that $6e^x \geq x^3$. This implies that for $0 < t < 1$

$$C^3 e^{-C(1-t)} \leq \frac{1}{6(1-t)^3}.$$

Hence, we get

$$|A(\epsilon, t, p, k)| \leq \frac{\beta}{12(1-t)^3}.$$

So, we get

$$\begin{aligned} J_\epsilon(t) = \|u^\epsilon(t) - u(t)\|^2 &= \sum_{p=1}^{\infty} |A(\epsilon, t, p, k) < u(1), \phi_p > |^2 \\ &\le \left(\frac{\beta}{12(1-t)^3}\right)^2 \sum_{p=1}^{\infty} | < u(1), \phi_p > |^2 \\ &= \left(\frac{\beta}{12(1-t)^3}\right)^2 \|u(1)\|^2. \end{aligned} \tag{12}$$

Since $\beta = \left(\frac{2}{a\ln(\frac{1}{\epsilon})}\right)^2$ and combining (10) and (12), we get

$$\begin{aligned} \|v^\epsilon(t) - u(t)\| &\le \|v^\epsilon(t) - u^\epsilon(t)\| + \|u^\epsilon(t) - u(t)\| \\ &\le \sqrt{\frac{e^{2k}}{2}\epsilon^{2-a} + \frac{\epsilon^2}{2}} + \frac{1}{3a^2(1-t)^3 \ln^2(\frac{1}{\epsilon})}\|u(1)\| \\ &\le \sqrt{\frac{e^{2k}}{2}\epsilon^{2-a} + \frac{\epsilon^2}{2}} + \frac{1}{3a^2(1-t)^3 \ln^2(\frac{1}{\epsilon})}E. \end{aligned}$$

The proof is completed. □

**Remark.** From Theorem 2, we find that $v^\epsilon$ is an approximation of the exact solution $u$. The approximation error depends continuously on the measurement error for fixed $0 \le t < 1$. However, as $t \to 1$ the accuracy of regularized solution becomes progressively lower. This is a common thing in the theory of ill-posed problems, if we do not have additional conditions on the smoothness of the solution. To retain the continuous dependence of the solution at $t = 1$, we introduce a stronger a priori assumption for the exact solution. Then, we have the following convergence result.

**Theorem 3.** *Let $u(t)$ and $v^\epsilon(t)$ be the functions defined by (5) and (7) respectively. Suppose that there is a positive constant $E_2$ such that*

$$\sum_{p=1}^{\infty} (\lambda_p + k^2)^3 | < u(1), \phi_p > |^2 < E_2^2.$$

*Let $\beta$ be the parameter regularization as in Theorem 2. Then for $t \in [0, 1]$ one has*

$$\|u(t) - v^\epsilon(t)\|^2 \le \sqrt{\frac{e^{2k}}{2}\epsilon^{2-a} + \frac{\epsilon^2}{2}} + \frac{1}{8a^2 2\ln^2(\frac{1}{\epsilon})}E_2.$$

*Proof.* From (11), we obtain

$$\begin{aligned} J_\epsilon(t) = \|u^\epsilon(t) - u(t)\|^2 &= \sum_{p=1}^{\infty} |A(\epsilon,t,p,k) < u(1), \phi_p > |^2 \\ &\leq \frac{1}{4}\beta^2 \sum_{p=1}^{\infty} C^6 | < u(1), \phi_p > |^2 \\ &= \frac{\beta^2}{4} \sum_{p=1}^{\infty} (\lambda_p + k^2)^3 | < u(1), \phi_p > |^2. \end{aligned} \tag{13}$$

It follows from (10) and (13) that

$$\begin{aligned} &\|v^\epsilon(t) - u(t)\| \leq \|v^\epsilon(t) - u^\epsilon(t)\| + \|u^\epsilon(t) - u(t)\| \\ &\qquad \leq \sqrt{\frac{e^{2k}}{2}\epsilon^{2-a} + \frac{\epsilon^2}{2}} + \frac{1}{2a^2 \ln^2(\frac{1}{\epsilon})} \sqrt{\sum_{p=1}^{\infty} (\lambda_p + k^2)^3 | < u(1), \phi_p > |^2}. \end{aligned}$$

The proof is completed. ∎

## 4. A numerical example

In this section, some examples are devised for verifying the validity of the proposed method. The operator is taken $\frac{-\partial^2}{\partial x^2}$ with the domain $D(A) = H_0^1(0,\pi) \in H = L^2(0,\pi)$. We also use $\|.\|$ as the norm in $L^2(0,\pi)$.

**Example 1.** We consider the problem

$$\begin{cases} u_{xx} + u_{tt} = 3u, (x,t) \in (0,\pi) \times (0,1) \\ u(0,t) = u(\pi,t) = 0, t \in (0,1) \\ u_t(x,0) = 0, (x,t) \in (0,\pi) \times (0,1) \\ u(x,0) = \dfrac{\sin(x)}{4}, 0 < x < \pi. \end{cases} \tag{14}$$

The exact solution to this problem is

$$u(x,t) = \frac{e^{2t} + e^{-2t}}{8} \sin x.$$

Let $t = 1$, we get $u(x,1) = 0.940548922770908 \sin x$.
Let $g_m$ be the measured data

$$g_m(x) = \frac{1}{4}\sin(x) + \frac{1}{m}\sin(mx).$$

So that the data error, at the $t = 0$ is

$$F(m) = \|g_m - g\| = \sqrt{\int_0^\pi \frac{1}{m^2}\sin^2(mx)dx} = \sqrt{\frac{\pi}{2}}\frac{1}{m} \le \epsilon.$$

The solution of (14), corresponding to $g_m$, is

$$u^m(x,t) = \frac{e^t + e^{-t}}{2}\sin x + \frac{e^{\sqrt{m^2+3}t} + e^{-\sqrt{m^2+3}t}}{2m}\sin mx.$$

The error at $t = 1$ is

$$\begin{aligned} O(m) := \|u^m(.,1) - u(.,1)\| &= \sqrt{\int_0^\pi \frac{(e^{\sqrt{m^2+3}} + e^{-\sqrt{m^2+3}})^2}{4m^2}\sin^2(mx)\,dx} \\ &= \frac{(e^{2\sqrt{m^2+3}} + e^{-2\sqrt{m^2+3}} + 2)}{4m^2}\sqrt{\frac{\pi}{2}}. \end{aligned}$$

Then, we notice that

$$\lim_{m\to\infty} F(m) = \lim_{m\to\infty}\frac{1}{m}\sqrt{\frac{\pi}{2}} = 0, \tag{15}$$

$$\lim_{m\to\infty} O(m) = \lim_{m\to\infty}\frac{(e^{2\sqrt{m^2+3}} + e^{-2\sqrt{m^2+3}} + 2)}{4m^2}\sqrt{\frac{\pi}{2}} = \infty. \tag{16}$$

From the two equalities above, we see that (14) is an ill-posed problem. Hence, the Cauchy problem (14) cannot be solved by using classical numerical methods and it needs regularization techniques.

By applying the method given by equation (7), we have the approximate solution

$$w^\epsilon(x,t) = \sum_{p=1}^\infty \left[\cosh\left(\sqrt{\frac{p^2+3}{1+\beta p^2}}\right) < g_m(x), \sin px > \right]\sin px. \tag{17}$$

Then

$$\begin{aligned} w^\epsilon(x,1) &= \sum_{p=1}^\infty \left[\cosh\left(\sqrt{\frac{p^2+3}{1+\beta p^2}}\right) < g_m(x), \sin px > \right]\sin px \\ &= \frac{\cosh\left(\sqrt{\frac{4}{1+\beta}}\right)}{4}\sin x + \frac{\cosh\left(\sqrt{\frac{m^2+3}{1+\beta m^2}}\right)}{2}\sin(mx) \end{aligned}$$

$$\|w^\epsilon(.,1) - u(.,1)\| = \frac{\pi}{2}\left[\left(\frac{\cosh(\sqrt{\frac{4}{1+\beta}}) - \cosh(2)}{4}\right)^2 + \cosh^2\left(\sqrt{\frac{m^2+3}{1+\beta m^2}}\right)\right].$$

We have the following table for

1. $\epsilon = 10^{-2}\sqrt{\frac{\pi}{2}}$ corresponding to $m = 10^{20}$.

2. $\epsilon = 10^{-3}\sqrt{\frac{\pi}{2}}$ corresponding to $m = 10^{20}$.
3. $\epsilon = 10^{-4}\sqrt{\frac{\pi}{2}}$ corresponding to $m = 10^{50}$.

TABLE 1

| $\epsilon$ | $w^\epsilon$ | $\|w^\epsilon - u\|$ |
|---|---|---|
| $10^{-2}\sqrt{\frac{\pi}{2}}$ | $0.929362864692100\sin(x)$ $+3.786855438\times 10^{-17}\sin(10^{20}x)$ | 0.0140196447310024 |
| $10^{-3}\sqrt{\frac{\pi}{2}}$ | $0.939414328021399\sin(x)$ $+9.255956190\times 10^{-9}\sin(10^{20}x)$ | 0.00142200363973089 |
| $10^{-4}\sqrt{\frac{\pi}{2}}$ | $0.940435300951564\sin(x)$ $+3.104968144\times 10^{-12}\sin(10^{50}x)$ | 0.000142403832491343 |

From Table 1, we note that the results become less accurate when the error level $\epsilon$ increases which indicates that the method is useful. For $m$ large, we find that the numerical results become less accurate. To obtain better results, we should choose $m$ which is suitable. However, if $m$ is not large, the method is not effective.

**Example 2.** For the reader, we make a comparison between the method in this paper with the method in [7]. In fact, we consider the problem with $k = 0$ as follows

$$\begin{cases} u_{xx} + u_{tt} = 0, (x,t) \in (0,\pi)\times(0,1) \\ u(0,t) = u(\pi,t) = 0, t \in (0,1) \\ u_t(x,0) = 0, (x,t) \in (0,\pi)\times(0,1) \\ u(x,0) = \sin(x), 0 < x < \pi \end{cases} \tag{18}$$

The exact solution to this problem is

$$u(x,t) = \cosh(t)\sin x.$$

Let $g_m$ be the measured data

$$g_m(x) = \sin(x) + \frac{1}{e^{\frac{m}{2}}}\sin(mx).$$

So that the data error, at the $t = 0$ is

$$F(m) = \|g_m - g\| = \sqrt{\int_0^\pi \frac{1}{e^m}\sin^2(mx)dx} = \sqrt{\frac{\pi}{2}}\frac{1}{e^m} \le \epsilon.$$

The solution of (18), corresponding to $g_m$, is

$$u^m(x,t) = \cosh(t)\sin x + \frac{\cosh(mt)}{m}\sin mx.$$

The error at $t = 1$ is

$$O(m) := \|u^m(.,1) - u(.,1)\| = \sqrt{\int_0^\pi \frac{(e^m + e^{-m})^2}{4e^m} \sin^2(mx)\, dx} = \frac{(e^{2m} + e^{-2m} + 2)}{4e^m}\sqrt{\frac{\pi}{2}}.$$

Then, we notice that

$$\lim_{m\to\infty} F(m) = \lim_{n\to\infty} ||g_m - g|| = \lim_{m\to\infty} \frac{1}{e^m}\sqrt{\frac{\pi}{2}} = 0, \tag{19}$$

$$\lim_{m\to\infty} O(m) = \lim_{m\to\infty} \|u^m(.,1) - u(.,1)\| = \lim_{m\to\infty} \frac{(e^{2m} + e^{-2m} + 2)}{4e^m}\sqrt{\frac{\pi}{2}} = \infty. \tag{20}$$

From the two equalities above, we see that (18) is an ill-posed problem.

By applying the Quasi-reversibility method in this paper, we have the approximate solution

$$w_\epsilon(x,t) = \sum_{n=1}^\infty \left[\cosh\left(\sqrt{\frac{m^2}{1+\beta m^2}}t\right) < g_m(x), \sin nx >\right] \sin nx. \tag{21}$$

Letting $t = \frac{1}{3}$, we have

$$w_\epsilon(x, \frac{1}{3}) = \sum_{n=1}^\infty \left[\cosh\left(\frac{1}{3}\sqrt{\frac{m^2}{1+\beta m^2}}\right) < g_m(x), \sin nx >\right] \sin nx$$
$$= \cosh\left(\sqrt{\frac{1}{9+9\beta}}\right)\sin x + \cosh\left(\sqrt{\frac{m^2}{9+9\beta}}\right)\sin mx.$$

The error is given in Table 2.

TABLE 2. The error of the method in this paper.

| $\epsilon$ | $w_\epsilon$ | $b_\epsilon = \|w_\epsilon(.,\frac{1}{3}) - u(.,\frac{1}{3})\|$ |
|---|---|---|
| $\epsilon_1 = 10^{-3}\sqrt{\frac{\pi}{2}}$ | $1.05600103226174\sin(x)$ $+4.357617684 \times 10^{-214}\sin(10^3 \times x)$ | 0.0000887792190498364 |
| $\epsilon_2 = 10^{-4}\sqrt{\frac{\pi}{2}}$ | $1.05606477621057\sin(x)$ $+1.435733048 \times 10^{-2159}\sin(10^4 \times x)$ | 0.00000888802681288146 |
| $\epsilon_3 = 10^{-8}\sqrt{\frac{\pi}{2}}$ | $1.05607186775902\sin x$ $+7.902618100 \times 10^{-217143153}\sin(10^8 \times x)$ | $8.88850386184156 \times 10^{-11}$ |

By applying the non-local method in [7], we have the approximate solution

$$U_\epsilon(x,t) = \sum_{n=1}^\infty \left[\left(\frac{e^{nt} + e^{-nt}}{1 + \alpha e^{an} + \alpha e^{-an}}\right) < g_m(x), \sin nx >\right] \sin nx. \tag{22}$$

Choose $\alpha = \epsilon$, $a = 2$, $t = 1/3$, we have

$$U_\epsilon(x, 1/3) = \sum_{n=1}^{\infty} \left[ \left( \frac{e^{\frac{n}{3}} + e^{-\frac{n}{3}}}{2 + \epsilon e^{2n} + \epsilon e^{-2n}} \right) < g_m(x), \sin nx > \right] \sin nx \tag{23}$$

$$= \frac{e^{\frac{1}{3}} + e^{-\frac{1}{3}}}{2 + \epsilon e^{2} + \epsilon e^{-2}} \sin x + \frac{e^{\frac{m}{3}} + e^{-\frac{m}{3}}}{(2 + \epsilon e^{2m} + \epsilon e^{-2m}) e^{m/2}} \sin mx. \tag{24}$$

The error is given in Table 3.

TABLE 3. The error of the method in [7].

| $\epsilon$ | $U_\epsilon$ | $b_\epsilon = \Vert U_\epsilon(., \frac{1}{3}) - u(., \frac{1}{3}) \Vert$ |
|---|---|---|
| $\epsilon_1 = 10^{-3}\sqrt{\frac{\pi}{2}}$ | $1.05111563368012 \sin(x)$ <br> $+8.522413265 \times 10^{-939} \sin(10^3 \times x)$ | 0.000621171832 |
| $\epsilon_2 = 10^{-4}\sqrt{\frac{\pi}{2}}$ | $1.05607186782994 \sin(x)$ <br> $+1.812094591 \times 10^{-10347} \sin(10^4 \times x)$ | 0.000567121255053086 |
| $\epsilon_3 = 10^{-8}\sqrt{\frac{\pi}{2}}$ | $1.05607181803390 \sin x$ <br> $+1.428196126 \times 10^{-94097130} \sin(10^8 \times x)$ | $6.24100809143283 \times 10^{-8}$ |

Looking at Tables 2 and 3 a comparison between the two methods shows that the error results of Table 2 are smaller than the errors in Table 3. For the same parameter regularization, the error in Table 2 converges to zero more quickly many times than the Table 3. This shows that our approach has a nice regularizing effect and gives a better approximation in comparison to the method in paper [7]. However, in comparison to [7] the method proposed has some serious limitations from the numerical point of view, such as: it can deal with simple shapes, e.g. rectangles, circles, only and it cannot be extended easily to arbitrary irregular domains. In addition, writing down (17) implies you can evaluate the inner product $< g_m(x), \sin(mx) >$, which is not easy if $g_m$ is random noisy perturbation of $g$.

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