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QUASI-DIAGONAL OPERATORS

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Abstract. Let H be a separable complex Hilbert space and let $B(H)$ denote the algebra of all bounded linear operators on H. If T is a quasi-normal Fredholm operator we prove that $TT^* \in (QD)(P_n)$ if and only if $T^*T \in (QD)(P_n)$. We also show that if T is quasi-normal and $T(T^*T)$ is quasi-diagonal with respect to any sequence (P_n) in $PF(H)$, such that $P_n \to I$ strongly, then $T = N + K$, where N is a normal operator and K is a compact operator.

1. Introduction

Let $B(H)$ be the algebra of all bounded linear operators acting in a separable Hilbert space H and let $PF(H)$ denote the set of all finite rank (orthogonal) projections on H. An operator T is said to be quasi-diagonal (block-diagonal), if there exists an increasing sequence $(P_n)_{n\in N}$ in $PF(H)$ such that $P_n \to I$ strongly, as $n \to \infty$, and $\lim_{n\to\infty} \|TP_n - P_nT\| = 0$ ($TP_n = P_nT$ for all $n = 1, 2, \ldots$, respectively).

The class of quasi-diagonal operators is denoted by (QD) whereas the class of block-diagonal operators is denoted by (BD). Denote by $A(H) = B(H)/K(H)$ the quotient algebra, where $K(H)$ is the ideal of all compact operators and let $\pi : B(H) \to A(H)$ be the canonical projection. $A(H)$ is a Banach algebra with respect to the norm $\|\pi(T)\| = inf\{\|T - K\| : K \in K(H)\}$. π is a continuous linear map and $A(H)$ is a C^*–algebra with respect to the involution $^* : \pi(T) \to [\pi(T)]^* = \pi(T^*)$, is called a Calkin Algebra.

We say that an operator $T \in B(H)$ is Fredholm if $\pi(T)$ is invertible element in the Calkin algebra $A(H)$. Denote by $F(H)$ the set of all Fredholm operators.

Further, we say that an operator T is essentially unitary (essentially normal) operator, if $\pi(T)$ is unitary element (normal element) in $A(H)$.

The classes (QD) and (BD) were introduced and studied by P.R. Halmos in [3], and later on by many authors including R.A. Smucker, G.R. Luecke,

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D.A. Herrero, etc. From the definition of quasi-diagonal (block-diagonal) operators, it is easy to see that these classes are invariant under unitary transformations. However, operators similar to quasi-diagonal operators may fail to be quasi-diagonal, see [8]. R.A. Smucker has found a weaker condition for quasi-diagonality. He has shown that if (K_n) is a sequence (not necessary increasing) of compact operators converging strongly to the identity operator I, such that $\|K_n - K_n^2\| \to 0, \|K_n - K_n^*\| \to 0$ and $\|TK_n - K_nT\| \to 0$ then $T \in (QD)$, see [7], [1].

Let $(P_n)_{n\in N}$ be in $PF(H)$ such that $P_n \to I$ strongly. Denote by $(QD)(P_n) = \{T \in B(H) : \|TP_n - P_nT\| \to 0, n \to \infty\}$. This means that $(QD)(P_n)$ is the subset of (QD) containing those quasi-diagonal operators that are quasi-diagonal with respect to the same sequence (P_n) of finite (orthogonal) projections. For the properties of the class $(QD)(P_n)$ see [4].

In this article we show that if T is a quasi-normal and Fredholm operator, then $TT^* \in (QD)(P_n)$ if and only if $T^*T \in (QD)(P_n)$. We also show that if T is quasi-normal and $T(T^*T)$ is quasi-diagonal with respect to some sequence (P_n) in $PF(H)$, such that $P_n \to I$ strongly, then $T = N + K$, where N is a normal operator and K is a compact operator. Further, we show that (QD) is invariant under certain similarities.

2. Quasi-diagonal operators

Proposition 2.1. *If $T \in (QD)(P_n)$ then $\sqrt{T^*T} \in (QD)(P_n)$.*

Proof. If $T \in (QD)(P_n)$, and since $(QD)(P_n)$ is a C^*–algebra then $T^*T \in (QD)(P_n)$. Further, it is well known from the general theory of operators that

$$\sqrt{T^*T}x = \lim_{n\to\infty} p_n(T^*T)x,$$

for all x in H, where $(p_n(t))$ is a sequence of polynomials. Since $p_k(T^*T) \in (QD)(P_n)$ for every $k \in \mathbb{N}$ (see [4]), then

$$\begin{aligned}\|p_k(T^*T)P_nx - P_np_k(T^*T)x\| &= \|(p_k(T^*T)P_n - P_np_k(T^*T))x\| \\ &\le \|p_k(T^*T)P_n - P_np_k(T^*T)\|\end{aligned}$$

for all $x \in H, \|x\| \le 1$.

Now, $p_k(T^*T) \in (QD)(P_n)$ implies that for every $\varepsilon > 0$, there exists $n_0(\varepsilon)$ such that for all $n \ge n_0(\varepsilon)$, we have

$$\|p_k(T^*T)P_nx - P_np_k(T^*T)x\| \le \|p_k(T^*T)P_n - P_np_k(T^*T)\| \le \varepsilon.$$

Taking limits on k in both sides of the last inequality we have

$$\|\sqrt{T^*T}P_nx - P_n\sqrt{T^*T}x\| \le \varepsilon$$

for all $x \in H, \|x\| \leq 1$, and consequently

$$\|\sqrt{T^*T}P_n - P_n\sqrt{T^*T}\| \leq \varepsilon.$$

This means that $\sqrt{T^*T} \in (QD)(P_n)$. □

Theorem 2.2. *If T, SS^* are from the class $(QD)(P_n)$ and S is invertible, then $S^{-1}TS^{*-1}$ is an element in the class $(QD)(Q_n)$, where $Q_n = UP_nU^*$ and U is the unitary operator from the polar form of the operator S^*.*

Proof. Since, $SS^* \in (QD)(P_n)$ then $\sqrt{SS^*} \in (QD)(P_n)$, which implies that

$$(\sqrt{SS^*})^{-1}T(\sqrt{SS^*})^{-1} \in (QD)(P_n)$$

see [4]

$$U(\sqrt{SS^*})^{-1}T(\sqrt{SS^*})^{-1}U^* \in (QD)(UP_nU^*)$$

where U is from the polar form of the operator $S^* = U\sqrt{SS^*}$. From the above relations we have

$$(\sqrt{SS^*}U^*)^{-1}T(U\sqrt{SS^*})^{-1} = S^{-1}TS^{*-1} \in (QD)(UP_nU^*).$$

□

Theorem 2.3. *Let T be a Fredholm operator with $\operatorname{ind} T = 0$ and let $T = V\sqrt{T^*T}$ be the polar form of the operator T. If the range $R(\sqrt{T^*T})$ is a closed set, then if T is in the class $(QD)(P_n)$ and V is in the class $(QD)(P_n)$.*

Proof. Because $T \in F(H)$, $ind\, T = \dim\ker T - \dim\ker T^* = 0$ then it follows that there exists $K \in K(H)$ such that $T + K$ is invertible operator. This implies that

$$T + K = U\sqrt{(T+K)^*(T+K)} \tag{1}$$

where U is a unitary operator. Further more, since $(QD)P_n$ is a C^*-algebra then $T + K, (T+K)^*(T+K) \in (QD)P_n$. As in the proof of Proposition 2.1

$$\sqrt{(T+K)^*(T+K)}x = \lim_{n\to\infty} p_n\big((T+K)^*(T+K)\big)x$$

for some sequence $p_n(t)$ of polynomials. Therefore,

$$p_n\big((T+K)^*(T+K)\big) \in (QD)(P_n)$$

and

$$\sqrt{(T+K)^*(T+K)} \in (QD)(P_n).$$

From these facts we have

$$U = (T+K)(\sqrt{(T+K)^*(T+K)})^{-1} \in (QD)(P_n). \tag{2}$$

Further

$$(T + K) = (V\sqrt{T^*T} + K) \in (QD)(P_n).$$

Therefore

$$V\sqrt{T^*T}+K=U\sqrt{(T+K)^*(T+K)}\in(QD)(P_n). \tag{3}$$

Now, since $\sqrt{T^*T}$ is self-adjoint operator and $R(\sqrt{T^*T})$ is closed set we conclude that $\sqrt{T^*T}$ is Fredholm operator (because $\ker\sqrt{T^*T}=\ker T$). It is obvious that

$$\begin{aligned} ind\sqrt{T^*T} &= \dim\ker\sqrt{T^*T}-\dim\ker(\sqrt{T^*T})^* \\ &= \dim\ker T-\dim\ker T=0, \end{aligned}$$

which implies that $\sqrt{T^*T}$ is Fredholm operator with index zero. Therefore, there exists a compact operator K_1, such that $\sqrt{T^*T}+K_1$ is invertible. Now, from (3) we have that

$$V\sqrt{T^*T}=U\sqrt{(T+K)^*(T+K)}-K,$$

is a Fredholm operator with index zero. Finally

$$\begin{aligned} V\sqrt{T^*T}+VK_1 &= U\sqrt{(T+K)^*(T+K)}+VK_1-K, \\ V(\sqrt{T^*T}+K_1) &= U\sqrt{(T+K)^*(T+K)}+VK_1-K, \\ V(\sqrt{T^*T}+K_1) &= U\sqrt{(T+K)^*(T+K)}+K_2, \end{aligned}$$

where $K_2=VK_1-K\in K(H)$. From the last equality we have

$$V=(U\sqrt{(T+K)^*(T+K)}+K_2)(\sqrt{T^*T}+K_1)^{-1},$$

and hence $V\in(QD)P_n)$. □

Remark 2.4. If $T\in\bigcap(QD)(P_n)$ and if $R(\sqrt{T^*T})$ is closed, then $V\in\bigcap(QD)(P_n)$, where V is taken from the polar form of the operator $T=V\sqrt{T^*T}$.

From the Remark 2.4. we observe that T is a thin operator if and only if V is a thin operator.

A bounded operator T is called quasi-normal if T commutes with the operator T^*T.

Theorem 2.5. *If T is quasi-normal and Fredholm operator, then $TT^*\in(QD)(P_n)$ if and only if $T^*T\in(QD)(P_n)$.*

Proof. For

$$\lambda\geq r(T^*T)=\|T^*T\|=\|TT^*\|=r(TT^*)=\|T\|^2$$

it is easy to see that the equality

$$(\lambda I-TT^*)^{-1}=\frac{I}{\lambda}+\frac{1}{\lambda}T(\lambda I-T^*T)^{-1}T^* \tag{4}$$

holds true. Now, the quasi-normality of T implies that

$$T(\lambda I - T^*T)^{-1} = (\lambda I - T^*T)^{-1}T,$$

hence from (4) we assume

$$(\lambda I - TT^*)^{-1} = \frac{I}{\lambda} + \frac{1}{\lambda}(\lambda I - T^*T)^{-1}TT^*. \tag{5}$$

Now, since T is a Fredholm operator, TT^* is a Fredholm operator of index zero, consequently there exists compact operator K, such that $TT^* + K$ is an invertible operator. Further more, from equality (5) we get

$$(\lambda I - TT^*)^{-1} = \frac{I}{\lambda} + \frac{1}{\lambda}(\lambda I - T^*T)^{-1}(TT^* + K) - \frac{1}{\lambda}(\lambda I - T^*T)^{-1}K$$
$$(\lambda I - TT^*)^{-1} = \frac{I}{\lambda} + \frac{1}{\lambda}(\lambda I - T^*T)^{-1}(TT^* + K) - K_1,$$

where $K_1 = \frac{1}{\lambda}(\lambda I - T^*T)^{-1}K \in K(H)$. Thus

$$(\lambda I - T^*T)^{-1} = \lambda\big((\lambda I - TT^*)^{-1} - \frac{I}{\lambda} + K_1\big)(TT^* + K)^{-1} \in (QD)(P_n).$$

Because $(QD)(P_n)$ is a C^*-algebra, $(\lambda I - T^*T) \in (QD)(P_n)$ and finally $T^*T \in (QD)(P_n)$.

□

Theorem 2.6. *Let T be quasi-normal operator. If $T(T^*T) \in \bigcap(QD)(P_n)$, then T is an essentially normal operator.*

Proof. Let T be quasi-normal operator and let $T(T^*T) \in \bigcap(QD)(P_n)$, then $T(T^*T)$ is uniformly quasi-diagonal, which implies that

$$T(T^*T) = \lambda I + K_1 \tag{6}$$

see [5]. Now, let $\pi : B(H) \to B(H)/K(H)$, be the canonical map. Because T is quasi-normal operator we have

$$\pi(T)\pi(T^*T) = \pi(T^*T)\pi(T) = \lambda\pi(I). \tag{7}$$

From the above equality, we conclude that $\pi(T^*T)$ is invertible element in $B(H)/K(H)$. On the other hand

$$\pi(T^*T)(\pi(T)\pi(T^*))^{-1} = \pi(T^*T)(\pi(T^*)^{-1}\pi(T)^{-1})$$
$$= \pi(T^*)^{-1}\pi(T^*T)\pi(T)^{-1} = \pi(T^*)^{-1}\pi(T^*)\pi(T)\pi(T)^{-1} = \pi(I).$$

Therefore

$$(\pi(T)\pi(T^*))^{-1} = (\pi(T^*)\pi(T))^{-1}$$

and thus

$$\pi(T)\pi(T^*) = \pi(T^*)\pi(T),$$

thus proving that $\pi(T)$ is a normal element in $B(H)/K(H)$, hence T is essentially normal operator. □

Corollary 2.7. *If a quasi-normal operator T is quasi-diagonal operator with respect to some sequence (P_n) in $PF(H)$, such that $P_n \to I$ strongly, then $T = N + K$, where N is a normal operator and K is a compact operator.*

Proof. The proof of this corollary is a direct consequence of Theorem 2.6. and the fact that every essentially-normal quasi-diagonal operator is in the class $(N + K)(H)$, see Theorem 6.5 in [6]. □

Theorem 2.8. *If the operators T, SS^* are in the class $(QD)(P_n)$ and S is a quasi-normal operator, then*

(1) $S^{-1}TS \in (QD)$
(2) $(S^*)^{-1}TS^{-1} \in (QD)$.

Proof. The result in (1) was proven in [4]. Since $T, SS^* \in (QD)(P_n)$ and S is quasi-normal operator from Theorem 2.5, we conclude that $S^*S = S^*(S^*)^* \in (QD)(P_n)$. Now, by Theorem 2.2. we have $(S^*)^{-1}TS^{-1} \in (QD)$. Hence (2) holds true. □

Corollary 2.9. *If T is a quasi-diagonal operator and $S = \lambda U + K$ is an invertible operator, then*

(1) $(\lambda U + K)^{-1}T(\lambda U + K) \in (QD)$
(2) $(\lambda U + K)^{-1}T(\overline{\lambda}U^* + K^*)^{-1} \in (QD)$
(3) $(\overline{\lambda}U^* + K^*)^{-1}T(\lambda U + K)^{-1} \in (QD)$.

Proof. The proof of the Corollary is a direct consequence of the Theorem 2.8, Theorem 2.5 and the fact that if $S = \lambda U + K$, then

$$SS^* = (\lambda U + K)(\overline{\lambda}U^* + K^*) = |\lambda|^2 I + \lambda U K^* + \overline{\lambda} K U^* + KK^*,$$
$$S^*S = (\overline{\lambda}U^* + K^*)(\lambda U + K) = |\lambda|^2 I + \overline{\lambda} U^* K + \lambda K^* U + K^* K.$$

From the above equalities we see that SS^* and S^*S are thin operators. Therefore $SS^*, S^*S \in \bigcap(QD)(P_n)$. □

Theorem 2.10. *Let $T \in (QD)(P_n)$. If $\lambda U^* + \overline{\lambda}U \in (QD)(P_n)$, where U is a unitary operator, $\lambda \in \mathbb{C}$, then*

$$T' = (\lambda I + U + K)^{-1}T(\lambda I + U + K) \in (QD).$$

Proof. Let $T \in (QD)(P_n)$ and let $\lambda U^* + \overline{\lambda}U \in (QD)(P_n)$. Denote by

$$K_n = (\lambda I + U + K)^{-1}P_n(\lambda I + U + K).$$

Since P_n are orthogonal projections of finite rank, K_n are compact operators with finite rank. Now, $K_n \to I$ strongly because $P_n \to I$ strongly and obviously $K_n^2 = K_n$. Further, to prove our result, we need only to show that

$$\|K_n - K_n^*\| \to 0, n \to \infty$$

and
$$\|T'K_n - K_nT'\| \to 0, n \to \infty$$
see [7]. From the inequality
$$\|T'K_n - K_nT'\| =$$
$$= \|(\lambda I + U + K)^{-1}TP_n(\lambda I + U + K) - (\lambda I + U + K)^{-1}P_nT(\lambda I + U + K)\|$$
$$\leq \|(\lambda I + U + K)^{-1}\| \cdot \|TP_n - P_nT\| \cdot \|\lambda I + U + K\|$$
we observe that
$$\|T'K_n - K_nT'\| \to 0, n \to \infty.$$
Next,
$$\|K_n - K_n^*\|$$
$$= (\lambda I + U + K)^{-1}P_n(\lambda I + U + K) - (\lambda I + U + K)^*P_n(\lambda I + U + K)^{*-1}\|$$
$$\leq \|(\lambda I + U + K)^{-1}\| \cdot \|P_n(\lambda I + U + K)(\lambda I + U + K)^*$$
$$-(\lambda I + U + K)(\lambda I + U + K)^*\| \cdot \|\lambda I + U + K\|$$
$$\leq M\|P_n(\lambda I + U + K)(\lambda I + U + K)^* - (\lambda I + U + K)(\lambda I + U + K)^*P_n\|.$$
Now, it is easy to see that the right hand of the last equality tends to zero, as $n \to \infty$. □

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