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THEBAULT'S PENCIL OF CIRCLES IN AN ISOTROPIC PLANE

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ABSTRACT. In the Euclidean plane Griffiths's and Thébault's pencil of the circles are generally different. In this paper it is shown that in an isotropic plane the pencils of circles, corresponding to the Griffiths's and Thébault's pencil of circles in the Euclidean plane, coincide.

In a (scalene) triangle of the Euclidean plane there exists one pencil of circles, to which belong the circumscribed circle, Euler circle, polar circle and orthocentroidal circle of that triangle as well as the circumscribed circle of its tangential triangle. The potential axis of this pencil is the orthic line of that triangle, and its central line is the Euler line of that triangle. It is the so called Griffiths's pencil of the circles of the considered triangle. All these facts are well known (see for example Johnson [5]).

In [8] Thébault considered the set of circles with regard to which the powers of the vertices of a given triangle are inversely proportional to the corresponding lengths of the sides of that triangle. Thus one obtains the pencil of circles, to which the circumscribed circle of the triangle also belongs. The central line of this pencil is the line connecting the center of the circumscribed circle and inscribed circle of the triangle, while the potential line is the antiorthic line of that triangle.

Griffiths's and Thébault's pencil of circles are generally different.

We will show that the situation in an isotropic plane is different, indeed.

Each allowable triangle in an isotropic plane can be set, by a suitable choice of coordinates, in the so called *standard position*, i.e. that its circumscribed circle has the equation $y = x^2$, and its vertices are of the form $A = (a, a^2)$, $B = (b, b^2)$, $C = (c, c^2)$ where $a + b + c = 0$. With the labels $p = abc$, $q = bc + ca + ab$ it can be shown that for example the equalities $a^2 = bc - q$, $(c - a)(a - b) = 2q - 3bc$ are valid (see Sachs [6], Strubecker [7] and [3]).

In [1] it is shown that the circumscribed circle $\mathcal{K}_c$, Euler circle $\mathcal{K}_e$, polar circle $\mathcal{K}_p$, orthocentroidal circle $\mathcal{K}_o$ of the standard triangle ABC and the

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circumscribed circle $\mathcal{K}_t$ of its tangential triangle are given by the following equations

$$\begin{aligned}
&\mathcal{K}_c && \dots && y = x^2, \\
&\mathcal{K}_e && \dots && -\frac{1}{2}y = x^2 + \frac{1}{2}q, \\
&\mathcal{K}_p && \dots && -2y = x^2 + q, \\
&\mathcal{K}_o && \dots && -y = x^2 + \frac{2}{3}q, \\
&\mathcal{K}_t && \dots && \frac{1}{4}y = x^2 + \frac{1}{4}q,
\end{aligned} \tag{1}$$

and these circles belong to one pencil of circles, whose potential axis is the orthic line of the triangle ABC which, owing to [3], has the equation

$$\mathcal{H} \quad \dots \quad y = -\frac{1}{3}\,q. \tag{2}$$

The singular circle $\mathcal{K}_s$ with the equation

$$\mathcal{K}_s \quad \dots \quad x^2 + \frac{1}{3}\,q = 0, \tag{3}$$

also belongs to this pencil, however this circle degenerates into a pair of isotropic lines with the equations $x = \sqrt{-\frac{1}{3}\,q}$ and $x = -\sqrt{-\frac{1}{3}\,q}$.

Let us now find Thébault's circles of the standard triangle ABC. According to [4] Corollary 4 the power of the point (x_o, y_o) with respect to the circle $ty = x^2 + ux + v$ is equal to

$$x_o^2 + ux_o + v - ty_o.$$

We want to find the circle with regard to which the points $A = (a, a^2)$, $B = (b, b^2)$, $C = (c, c^2)$ have the powers of the form $k \cdot AB \cdot AC$, $k \cdot BC \cdot BA$, $k \cdot CA \cdot CB$, i.e. $k(a-b)(a-c)$, $k(b-c)(b-a)$, $k(c-a)(c-b)$. Because of that we obtain the equalities

$$\begin{aligned}
a^2 + au + v - a^2 t &= k(a-b)(a-c), \\
b^2 + bu + v - b^2 t &= k(b-c)(b-a), \\
c^2 + cu + v - c^2 t &= k(c-a)(c-b).
\end{aligned}$$

The solution of this system of equations by u, v, t is given by $u = 0$, $v = kq$, $t = 1 - 3k$. This yields Theorem 1.

Theorem 1. *The circle, with regard to which the vertices A, B, C of the standard triangle have the powers $k \cdot AB \cdot AC$, $k \cdot BC \cdot BA$, $k \cdot CA \cdot CB$, has the equation*

$$(1 - 3k)y = x^2 + kq. \tag{4}$$

With $k = \infty$, $k = 0$, $k = \frac{1}{2}$, $k = \frac{1}{4}$, $k = 1$, $k = \frac{2}{3}$, $k = \frac{1}{3}$ the equation (4) includes (2), and then successively five equations (1) and finally the equation (3). It means that in an isotropic plane the pencils of circles, corresponding to the Griffiths's and Thébault's pencil of circles in a Euclidean plane, coincide. The orthic line and the antiorthic line of the triangle ABC also coincide, which is otherwise proved in [2], while here the role of the central line of the pencil and the Euler line of the triangle ABC have the isotropic line through the centroid $G = \left(0, -\frac{2}{3}\,q\right)$ with the equation $x = 0$.

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