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UNIFORM CONVERGENCE OF FOURIER SERIES ON COMPACT SUBSETS

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Abstract. In this paper, the speed of approximation of $\omega_n(B;f,z)$ to zero has been calculated by using analytic and geometric properties of the boundary of the given region, where B is a subset of G (a finite simpy connected domain bounded by a Jordan curve) and $\omega_n = |f(z) - S_n(f,z)|\,,\ z \in B$.

1. Statement of the problem and main results

Let $G \subset \mathbb{C}$ be a finite simply connected domain bounded by a Jordan curve $L = \partial G$; $h(z)$ be a weight function on G, that is positive and measurable on G. Let $K_n(z) := K_n(h,z) = a_n z^n + \dots,\ a_n > 0,\ n = 0,1,2,\dots,$ be the sequence of orthonormal polynomials in G, with respect to the inner product

$$\langle f, g\rangle := \iint_G h(z) f(z)\overline{g(z)} dm(z)$$

where $dm(z)$ denotes two dimensional Lebesgue measure.

Define the space $A_2 := A_2(h,G)$ as the space of square integrable analytic functions with the norm $\|.\|_{A_2}$ given by

$$\|f\|_{A_2} := \left(\iint_G h(z)\,|f(z)|^2\,dm(z)\right)^{\frac{1}{2}}. \tag{1.1}$$

Fourier coefficients of the function $f \in A_2$ are defined by $a_n(f) := \langle f, K_n\rangle\ n = 0,1,2,3,\dots,$ and correspond to f following series

$$\sum_{n=0}^{\infty} a_n(f) K_n(z). \tag{1.2}$$

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The convergence of the series in (1.2) depends on the completeness of orthonormal polynomials system with respect to the norm in (1.1). It is well known that, if the weight function is bounded above and below by positive constants then the system of orthonormal polynomials is complete with respect to the norm $\|.\|_{A_2}$ (see [14]). In this paper, weight function will be taken as follows:

$$h(z) = |D(z)|^2 , \tag{1.3}$$

where $D \in A(\overline{G})$ (that is, D is an analytic function inside G and continuous on $\overline{G}$) and $D(z) \neq 0, \forall z \in \overline{G}$. It is clear from (1.3) that $h(z)$ satisfies completeness condition explained above. Let us denote the $n-th$ partial sum of (1.2) by

$$S_n(f,z) := \sum_{k=0}^{n} a_k(f)K_k(z), \quad n = 0,1,2,\dots,$$

and define $\omega_n(B,f;z)$ by

$$\omega_n(B,f;z) := |f(z) - S_n(f,z)| \tag{1.4}$$

where $z \in B \Subset G$.

Throughout this paper, $c, c_1, \dots$ are positive and $\varepsilon, \varepsilon_1, \dots$ are sufficiently small positive constants, in general depending on G.

We say that,

(i) $G \in C(k,\alpha),\ k = 1,2,3,\dots,\ 0 < \alpha \leq 1$, if $L = \partial G$ has a natural parametrization $z = z(s)$, where s is arc length, and the function $z = z(s)$ is k-times continuously differentiable with $z^{(k)}(s) \in Lip_\alpha$.

(ii) $G \in C_\theta$ if L has continuous tangent $\theta(s) := \theta(z(s))$ at every point $z(s)$.

Suetin proved that (see, [17]) if $L \in C(k+1,\alpha),\ k = 0,1,2\dots,\ 0 < \alpha < 1$ and $h(z)$ satisfies (1.3) with $D^{(k)} \in Lip_\alpha$ then

$$\delta(B)^{(k+3)}\omega_n(B,f;z) \leq cE_n(f,A_2)\ n^{-(k+\alpha)},\ \forall z \in B \Subset G \tag{1.5}$$

where $\delta(B) := dist(B,L)$ and

$$E_n(f,A_2) := \inf_{P_n} \left(\iint_G h(z)\,|f(z) - P_n(z)|^2\,dm(z) \right)^{\frac{1}{2}}$$

denotes the best approximation in A_2 by algebraic polynomials of degree no more than n.

It is clear from (1.5) that $\omega_n(B,f;z)$ tends to zero uniformly when $n \to \infty$ and the speed of approximation depends on not only the properties of h and G but also depends the distance of B to the boundary.

This type of calculations gained interest after Gaier's paper. In 1997 D. Gaier [9, Res.Prob. 97-1] asked the question: "How fast is the convergence of Bieberbach polynomials to Riemann function on $B \Subset G$?" This question is investigated in [3] and [10] for various regions of the complex plane.

In this paper we considered this problem for Fourier series in region with cusps on the boundary.

Now, Let us give some definitions before explaining our results:

Definition 1. [13, page 97] *The Jordan arc (or curve) L is called K–quasiconformal ($K \geq 1$) if there exists a K–quasiconformal mapping f of a domain $H \supset L$ such that $f(L)$ is a line segment (or circle).*

If $H = \mathbb{C}$, this definition is called the global definition of the K–quasiconformal arc (or curve). At the same time, H can be chosen as a neighborhood of the curve. In this case, it is called the local definition of K–quasiconformal arc (or curve) (see [1]). In [6] and [7] the global definition of the K– quasiconformal arc(or curve) has been considered.

The local definition will be used in this work because of the quasiconformality coefficient of the curve can be determined easily for some complex region. For example, with the help of [15], if $L \in C_\theta$ then $K = 1+\varepsilon$, $\forall\varepsilon > 0$, and if L is an analytic curve then $K = 1$.

Definition 2. *We say that $G \in PQ(K,p)$, $K \geq 1$, $p \geq 1$, if $L = \partial G$ consists of a finite number of K_j–quasiconformal L_j–arcs connecting the points $\{z_j\}_{j=2}^m \subset L$, $K := \max_{2\leq j\leq m}\{K_j\}$ and L is locally K–quasiconformal at $z_1 \in L$ and two quasiconformal arcs $L_j, L_{j+1} \subset L$ meeting at z_j form x^p-type interior zero angles that there is a neighborhood of z_j, $j = 2,\ldots,m$ such that the following conditions are satisfied for every $z = (x,y) \in L_j(L_{j+1})$ then*

$$c_1x^p \leq y \leq c_2x^p(-c_2x^p \leq y \leq -c_1x^p)$$

for some constants $-\infty < c_1 < c_2 < +\infty$.

It is clear from Definition 2 that G may have $m-1$ number x^p-type interior zero angles. If $p = 1$ then G is bounded by a K–quasiconformal curve and it is denoted by $G \in Q(K,1)$.

Especially, if $L_j \in C_\theta$, $j = 1,2,\ldots,m$ then $G \in Q(1+\varepsilon,p)$ for every $\varepsilon > 0$.

In Definition 2 the parameters K and p are analytic and geometric properties of the region respectively.

The following theorem shows how fast the speed of approximation depends on these parameters. Also, it gives an extension of the result of Suetin given in (1.5) to more general region.

Theorem 1. *Let $G \in PQ(K,p)$, $K \geq 1$ for some p, $1 \leq p < 2$, $h(z)$ defined by* (1.3) *with $D \in Lip\,\alpha$, $0 < \alpha \leq 1$ and $B \Subset G$. Then, for all*

$f \in A_2$ and $z \in B \Subset G$,

$$\delta(B)^{\frac{5}{2}}\omega_n(B,f;z) \leq cE_n(f,A_2)\begin{cases} n^{-\gamma}, & \text{if } \alpha > \frac{1}{2K^6}, 1 \leq p < 1+\frac{K^2-1}{K^2+1}, \\ & \text{or } \alpha > \frac{2-p}{2pK^4},\ p \geq 1+\frac{K^2-1}{K^2+1}, \\ n^{-\eta}, & \text{otherwise}, \end{cases} \tag{1.6}$$

where $\gamma < \frac{1}{2K^2}\min\left\{\frac{2-p}{p}, \frac{1}{K^2}\right\}$ and $0 < \eta < \alpha K^2$.

The following two results are relatively simple consequences of Theorem 1.

Corollary 1. *Let $G \in Q(K,1)$, $K \geq 1$, $h(z)$ defined by (1.3) with $D \in Lip\alpha$, $0 < \alpha \leq 1$, and $B \Subset G$. Then, for all $f \in A_2$ and $z \in B \Subset G$,*

$$\delta(B)^{\frac{5}{2}}\omega_n(B,f;z) \leq cE_n(f,A_2)\begin{cases} n^{-\gamma}, & \alpha > \frac{1}{2K^6}, \\ n^{-\eta}, & \text{otherwise}, \end{cases} \tag{1.7}$$

for every $\gamma < \frac{1}{2K^4}$ and $0 < \eta < \alpha K^2$.

Corollary 2. *Let $L_j \in C_\theta$ (or analytic curve) in Definition 2 then (1.6) is satisfied with $\gamma < \frac{2-p}{2p}$ and $0 < \eta < \alpha$.*

It is clear from (1.6) that

$$f(z) = \sum_{n=0}^{\infty} a_n K_n(z)$$

for every $z \in B \Subset G$ and this convergence depends on the distance of B to the boundary L as a power $\frac{5}{2}$.

In order to obtain the speed of approximation in Theorem 1 the quasiconformality coefficient of the region must be known. But, generally it is not easy to calculate of this parameter for a given region.

Is it possible to calculate the speed of approximation of $\omega_n(B,f;z)$ by using other properties of the region? Before the answer, let us give following definition:

Definition 3. [2] *We say that $G \in Q(v)$, $0 < v < 1$ if*

(i) *$L = \partial G$ is qusiconformal curve,*

(ii) *For every $z \in L$, there exists a unique $r > 0$ and $0 < v < 1$ such that a closed circular sector*

$$S(z;r,v) := \left\{\xi : \xi = z + re^{i\theta},\ 0 \leq \theta_0 < \theta < \theta_0 + v\right\}$$

of radius r and opening $v\pi$ lies in $\overline{\Omega}$ with vertex at z.

It is well known that each quasiconformal curve satisfies the condition (ii). Nevertheless, this condition imposed on L gives a new geometric characterization of the curve. For example, if the region G^* is defined by

$$G^* = \left\{ z : z = re^{i\theta},\ 0 < r < 1,\ \frac{\pi}{2} < \theta < 2\pi \right\}$$

then the quasiconformality coefficient of G^* is not easy to obtain, whereas $G^* \in Q(\frac{1}{2})$.

The following theorem shows we can still estimate the speed of approximation even if we only have the information about v.

Theorem 2. *Let $G \in Q(v),\ 0 < v < 1,\ h(z)$ defined by (1.3) with $D \in Lip\alpha,\ 0 < \alpha \le 1$ and $B \Subset G$. Then, for all $f \in A_2,\ z \in B \Subset G$,*

$$\delta(B)^{\frac{5}{2}} \omega_n(B, f; z) \le cE_n(f, A_2) \begin{cases} n^{-\gamma}, & \alpha > \frac{1}{4(2-v)}, \\ n^{-\eta}, & otherwise, \end{cases} \tag{1.8}$$

where $\gamma < \frac{v}{4(2-v)}$ and $\eta < v\alpha$.

Remark 1. If $G^* = \left\{ z : z = re^{i\theta},\ 0 < r < 1,\ \frac{\pi}{2} < \theta < 2\pi \right\}$ then, (1.8) is satisfied for every $\gamma < \frac{1}{12}$ and $\eta < \frac{\alpha}{2}$.

2. Some known results

The notation $a \prec b$ will be used if there exists a positive constant c such that $a < cb$ and $a \asymp b$ will be used if $a \prec b$ and $b \prec a$. For an arbitrary $z_0 \in B \Subset G$ let us denote by $w = \varphi(z, z_0)$ the conformal mapping of G to $D := \{w : |w| < 1\}$ with the normalization

$$\varphi(z_0, z_0) = 0, \varphi'(z_0, z_0) > 0$$

and $\psi := \varphi^{-1}$ is the inverse mapping.

Let $w := \Phi(z)$ be a conformal mapping of $\Omega := ext\overline{G}$ to $\Delta := \{w : |w| > 1\}$ normalized by $\Phi(\infty) = \infty,\ \Phi'(\infty) > 0$.

For $u > 0$ the level curve(interior or exterior) of G is defined as follows:

$$L_u := \{z : |\varphi(z, z_0)| = u, \text{ if } u < 1 \text{ or } |\Phi(z)| = u, \text{ if } u > 1\}$$

and $G_u := intL_u,\ \Omega_t := extL_u$.

The region H in the Definition 1 can be chosen as $G_{R_0} - G_{r_0}$ for a certain number $1 < R_0 \le 2$, depending on φ, Φ, f and $r_0 = R_0^{-1}$ (see, [3]).

Also, $c(K)$–quasiconformal reflection $\alpha^*(.)$ across L can be found (see, [5, page 76] and [13, page 98]) satisfying

$$|z_1 - \alpha^*(z)| \asymp |z_1 - z|,\ z_1 \in L,\ z \in H. \tag{2.1}$$

Lemma 1. [3] *Let L be a K–quasiconformal curve. Assume that*

$$z_1 \in L,\ z_2, z_3 \in G \cap \{z : |z_1 - z| \prec d(z_1, L_{R_0})\},\ w_j = \varphi(z_j)$$

$$(\ or\ \ z_2, z_3 \in \Omega \cap \{z : |z_1 - z| \prec d(z_1, L_{r_0})\},\ w_j = \Phi(z_j))$$

$j = 1, 2, 3$. *Then, the following statements are true*

i) *the relations $|z_1 - z_2| \prec |z_1 - z_3|$ and $|w_1 - w_2| \prec |w_1 - w_3|$ are equivalent,*

ii) *if $|z_1 - z_2| \prec |z_1 - z_3|$ then*

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{\frac{1}{K^2}} \prec \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \prec \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^2}.$$

Consequently if $z_3 \in L_{R_0} (z_3 \in L_{r_0})$ then

$$|w_1 - w_2|^{K^2} \prec |z_1 - z_2| \prec |w_1 - w_2|^{\frac{1}{K^2}}. \tag{2.2}$$

Lemma 2. [6] *Let L be a K–quasiconformal curve. Then, for every $z \in L$ and $z_0 \in G$, there exists an arc $\gamma(z_0, z)$ in G joining z_0 to z and having the following properties:*

i) *$d(\xi, L) \asymp |\xi - z|$ for every $\xi \in \gamma(z_0, z)$;*

ii) *$mes\, \widetilde{\gamma}(\xi_1, \xi_2) \prec |\xi_1 - \xi_2|$ for every pair $\xi_1, \xi_2 \in \gamma(z_0, z)$, where $\widetilde{\gamma}$ is the sub-arc of γ.*

Lemma 3. [6] *Let L be a K–quasiconformal curve. Then, for every rectifiable arc $\gamma \subset G$,*

$$mes\ \gamma \asymp mes\ \alpha^*(\gamma).$$

Lemma 4. [3] *Let L be a K-quasiconformal curve and $G_\varepsilon := \{z \in G : d(z, L) < \varepsilon\}$. Then, for $\forall \varepsilon > 0$,*

$$mes\ \varphi(G_\varepsilon) \prec \varepsilon^{\frac{1}{\delta}} \tag{2.3}$$

where $\delta := \min\{2, K^2\}$.

3. Integral representation of $\varphi(., z_0)$ when $G \in PQ(K, p)$ and auxiliary results

Suppose $G \in PQ(K, p)$ is given. Without loss of generality we may assume that $m = 2$ in Definition 2 $z_1 = 1,\ z_2 = -1$ and $(-1, 1) \subset G$. Let us denote $L = L^1 \cup L^2$ where

$$L^1 := \{z \in L : \operatorname{Im} z \geq 0\},\ L^2 := \{z \in L : \operatorname{Im} z \leq 0\}$$

and they are connecting the points $z_1 = 1,\ z_2 = -1$. The boundary of the domain is locally K–quasiconformal at z_1 and has interior zero angles at $z_2 = -1$ in the sense of Definition 2. Since each $L^j,\ j = 1, 2$ is K_j–quasiconformal there is $\alpha_j^*(.)$ quasiconformal reflection across L^j. Denote

$$\gamma^1 := \{z = x + iy : y = c_2(x+1)^p\}, \quad \gamma^2 := \{z = x + iy : y = -c_2(x+1)^p\},$$

where c_2 is taken from Definition 2.

According to Lemma 2 and [8, Lemma 4.2] for all $\xi_1, \xi_2 \in \gamma^j,\ j = 1, 2$, we have

$$mes\ \gamma^j(\xi_1, \xi_2) \prec |\xi_1 - \xi_2|. \tag{3.1}$$

For $n > N(R_0)$ big enough, and an arbitrary small $\varepsilon < 1$, let us choose $R = 1 + cn^{\varepsilon-1}$ such that $1 < R < R_0$. Let us denote intersection of γ^j and L_R by z^j. These points divide L_R into two parts;

$$L_R^1 := L_R^1(z^2, z^1), \quad L_R^2 := L_R^2(z^1, z^2)$$

and $L_R = \bigcup_{j=1}^2 L_R^j$.

Let us denote $\gamma^j(R) := \gamma^j \cap (intL_R)$ and set

$$\Gamma_R := \gamma^1(R) \cup \gamma^2(R) \cup L_R^1 \quad \text{and} \quad U := int\,(\Gamma_R \cup L).$$

The function $w = \varphi(z, z_0)$ can be extended to U as follows:

$$\widetilde{\varphi}(z, z_0) := \begin{cases} \varphi(z), & z \in \overline{G}, \\ \frac{1}{\overline{\varphi(\alpha_j^*(z), z_0)}}, & z \in U, \end{cases}$$

and integral representation of $\varphi(z, z_0)$ is obtained by using the Cauchy-Pompeiu formulas (see [13, page 148]):

$$\varphi(z, z_0) := \frac{1}{2\pi i} \int_{\Gamma_R} \frac{\widetilde{\varphi}(\xi, z_0)}{\xi - z} d\xi - \frac{1}{\pi} \iint_U \frac{\widetilde{\varphi}_{\overline{\xi}}(\xi, z_0)}{\xi - z} dm(\xi), \ z \in \overline{G}.$$

Then, using the above notations we obtain

$$\varphi(z, z_0) := \frac{1}{2\pi i} \int_{L_R} \frac{f(\xi, z_0)}{\xi - z} d\xi + \sum_{j=1}^{2} T_j(z, z_0) + A(z, z_0), \ z \in \overline{G}, \tag{3.2}$$

where

$$T_j(z, z_0) := \frac{1}{2\pi i} \int_{\gamma^j(R)} \frac{\widetilde{\varphi}(\xi, z_0) - \widetilde{\varphi}(-1, z_0)}{\xi - z} d\xi,$$

$$A(z, z_0) := -\frac{1}{\pi} \iint_U \frac{\widetilde{\varphi}_{\overline{\xi}}(\xi, z_0)}{\xi - z} dm(\xi)$$

and

$$f(\xi, z_0) := \begin{cases} \widetilde{\varphi}(\xi, z_0), & \xi \in L_R^1, \\ \varphi(-1, z_0), & \xi \in L_R^2. \end{cases}$$

Proposition 1. *Let $G \in PQ(K,p),\ K \geq 1,\ p \geq 1,\ z_0 \in B \Subset G$ and $\gamma \subset \Omega$ be a rectifiable Jordan arc with end point $z^* \in L$. Then,*

$$\left\|T'(.,z_0)\right\|_{A_2}^2 \prec \delta^{-1}(B)\,(mes\gamma)^{2-p} \tag{3.3}$$

where

$$T(z,z_0) := \int_\gamma \frac{\widetilde{\varphi}(\xi,z_0) - \widetilde{\varphi}(z^*,z_0)}{\xi - z} d\xi$$

Proof. From the assumption on γ we have (see, [3, Lemma 3.7])

$$\left|\widetilde{\varphi}(\xi,z_0) - \widetilde{\varphi}(z^*,z_0)\right|^2 \prec \delta^{-1}(B).\left|\xi - z^*\right|^{\frac{1}{2}}, \forall \xi \in \gamma.$$

So, (3.3) is obtained by using [4, Lemma 2.5] for $h(t) = t$ and

$$\vartheta(t) := \delta^{-\frac{1}{2}}(B)\sqrt{t}.$$

□

Proposition 2. *Let $U \subset \Omega$ and α^* is a quasiconformal reflection of L. Then,*

$$\left\|A'(.,z_0)\right\|_{A_2}^2 \prec (mes\ \varphi(\alpha^*(U),z_0)) \tag{3.4}$$

where

$$A(z,z_0) := \iint_U \frac{\widetilde{\varphi}_{\overline{\xi}}(\xi,z_0)}{(\xi - z)^2} dm(\xi).$$

Proof. The equation (3.4) is a consequence of the Hilbert transformation and Calderon-Zygmund inequality (see [5]) :

$$\begin{aligned}\left\|A'\right\|_{A_2}^2 &= \left\| \iint_U \frac{\widetilde{\varphi}_{\overline{\xi}}(\xi,z_0)}{(\xi - z)^2} dm(\xi) \right\|_{A_2}^2 \prec \left(\iint_U \left|\widetilde{\varphi}_{\overline{\xi}}(\xi,z_0)\right|^2 dm(\xi) \right) \\ &\asymp \left(\iint_U \left|\varphi'(\alpha^*(\xi),z_0)\right|^2 dm(\xi) \right) \prec \left(\iint_{\alpha^*(U)} \left|\varphi'(\xi,z_0)\right|^2 dm(\xi) \right) \\ &= mes\ \varphi\left(\alpha^*(U),z_0\right). \end{aligned} \tag{3.5}$$

□

The following is the Lemma which will play a central role in this work and the method used in [3] and [4] will be used in its proof.

Lemma 5. *Let $G \in PQ(K,p),\ K \geq 1,\ p \geq 1$ and $z_0 \in B \Subset G$. Then, there exists a polynomial $P_n(z,z_0)$ such that*

$$\left\|\varphi'(.,z_0) - P_n'(.,z_0)\right\|_{A_2} \prec \delta^{-\frac{1}{2}}(B) n^{-\gamma} \tag{3.6}$$

where $\gamma < \frac{1}{2K^2} \min\left\{\frac{2-p}{p}, \frac{1}{K^2}\right\}$.

Proof. First term in the integral representation of $\varphi(z,z_0)$ in (3.2) is an analytic function in $\overline{G}$, then, there is a polynomial $P_n(z,z_0)$, $\deg P_n \leq n$ (see [16, page 142]) such that

$$\left| \frac{1}{2\pi i} \int_{L_R} \frac{f(\xi,z_0)}{(\xi - z)^2} d\xi - P_n^{'}(z,z_0) \right| \prec \frac{1}{n} \tag{3.7}$$

for every $z \in \overline{G}$. So, we have

$$\left\| \varphi'(.,z_0) - P_n'(.,z_0) \right\|_{A_2} \prec \frac{1}{n} + \sum_{j=1}^{2} \left\| T_j' \right\|_{A_2} + \left\| A' \right\|_{A_2}. \tag{3.8}$$

Let us define $\alpha^*(z)$ and U as

$$\alpha^*(z) := \begin{cases} \alpha_1^*(z,z_0), & \operatorname{Im} z \geq 0, \\ \alpha_2^*(z,z_0), & \operatorname{Im} z \leq 0, \end{cases}$$

and $U := U_1 \cup U_2$ respectively such that

$$U_1 := \{z \in U : \operatorname{Im} z \geq 0\}, U_2 := \{z \in U : \operatorname{Im} z \leq 0\}.$$

Using Proposition 1 and Proposition 2 in (3.8) we have for $j = 1, 2$,

$$\left\| \varphi'(.,z_0) - P_n'(.,z_0) \right\|_{A_2}^2 \prec \frac{1}{n} + \delta^{-\frac{1}{2}}(B) \left(mes\ \gamma^j(R) \right)^{\frac{2-p}{2}} + \sum_{j=1}^{2} \left(mes\ \varphi\left(\alpha_j^*(U_j),z_0\right) \right)^{\frac{1}{2}}. \tag{3.9}$$

On the other hand, from Lemma 2, Lemma 3 and (3.1) we get

$$mes\ \gamma^j(R) \prec \left| z^j + 1 \right| \prec d^{\frac{1}{p}}(z^j, L^j) \prec n^{\frac{\varepsilon - 1}{pK^2}},\ j = 1, 2, \text{ and } \forall \varepsilon > 0. \tag{3.10}$$

For sufficiently small $\varepsilon_0 > 0$, let us denote

$$D_{\varepsilon_0}(-1) := \{\xi : |\xi + 1| \leq \varepsilon_0\},\ V_j := U_j \cap D_{\varepsilon_0}(-1),$$

and $\widetilde{V}_j := U_j - V_j$ such that $U_j = V_j \cup \widetilde{V}_j$. From [3, Lemma 3.8] and Lemma 4 we obtain

$$mes\ \varphi(\alpha_j^*(V_j),z_0) \prec \delta^{-1}(B) \left[d(z^j, L^j) \right]^{\frac{1}{K^2}} \prec \delta^{-1}(B) n^{\frac{\varepsilon - 1}{K^4}},$$

and

$$mes\ \varphi(\alpha_j^*(\widetilde{V}_j),z_0) \prec \delta^{-1}(B) n^{\frac{\varepsilon - 1}{K^4}},\quad j = 1, 2. \tag{3.11}$$

If we combine equation (3.10), (3.11) and (3.9) we obtained the desired result in (3.6). □

Corollary 3. *Assuming the conditions in Lemma 5 there is a polynomial* $Q_n\ (z,z_0)$ *such that* $Q_n(z_0,z_0)=0$, $Q_n^{'}(z_0,z_0)=\varphi'(z_0,z_0)$ *and*

$$\left\|\varphi'(.,z_0)-Q_n^{'}(.,z_0)\right\|_{A_2}\prec\delta^{-\frac{3}{2}}(B)n^{-\gamma} \tag{3.12}$$

where γ *in* (3.6).

Proof. Let us set,

$$Q_n(z,z_0):=P_n(z,z_0)-P_n(z_0,z_0)+(z-z_0)(\varphi'(z_0,z_0)-P_n^{'}(z_0,z_0)).$$

Then, $Q_n(z_0,z_0)=0,\ Q_n^{'}(z,z_0)=\varphi'(z_0,z_0)$. It is clear from Lemma 5 that

$$\left\|\varphi'(.,z_0)-Q_n^{'}(.,z_0)\right\|_{A_2}\prec(1+\delta^{-1}(B))\left\|\varphi'(.,z_0)-P_n^{'}(.,z_0)\right\|_{A_2} \tag{3.13}$$

and (3.6), (3.13) gives the proof of (3.12). □

Corollary 4. *Let* $G\in Q(v),\ 0<v<1,\ z_0\in B\Subset G$. *Then, there is a polynomial* $Q_n(z,z_0)$ *such that* $Q_n(z_0,z_0)=0,\ \ Q_n^{'}(z_0,z_0)=\varphi'(z_0,z_0)$ *and*

$$\left\|\varphi'(.,z_0)-Q_n^{'}(.,z_0)\right\|_{A_2(G)}\prec\delta^{-\frac{3}{2}}(B)n^{-\gamma} \tag{3.14}$$

where $\gamma<\frac{v}{4(2-v)}$.

Proof. We can follow the proof of Lemma 5 and Corollary 3 since L is a quasiconformal curve. So, there exists a polynomial $P_n(z,z_0),\ \deg P_n=n$ with $P_n(z_0,z_0)$ and $P_n'(z_0,z_0)=0$ such that

$$\left\|\varphi'(.,z_0)-P_n^{'}(.,z_0)\right\|_{A_2(G)}^2\prec\frac{1}{n}+mes\left(\varphi\left(\alpha^*(U)\right),z_0\right) \tag{3.15}$$

It is clear that, G satisfies the "v-wedge" condition since $G\in Q(v),\ 0<v<1$. Then, by [11] and [12], $\Psi\in Lip\,v$ and $\varphi\in Lip\,\frac{1}{2-v}$. Also, from [10, Corollary 1] we obtain

$$[mes\left(\varphi\left(\alpha^*(U)\right),z_0\right)]\prec\delta^{-1}(B)n^{-\frac{v}{2(2-v)}}.$$

From this fact and (3.15) we have

$$\left\|\varphi'(.,z_0)-P_n^{'}(.,z_0)\right\|_{A_2(G)}^2\prec\delta^{-1}(B)n^{-\frac{v}{2(2-v)}}$$

and choosing $Q_n(z,z_0)$ as in Corollary 3 the desired result is obtained. □

Lemma 6. *Let* $G\in PQ(K,p)$ *for some* $K\geq 1,\ p\geq 1$ *and* $h(z)$ *be defined by* (1.3) *with* $D\in Lip\,\alpha,\ 0<\alpha\leq 1$. *Then, for a polynomial* $T_n(z,z_0)$ *satisfying* $T_n^{'}(z,z_0)=\frac{\varphi'(z_0,z_0)}{D(z_0)}$ *we have*

$$\left\|\frac{\varphi'(.,z_0)}{D(.)}-T_n(.,z_0)\right\|_{A_2}\prec\delta^{-\frac{3}{2}}(B)\begin{cases}n^{-\gamma}, & \gamma\leq\eta,\\ n^{-\eta}, & \gamma>\eta,\end{cases}$$

where γ *is as in* (3.6) *and* $0 < \eta < \alpha K^2$.

Lemma 7. *Let* $G \in Q(v)$ *for some* $v,\ 0 < v < 1$ *and* $h(z)$ *defined by* (1.3) *with* $D \in Lip\,\alpha,\ 0 < \alpha \leq 1$. *Then, for a polynomial* $T_n(z, z_0)$ *satisfying* $T_n'(z, z_0) = \frac{\varphi'(z_0,z_0)}{D(z_0)}$ *we have*

$$\left\| \frac{\varphi'(., z_0)}{D(.)} - T_n(., z_0) \right\|_{A_2} \prec \delta^{-\frac{3}{2}}(B) \begin{cases} n^{-\gamma}, & \gamma \leq \eta, \\ n^{-\eta}, & \gamma > \eta, \end{cases}$$

where γ *is as in* (3.14) *and* $0 < \eta < \alpha v$.

Lemma 8. *Let* G *be a Jordan domain such that there exist polynomials*

$$T_n(z, z_0), T_n'(z_0, z_0) = \frac{\varphi'(z_0, z_0)}{D(z_0)}$$

satisfying the following properties:

$$\left\| \frac{\varphi'(., z_0)}{D(.)} - T_n(., z_0) \right\|_{A_2}^2 \prec \sigma_n(B) \tag{3.16}$$

for some sequence $\{\sigma_n(B)\}_{n=0}^{\infty}$ *with* $\sigma_n(B) \to 0, n \to \infty$ *for every* $z_0 \in B \Subset G$. *Then,*

$$\sum_{k=n}^{\infty} |K_k(z_0)|^2 = O(\delta^{-2}(B)\sigma_n(B)). \tag{3.17}$$

Proof. (Proof of Lemma 6, Lemma 7 and Lemma 8)

Using same process as in (see [4, Lemma 2.7 and Lemma 2.8]) we obtain the proof of Lemma 6, Lemma 7 and Lemma 8 respectively. □

4. Proof of theorems

It is well known (see [17]) that

$$E_n(f, A_2) := \inf_{P_n} \left(\iint_G h(z) \left| f(z) - P_n(z) \right|^2 dm(z) \right)^{\frac{1}{2}}$$
$$= \left(\iint_G h(z) \left| \sum_{k=n+1}^{\infty} a_k(f) K_k(z) \right|^2 dm(z) \right)^{\frac{1}{2}} = \left(\sum_{k=n+1}^{\infty} |a_k(f)|^2 \right)^{\frac{1}{2}}.$$

So, using the Minkowskii inequality we obtain

$$\omega_n(B,f;z) = |f(z) - S_n(f,z)| = \left|\sum_{k=n+1}^{\infty} a_k(f)K_k(z)\right|$$
$$\leq \left(\sum_{k=n+1}^{\infty} |a_k(f)|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{k=n+1}^{\infty} |K_k(z)|^2\right)^{\frac{1}{2}}$$
$$= E_n(f, A_2) \cdot \left(\sum_{k=n+1}^{\infty} |K_k(z)|^2\right)^{\frac{1}{2}} \tag{4.1}$$

Taking $\sigma_n(B)$ in Lemma 6 (in Lemma 7) and using Lemma 8 we obtain the second part of (4.1). So, Theorem 1 and Theorem 2 are proved.

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