SARAJEVO JOURNAL OF MATHEMATICS
Vol.6 (19) (2010), 217–228

P-ASYMPTOTICALLY EQUIVALENT IN PROBABILITY

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ABSTRACT. In this paper we present the following definitions P-asymptotically equivalent probability of multiple L and P-asymptotically probability regular. In addition to these definitions we asked and provide answers for the following questions.

(1) If $x \overset{Probability}{\approx} y$ then what type of four dimensional matrices transformation will satisfy the following $\mu(Ax) \overset{Probability}{\approx} \mu(Ay)$?
(2) If $[x]$ and $[y]$ are bounded double sequences that P-asymptotically converges at the same rate, then what are the necessary and sufficient conditions on the entries of any four dimensional matrix transformation A that will ensure that A sums $[x]$ and $[y]$ at the same P-asymptotic rate?
(3) What are the conditions on the entries of four dimensional matrices that ensure the preservation of P-asymptotically convergence in probability?

1. INTRODUCTION

In 1900 Pringsheim presented the following definition for convergence of double sequences.

Definition 1.1 (Pringsheim, [10]). *A double sequence $x = [x_{k,l}]$ has Pringsheim limit L (denoted by P-$\lim x = L$) provided that given $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > N$. We shall describe such an x more briefly as "P-convergent".*

Throughout this paper we shall only examine properties of four dimensional summability matrices via convergence in the Pringsheim sense. In addition the following is a list of definitions and notations that we shall use

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2000 *Mathematics Subject Classification.* Primary 42B15; Secondary 40C05.
Key words and phrases. P-convergent, P-asymptotical equivalent regular in probability.

This research was completed with the support of The Scientific and Technological Research Council of Turkey while the first author was a visiting scholar at Istanbul Commerce University, Istanbul, Turkey.

in this paper. Let us define $c_0^{''}$ as follows: a double sequence $[x]$ belongs to the set $c_0^{''}$ provided that $P-\lim_{k,l} x_{k,l}=0$. The four dimensional matrix transformation A is called an $c_0^{''}-c_0^{''}$ if Az is in the set $c_0^{''}$ whenever z is in $c_0^{''}$ and z is bounded.
$l^{''}=\{x_{k,l}:\sum_{k,l=1,1}^{\infty,\infty}|x_{k,l}|<\infty.\}$
$d_A=\{x_{k,l}:P-\lim_{m,m}\sum_{k,l=1,1}^{\infty,\infty}a_{m,n,k,l}x_{k,l}$ exists.$\}$
$E_\delta=\{$The set of all real double number sequences such that $x_{k,l}\geq\delta>0$ for all k and $l.\}$
$E_0=\{$The set of all nonnegative sequences which have at most a finite number of columns and/or rows with zero entries.$\}$

In 1936 Hamilton presented the following four dimensional matrix characterization of P-null sequences (i.e. double sequence that P-converges to zero).

Theorem 1.1 (Hamilton [2]). *A four dimensional matrix A is an $c_0^{''}-c_0^{''}$ if and only if*

(1) $\sum_{p,q=1,1}^{\infty,\infty}|a_{k,l,p,q}|<\infty$ *for all k,l;*
(2) *let $q=q_0$ then there exists $C_q(k,l)$ such that $a_{k,l,p,q}=0$ whenever $q>C_q(k,l)$ for all k, l, p;*
(3) *let $p=p_0$ then there exists $C_p(k,l)$ such that $a_{k,l,p,q}=0$ whenever $p>C_p(k,l)$ for all k, l, q;*
(4) $P-\lim_{k,l} a_{k,l,p,q}=0$ *for all p, and q.*

In 1980 Pobyvanets presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. Marouf in [4] extended Pobyvanets' notion of regularity to asymptotically conservative matrices and presented Silverman-Toeplitz characterization for asymptotically conservative matrices. Using this definition Patterson in [8] and [7] generalized Marouf definition of asymptotical equivalent sequences to double statistical asymptotical equivalent sequences, and asymptotically statistical regular. The extensions are as follow: First, two nonnegative double sequences $[x]$, and $[y]$ are said to be asymptotically equivalent, if

$$P-\lim_{k,l}\frac{x_{k,l}}{y_{k,l}}=1$$

(denoted by $x\overset{P}{\sim}y$). Second, two nonnegative sequences $[x]$ and $[y]$ are said to be asymptotically statistical equivalent of multiple L, provided that for every $\epsilon>0$,

$$\lim_n\frac{1}{n}\left\{\text{the number of } k\leq n:\left|\frac{x_k}{y_k}-L\right|\geq\epsilon\right\}=0$$

(denoted by $x \overset{S_L}{\sim} y$), and simply asymptotically statistical equivalent, if $L = 1$ which can be interpreted as the following

$$\left|\frac{x_k}{y_k} - L\right| < \epsilon \text{ for almost all } k \text{ (denoted } a.a.k),$$

and finally, a summability matrix A is asymptotically statistical regular provided that $Ax \overset{S_L}{\sim} Ay$ whenever $x \overset{S_L}{\sim} y$, $[x] \in E_0$, and $[y] \in E_\delta$ for some $\delta > 0$. These definitions were used to present a four dimensional matrix characterization of asymptotically equivalent double sequences. In addition, Patterson in [7] presented the following notions of double sequence: for each $[x] \in l''$ the remainder sequence $[Rx]$ is the double whose (m,n)-th term is $R_{m,n}x := \sum_{k,l \geq m,n} |x_{k,l}|$. Let $[x]$ be a convergent double sequence with limit L. Then the maximum remaining difference is given by $\rho_{m,n}x := \max_{k,l>m,n} |x_{k,l} - l|$. Also if $[x] \in l''$ let $[\mu(x)]$ denote the double sequence given by $\mu_{m,n}x := \sup_{k,l>m,n} |x_{k,l}|$. In this paper we shall generalize the above definitions in the following manner:

Definition 1.2. *Two nonnegative double sequences $[x]$ and $[y]$ are said to be P-asymptotically equivalent probability of multiple L provided that for every $\epsilon > 0$,*

$$P - \lim_{k,l} P\left(\left|\frac{x_{k,l}}{y_{k,l}} - L\right| < \epsilon\right) = 1$$

(denoted by $x \overset{Probability}{\approx} y$), and simply P-asymptotically equivalent probability if $L = 1$, which can be interpreted as $\left|\frac{x_{k,l}}{y_{k,l}} - L\right| < \epsilon$ converges in the Pringsheim sense in probability.

Definition 1.3. *A four dimensional summability matrix A is P-asymptotically probability regular provided that $Ax \overset{Probability}{\approx} Ay$ whenever $x \overset{Probability}{\approx} y$, $[x] \in E_0$, $[y] \in E_\delta$ for some $\delta > 0$, and $[x], [y] \in l''_\infty$.*

These definitions shall be used to present Silverman-Toeplitz type characterization of P-asymptotically equivalent probability of multiple L and present Silverman-Toeplitz type characterizations for the underlining sequence spaces.

2. Main Results

We begin our analysis by presenting and answering the following question. If $x \overset{Probability}{\approx} y$ then, what type of four dimensional matrices transformation will satisfy the following $\mu(Ax) \overset{Probability}{\approx} \mu(Ay)$?

Theorem 2.1. *Let A be a nonnegative $c_0''-c_0''$ summability matrix and let $[x]$ and $[y]$ be member of l'' such that $x \overset{Probability}{\approx} y$ with $[x] \in E_0$, and $[y] \in E_\delta$ for some $\delta > 0$ then $\mu(Ax) \overset{Probability}{\approx} \mu(Ay)$.*

Proof. Since $x \overset{Probability}{\approx} y$ there exists a P-null bounded double sequence $[z]$ such that $x \overset{Probability}{\approx} y(1+z)$. For each m and n the following holds:

$$\begin{aligned}
\frac{\mu_{m,n}(Ax)}{\mu_{m,n}(Ay)} = \frac{\sup_{p,q\geq m,n}(Ax)_{p,q}}{\sup_{p,q\geq m,n}(Ay)_{p,q}} &= \frac{\sup_{p,q\geq m,n}\sum_{k,l=1,1}^{\infty,\infty} a_{p,q,k,l}x_{k,l}}{\sup_{p,q\geq m,n}\sum_{k,l=1,1}^{\infty,\infty} a_{p,q,k,l}y_{k,l}} \\
&\overset{Probability}{\approx} \frac{\sup_{p,q\geq m,n}\sum_{k,l=1,1}^{\infty,\infty} a_{p,q,k,l}(y_{k,l}+z_{k,l}y_{k,l})}{\sup_{p,q\geq m,n}\sum_{k,l=1,1}^{\infty,\infty} a_{p,q,k,l}y_{k,l}} \\
&\leq \frac{\sup_{p,q\geq m,n}\left|\sum_{k,l=1,1}^{\infty,\infty} a_{p,q,k,l}(y_{k,l}+z_{k,l}y_{k,l})\right|}{\sup_{p,q\geq m,n}\sum_{k,l=1,1}^{\infty,\infty} a_{p,q,k,l}y_{k,l}} \\
&\leq 1 + \frac{\sup_{p,q\geq m,n}\sum_{k,l=1,1}^{\infty,\infty} a_{p,q,k,l}y_{k,l}\,|z_{k,l}|}{\sup_{p,q\geq m,n}\sum_{k,l=1,1}^{\infty,\infty} a_{p,q,k,l}y_{k,l}} \\
&\leq 1 + \frac{\sup_{p,q\geq m,n}\sum_{k,l=1,1}^{\infty,\infty} a_{p,q,k,l}y_{k,l}\,|z_{k,l}|}{\delta\sup_{p,q\geq m,n}\sum_{k,l=1,1}^{\infty,\infty} a_{p,q,k,l}}.
\end{aligned}$$

Since $[y]$ and $[z]$ are bounded real double sequences with $[z]$ is in c_0'' and A is a nonnegative $c_0''-c_0''$ matrix then the following holds:

$$\mathrm{P}-\lim_{m,n}\sup_{p,q\geq m,n}\sum_{k,l=1,1}^{\infty\infty} a_{p,q,k,l}y_{k,l}\,|z_{k,l}| = 0.$$

Hence

$$\frac{\mu_{m,n}(Ax)}{\mu_{m,n}(Ay)} \leq 1;\quad \text{in probability.}$$

In a similar manner we can establish the following

$$\frac{\mu_{m,n}(Ax)}{\mu_{m,n}(Ay)} \geq 1;\quad \text{in probability.}$$

Thus

$$\frac{\mu(Ax)}{\mu(Ay)} \overset{Probability}{\approx} 1$$

which implies

$$\mu(Ax) \overset{Probability}{\approx} \mu(Ay).$$

□

This theorem clearly presented the type of four dimensional matrices that will preserve P-asymptotically equivalent probability of multiple L. We now examine the question of rate-preserving. To that end we asked the following question. If $[x]$ and $[y]$ are bounded double sequences that P-asymptotically converges at the same rate, then what are the necessary and sufficient conditions on the entries of any four dimensional matrix transformation A that will ensure that A sums $[x]$ and $[y]$ at the same P-asymptotic rate? The following theorem provides the answer to our question.

Theorem 2.2. *If A is a nonnegative four dimensional summability matrix that maps bounded double sequences into l'' then the following are equivalent.*

(1) *If $[x]$ and $[y]$ are sequences such that $x \overset{Probability}{\approx} y, [x] \in E_0$, and $[y] \in E_\delta$ for some $\delta > 0$ then*

$$R(Ax) \overset{Probability}{\approx} R(Ay).$$

(2)

$$P - \lim_{m,n} P\left(\left|\frac{\sum_{p,q=m,n}^{\infty,\infty} a_{p,q,\alpha,\beta}}{\sum_{p,q=m,n}^{\infty,\infty}\sum_{k,l=1,1}^{\infty,\infty} a_{p,q,k,l}}\right| < \epsilon\right) = 1 \textit{ for each } \alpha \textit{ and } \beta.$$

Proof. We begin by showing that (2) implies (1). Without loss of generality let $L = 1$ and observe that $x \overset{Probability}{\approx} y$ implies $\mathrm{P}-\lim_{k,l}\left(\left|\frac{x_{k,l}}{y_{k,l}} - 1\right| \le \epsilon\right) = 1$. Thus $P\left((1-\epsilon)y_{k,l} \le x_{k,l} \le (1+\epsilon)y_{k,l}\right) = 1$. Let us consider the following

$$\begin{aligned}
R_{m,n}(Ax) &= \sum_{k,l=m,n}^{\infty,\infty} (Ax)_{k,l} \\
&\le \sum_{p,q=1,1}^{\bar{K}-1,\bar{L}-1} x_{p,q} \sum_{k,l=m,n}^{\infty,\infty} \max_{\{1\le p\le \bar{L}-1; 1\le q\le \bar{K}-1\}} \{a_{k,l,p,q}\} \\
&+ (1+\epsilon)\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=\bar{K},\bar{L}}^{\infty,\infty} a_{k,l,p,q} y_{p,q} + \sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=\bar{K},1}^{\infty,\bar{L}-1} a_{k,l,p,q} x_{p,q} \\
&+ \sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=1,\bar{L}}^{\bar{K}-1,\infty} a_{k,l,p,q} x_{p,q} \quad \text{in probability.}
\end{aligned}$$

Thus

$$\begin{aligned}R_{m,n}(Ax) \leq &\sum_{p,q=1,1}^{\bar{K}-1,\bar{L}-1} x_{p,q} \sum_{k,l=m,n}^{\infty,\infty} \max_{\{1\leq p\leq \bar{L}-1;1\leq q\leq \bar{K}-1\}}\{a_{k,l,p,q}\}\\ &+\sum_{p,q=\bar{K},1}^{\infty,\bar{L}-1} x_{p,q} \sum_{k,l=m,n}^{\infty,\infty} \sup_{\{\bar{K}\leq p<\infty;1\leq q\leq \bar{L}-1\}}\{a_{k,l,p,q}\}\\ &+\sum_{p,q=1,\bar{L}}^{\bar{K}-1,\infty} x_{p,q} \sum_{k,l=m,n}^{\infty,\infty} \sup_{\{\bar{L}\leq q<\infty;1\leq p\leq \bar{K}-1\}}\{a_{k,l,p,q}\}\\ &+(1+\epsilon)\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=\bar{K},\bar{L}}^{\infty,\infty} a_{k,l,p,q}y_{p,q} \text{ in probability.}\end{aligned}$$

When the above inequalities is observed the following is implied.

$$\begin{aligned}\frac{R_{m,n}(Ax)}{R_{m,n}(Ay)} \leq &\frac{\sum_{p,q=1,1}^{\bar{K}-1,\bar{L}-1} x_{p,q}\sum_{k,l=m,n}^{\infty,\infty}\max_{\{1\leq p\leq \bar{L}-1;1\leq q\leq \bar{K}-1\}}\{a_{k,l,p,q}\}}{\delta\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=1,1}^{\infty,\infty} a_{k,l,p,q}}\\ &+\frac{\sum_{p,q=\bar{K},1}^{\infty,\bar{L}-1} x_{p,q}\sum_{k,l=m,n}^{\infty,\infty}\sup_{\{\bar{K}\leq p<\infty;1\leq q\leq \bar{L}-1\}}\{a_{k,l,p,q}\}}{\delta\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=1,1}^{\infty,\infty} a_{k,l,p,q}}\\ &+\frac{\sum_{p,q=1,\bar{L}}^{\bar{K}-1,\infty} x_{p,q}\sum_{k,l=m,n}^{\infty,\infty}\sup_{\{\bar{L}\leq q<\infty;1\leq p\leq \bar{K}-1\}}\{a_{k,l,p,q}\}}{\delta\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=1,1}^{\infty,\infty} a_{k,l,p,q}}\\ &+1+\epsilon \text{ in probability.}\end{aligned}$$

Since A is nonnegative and (2) holds. Thus

$$P-\limsup_{m,n}\frac{R_{m,n}(Ax)}{R_{m,n}(Ay)} \leq 1+\epsilon \text{ in probability.}$$

In addition

$$\begin{aligned}R_{m,n}(Ax) = &\sum_{k,l=m,n}^{\infty,\infty}(Ax)_{k,l}\\ \geq &\sum_{k,l=m,n}^{\infty,\infty}\min_{\{1\leq p\leq \bar{L}-1;1\leq q\leq \bar{K}-1\}}\{a_{k,l,p,q}\}\sum_{p,q=1,1}^{\bar{K}-1,\bar{L}-1} x_{p,q}\\ &+\sum_{k,l=m,n}^{\infty,\infty}\inf_{\{\bar{K}\leq p<\infty;1\leq q\leq \bar{L}-1\}}\{a_{k,l,p,q}\}\sum_{p,q=\bar{K},1}^{\infty,\bar{L}-1} x_{p,q}\end{aligned}$$

$$+\sum_{k,l=m,n}^{\infty,\infty}\inf_{\{\bar{L}\le q<\infty;1\le p\le\bar{K}-1\}}\{a_{k,l,p,q}\}\sum_{p,q=1,\bar{L}}^{\bar{K}-1,\infty}x_{p,q}$$
$$+(1-\epsilon)\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=\bar{K},\bar{L}}^{\infty,\infty}a_{k,l,p,q}y_{p,q}\quad\text{in probability}$$

and

$$R_{m,n}(Ax)\ge\sum_{k,l=m,n}^{\infty,\infty}\min_{\{1\le p\le\bar{L}-1;1\le q\le\bar{K}-1\}}\{a_{k,l,p,q}\}\sum_{p,q=1,1}^{\bar{K}-1,\bar{L}-1}x_{p,q}$$
$$+\sum_{k,l=m,n}^{\infty,\infty}\inf_{\{\bar{K}\le p<\infty;1\le q\le\bar{L}-1\}}\{a_{k,l,p,q}\}\sum_{p,q=\bar{K},1}^{\infty,\bar{L}-1}x_{p,q}$$
$$+\sum_{k,l=m,n}^{\infty,\infty}\inf_{\{\bar{L}\le q<\infty;1\le p\le\bar{K}-1\}}\{a_{k,l,p,q}\}\sum_{p,q=1,\bar{L}}^{\bar{K}-1,\infty}x_{p,q}$$
$$+(1-\epsilon)\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=1,1}^{\infty,\infty}a_{k,l,p,q}y_{p,q}.$$
$$-(1-\epsilon)\sum_{k,l=m,n}^{\infty,\infty}\sup_{\{\bar{K}\le p<\infty;1\le q\le\bar{L}-1\}}\{a_{k,l,p,q}\}\sum_{p,q=\bar{K},1}^{\infty,\bar{L}-1}y_{p,q}$$
$$-(1-\epsilon)\sum_{k,l=m,n}^{\infty,\infty}\sup_{\{\bar{L}\le q<\infty;1\le p\le\bar{K}-1\}}\{a_{k,l,p,q}\}\sum_{p,q=1,\bar{L}}^{\bar{K}-1,\infty}y_{p,q}$$
$$-(1-\epsilon)\sum_{k,l=m,n}^{\infty,\infty}\max_{\{1\le p\le\bar{L}-1;1\le q\le\bar{K}-1\}}\{a_{k,l,p,q}\}\sum_{p,q=1,1}^{\bar{K}-1,\bar{L}-1}y_{p,q}$$
in probability.

These inequalities grant us the following

$$\frac{R_{m,n}(Ax)}{R_{m,n}(Ay)}\ge\frac{\sum_{k,l=m,n}^{\infty,\infty}\min_{\{1\le p\le\bar{L}-1;1\le q\le\bar{K}-1\}}\{a_{k,l,p,q}\}\sum_{p,q=1,1}^{\bar{K}-1,\bar{L}-1}x_{p,q}}{\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=1,1}^{\infty,\infty}a_{k,l,p,q}y_{p,q}}$$
$$+\frac{\sum_{k,l=m,n}^{\infty,\infty}\inf_{\{\bar{K}\le p<\infty;1\le q\le\bar{L}-1\}}\{a_{k,l,p,q}\}\sum_{p,q=\bar{K},1}^{\infty,\bar{L}-1}x_{p,q}}{\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=1,1}^{\infty,\infty}a_{k,l,p,q}y_{p,q}}$$

$$+\frac{\sum_{k,l=m,n}^{\infty,\infty}\inf_{\{\bar{L}\leq q<\infty;1\leq p\leq\bar{K}-1\}}\{a_{k,l,p,q}\}\sum_{p,q=1,\bar{L}}^{\bar{K}-1,\infty}x_{p,q}}{\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=1,1}^{\infty,\infty}a_{k,l,p,q}y_{p,q}}$$
$$+(1-\epsilon)$$
$$-(1-\epsilon)\frac{\sum_{k,l=m,n}^{\infty,\infty}\sup_{\{\bar{K}\leq p<\infty;1\leq q\leq\bar{L}-1\}}\{a_{k,l,p,q}\}\sum_{p,q=\bar{K},1}^{\infty,\bar{L}-1}y_{p,q}}{\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=1,1}^{\infty,\infty}a_{k,l,p,q}y_{p,q}}$$
$$-(1-\epsilon)\frac{\sum_{k,l=m,n}^{\infty,\infty}\sup_{\{\bar{L}\leq q<\infty;1\leq p\leq\bar{K}-1\}}\{a_{k,l,p,q}\}\sum_{p,q=1,\bar{L}}^{\bar{K}-1,\infty}y_{p,q}}{\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=1,1}^{\infty,\infty}a_{k,l,p,q}y_{p,q}}$$
$$-(1-\epsilon)\frac{\sum_{k,l=m,n}^{\infty,\infty}\max_{\{1\leq p\leq\bar{L}-1;1\leq q\leq\bar{K}-1\}}\{a_{k,l,p,q}\}\sum_{p,q=1,1}^{\bar{K}-1,\bar{L}-1}y_{p,q}}{\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=1,1}^{\infty,\infty}a_{k,l,p,q}y_{p,q}}$$

in probability.

Thus
$$\frac{R_{m,n}(Ax)}{R_{m,n}(Ay)}\geq 1-\epsilon \text{ in probability.}$$

Thus
$$\frac{R(Ax)}{R(Ay)}\overset{Probability}{\approx}1.$$

and yields $R(Ax)\overset{Probability}{\approx}R(Ay)$. To show that (1) implies (2) we can fix $\bar{K}$ and $\bar{L}$ as positive integers and define x and y as follows
$$x_{k,l}:=\begin{cases}0, & \text{if } p<\bar{K} \text{ or } q<\bar{L}\\ 1, & \text{otherwise,}\end{cases}$$

and $y_{p,q}:=1$ for all p and q. This implies that
$$\begin{aligned}R_{m,n}(Ax)&=\sum_{k,l=m,n}^{\infty,\infty}(Ax)_{k,l}\\&=\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=\bar{K}+1,\bar{L}+1}^{\infty,\infty}a_{k,l,p,q}\\&=\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=1,1}^{\infty,\infty}a_{k,l,p,q}y_{p,q}-\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=\bar{K},1}^{\infty,\bar{L}-1}a_{k,l,p,q}\\&-\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=1,\bar{L}}^{\bar{K}-1,\infty}a_{k,l,p,q}-\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=1,1}^{\bar{K}-1,\bar{L}-1}a_{k,l,p,q}\end{aligned}$$

Thus

$$\begin{aligned}\frac{R_{m,n}(Ax)}{R_{m,n}(Ay)} &= 1 - \frac{\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=\bar{K},1}^{\infty,\bar{L}-1} a_{k,l,p,q}}{\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=1,1}^{\infty,\infty} a_{k,l,p,q}} \\ &\quad - \frac{\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=1,\bar{L}}^{\bar{K}-1,\infty} a_{k,l,p,q}}{\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=1,1}^{\infty,\infty} a_{k,l,p,q}} - \frac{\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=1,1}^{\bar{K}-1,\bar{L}-1} a_{k,l,p,q}}{\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=1,1}^{\infty,\infty} a_{k,l,p,q}} \\ &\leq 1 - \frac{\sum_{k,l=m,n}^{\infty,\infty} a_{\bar{K},\bar{L}-1,p,q}}{\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=1,1}^{\infty,\infty} a_{k,l,p,q}} - \frac{\sum_{k,l=m,n}^{\infty,\infty} a_{\bar{K}-,\bar{L},p,q}}{\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=1,1}^{\infty,\infty} a_{k,l,p,q}} \\ &\quad - \frac{\sum_{k,l=m,n}^{\infty,\infty} a_{\bar{K}-1,\bar{L}-1,p,q}}{\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=1,1}^{\infty,\infty} a_{k,l,p,q}}.\end{aligned}$$

Therefore

$$\begin{aligned}P-\liminf_{m,n}\frac{R_{m,n}(Ax)}{R_{m,n}(Ay)} &\leq 1 - P - \limsup_{m,n}\frac{\sum_{k,l=m,n}^{\infty,\infty} a_{\bar{K},\bar{L}-1,p,q}}{\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=1,1}^{\infty,\infty} a_{k,l,p,q}} \\ &\quad - P - \limsup_{m,n}\frac{\sum_{k,l=m,n}^{\infty,\infty} a_{\bar{K}-1,\bar{L},p,q}}{\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=1,1}^{\infty,\infty} a_{k,l,p,q}} \\ &\quad - P - \limsup_{m,n}\frac{\sum_{k,l=m,n}^{\infty,\infty} a_{\bar{K}-1,\bar{L}-1,p,q}}{\sum_{k,l=m,n}^{\infty,\infty}\sum_{p,q=1,1}^{\infty,\infty} a_{k,l,p,q}}.\end{aligned}$$

Since each nonconstant element of the last inequality has P-limit zero we obtain the following

$$\frac{R(Ax)}{R(Ay)} \overset{Probability}{\approx} 1.$$

□

We shall now examine the concepts of convergence in the Pringsheim sense in probability. However, it is necessary to restrict our attention to bounded double sequences since a P-convergent double sequence is not necessarily bounded. Our goal now is to establish Robison and Hamilton type characterization for P-asymptotically regular in probability. To accomplish this goal we will provide answer to the following question. What are the conditions on the entries of four dimensional matrices that ensure the preservation of P-asymptotically convergence in probability? Let $B_{\alpha,\beta}$ the following set:

$$\{(k,l) : \{\alpha+1 \leq k < \infty \cap 1 \leq l\infty\} \cup \{1 \leq k < \infty \cap \beta+1 \leq l\infty\}\}.$$

Theorem 2.3. *In order for a four dimensional summability matrix A to be P-asymptotically regular in probability it is necessary and sufficient that for each fixed positive integer (α,β)*

(1)

$$\sum_{k,l=1,1}^{\alpha,\beta} a_{m,n,k,l} \textit{ is bounded for each } (m,n),$$

(2)

$$P-\lim_{m,n} P\left(\left|\frac{\sum_{(k,l)\in B_{\alpha,\beta}} a_{m,n,k,l}}{\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}}\right| < \epsilon\right) = 1.$$

Proof. The necessary part of this theorem is clearly similar to that of the last theorem. Thus it is omitted. Let us now consider the sufficient part. Let $\epsilon > 0$, $x \overset{Probability}{\approx} y$, $[x] \in E_0$, and $[y] \in E_\delta$ for some $\delta > 0$. These conditions imply that

$$\begin{aligned} \mathrm{P}-\lim_{k,l} P\,&((L-\epsilon)y_{k+\alpha,l+\beta} \leq x_{k+\alpha,l+\beta} \\ &\leq (L+\epsilon)y_{k+\alpha,l+\beta}) = 1,\ k,\ \text{for } \alpha,\beta = 1,2,\ldots\ . \end{aligned} \tag{2.1}$$

Let us partition $(Ax)_{m,n}$ as follows:

$$\begin{aligned} (Ax)_{m,n} = &\sum_{k,l=1,1}^{\alpha,\beta} a_{m,n,k,l}x_{k,l} + \sum_{k,l=1,\beta+1}^{\alpha+1,\infty} a_{m,n,k,l}x_{k,l} \\ &+ \sum_{k,l=\alpha+1,1}^{\infty,\beta+1} a_{m,n,k,l}x_{k,l} + \sum_{k,l=\alpha+1,\beta+1}^{\infty,\infty} a_{m,n,k,l}x_{k,l} \end{aligned}$$

and let us denote the above sums as follows

$$\sum_{x}^{1} = \sum_{k,l=1,1}^{\alpha,\beta} a_{m,n,k,l}x_{k,l}, \sum_{x}^{2} = \sum_{k,l=1,\beta+1}^{\alpha+1,\infty} a_{m,n,k,l}x_{k,l}, \sum_{x}^{3} = \sum_{k,l=\alpha+1,1}^{\infty,\beta+1} a_{m,n,k,l}x_{k,l},$$

and

$$\sum_{x}^{4} = \sum_{k,l=\alpha+1,\beta+1}^{\infty,\infty} a_{m,n,k,l}x_{k,l}.$$

Similar to the above notation let us denote (Ay) as follows:

$$(Ay)_{m,n} = \sum_{y}^{1} + \sum_{y}^{2} + \sum_{y}^{3} + \sum_{y}^{4}.$$

Also let us consider the following

$$\begin{aligned}\frac{(Ax)_{m,n}}{(Ay)_{m,n}} &= \frac{\sum_x^1 + \sum_x^2 + \sum_x^3 + \sum_x^4}{\sum_y^1 + \sum_y^2 + \sum_y^3 + \sum_y^4} \\ &= \frac{\frac{\sum_x^1}{\sum_y^4} + \frac{\sum_x^2}{\sum_y^4} + \frac{\sum_x^3}{\sum_y^4} + \frac{\sum_x^4}{\sum_y^4}}{\frac{\sum_y^1}{\sum_y^4} + \frac{\sum_y^2}{\sum_y^4} + \frac{\sum_y^3}{\sum_y^4} + 1}.\end{aligned}$$

Inequality (2.1) implies that

$$P - \lim_{m,n} P\left(\left|\frac{\sum_{k,l=\alpha+1,\beta+1}^{\infty,\infty} a_{m,n,k,l}x_{k,l}}{\sum_{k,l=\alpha+1,\beta+1}^{\infty,\infty} a_{m,n,k,l}y_{k,l}} - L\right| < \epsilon\right) = 1.$$

Since $[x] \in E_0$, $[y] \in E_\delta$, $[x] \in l''_\infty$ and condition (2) holds we obtain the following

$$P - \lim_{m,n} P\left(\left|\frac{\sum_x^1}{\sum_y^4}\right|\right) = 1$$

$$P - \lim_{m,n} P\left(\left|\frac{\sum_x^2}{\sum_y^4}\right|\right) = 1$$

$$P - \lim_{m,n} P\left(\left|\frac{\sum_x^3}{\sum_y^4}\right|\right) = 1$$

$$P - \lim_{m,n} P\left(\left|\frac{\sum_y^1}{\sum_y^4}\right|\right) = 1$$

$$P - \lim_{m,n} P\left(\left|\frac{\sum_y^2}{\sum_y^4}\right|\right) = 1$$

and

$$P - \lim_{m,n} P\left(\left|\frac{\sum_y^3}{\sum_y^4}\right|\right) = 1.$$

Thus

$$\frac{(Ax)_{m,n}}{(Ay)_{m,n}} \overset{Probability}{\approx} L.$$

This implies that $Ax \overset{Probability}{\approx} Ay$ whenever $x \overset{Probability}{\approx} y$, $[x] \in E_0$, $[y] \in E_\delta$ for some $\delta > 0$, and $[x], [y] \in l''_\infty$. □

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(Received: November 17, 2009)
(Revised; May 17, 2010)

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