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INEQUALITIES APPLICABLE TO MIXED VOLTERRA-FREDHOLM TYPE INTEGRAL EQUATIONS

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Abstract. In this paper we establish some new integral inequalities with explicit estimates which can be used as tools in the study of some basic properties of solutions of mixed Volterra-Fredholm type integral equations. Discrete analogues of the main results and some applications of one of our results are also given.

1. Introduction

The inequalities with explicit estimates serve as an important tool in the qualitative study of various types of differential, integral and finite difference equations. The extensive surveys of such inequalities may be found in the monographs [6-8], see also the relevant references cited therein. In the study of many basic models in epidemiology and parabolic equations which describe diffusion or heat transfer phenomena, the integral equations of the form

$$u(t,x) = f(t,x) + \int_0^t \int_B k(t,x,s,y)\, g(u(s,y))\, dyds, \tag{1.1}$$

occur in a natural way, see [3-5, 11] where B is a closed subset of R^m. The equation (1.1) appears to be Volterra type in t, and of Fredholm type with respect to x and hence it can be viewed as a mixed Volterra-Fredholm type integral equation (see [1-4, 9,11,12]). It is easy to observe that the integral inequalities with explicit estimates available in the literature are not directly applicable to study the qualitative properties of solutions of equations of the form (1.1). Motivated by the desire to widen the scope of such inequalities, in the present paper we offer some fundamental inequalities which can be used as tools for handling equations of the form (1.1) and other variants. Discrete analogues of the main results and some applications to illustrate the usefulness of one of our results are also given.

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2. Statement of Results

Let R be the set of real numbers, $N_0 = \{0, 1, 2, \ldots\}$, $R_+ = [0, \infty)$, $R_1 = [1, \infty)$ and $B = \prod_{i=1}^{m} [c_i, d_i] \subset R^m$ $(c_i < d_i)$. Let $S = \{(t, x, s, y) : 0 \le s \le t < \infty; x, y \in B\}$ for $s, t \in R_+$, $E = R_+ \times B$ and the partial derivative of a function $h(t, x, s, y)$ defined on S, with respect to the first variable is denoted by $D_1 h(t, x, s, y)$. For any function u defined on B, we denote by $\int_B u(y)\, dy$ the m-fold integral $\int_{c_1}^{d_1} \ldots \int_{c_m}^{d_m} u(y_1, \ldots y_m)\, dy_m \ldots dy_1$. Let $N_i[\alpha_i, \beta_i] = \{\alpha_i, \alpha_i + 1, \ldots, \beta_i\}$ $(\alpha_i < \beta_i)$, $\alpha_i, \beta_i \in N_0$ for $i = 1, \ldots, m$ and $G = \prod_{i=1}^{m} N_i[\alpha_i, \beta_i] \subset R^m$. Let $\Omega = \{(n, x, s, y) : 0 \le s \le n < \infty; x, y \in G\}$ for $s, n \in N_0$, $H = N_0 \times G$ and for the functions z and r defined respectively on N_0 and Ω we define the operators Δ and Δ_1 by $\Delta z(n) = z(n+1) - z(n)$ and $\Delta_1 r(n, x, s, y) = r(n+1, x, s, y) - r(n, x, s, y)$. For any function w defined on G we denote the m-fold sum over G with respect to variable $y = (y_1, \ldots, y_m) \in G$ by $\sum_G w(y) = \sum_{y_1=\alpha_1}^{\beta_1} \ldots \sum_{y_m=\alpha_m}^{\beta_m} w(y_1, \ldots, y_m)$. Clearly, $\sum_G w(y) = \sum_G w(x)$ for $x, y \in G$. We denote by $C(S_1, S_2)$ and $D(S_1, S_2)$ respectively the class of continuous and discrete functions from the set S_1 to the set S_2. We use the usual conventions that empty sums and products are taken to be 0 and 1 respectively and assume that all the integrals, sums and products involved exist and are finite.

Our main results are given in the following theorem.

Theorem 1. *Let* $u(t, x) \in C(E, R_+)$, $k(t, x, s, y)$, $D_1 k(t, x, s, y) \in C(S, R_+)$ *and* $c \ge 0$ *is a real constant.*

(a_1) *If*

$$u(t, x) \le c + \int_0^t \int_B k(t, x, s, y)\, u(s, y)\, dy ds, \tag{2.1}$$

for $(t, x) \in E$, *then*

$$u(t, x) \le c \exp\left(\int_0^t A(\sigma, x)\, d\sigma\right), \tag{2.2}$$

for $(t, x) \in E$, *where*

$$A(t, x) = \int_B k(t, x, t, y)\, dy + \int_0^t \int_B D_1 k(t, x, s, y)\, dy ds, \tag{2.3}$$

for $(t, x) \in E$.

(a_2) *Let* $g \in C(R_+, R_+)$ *be a nondecreasing function,* $g(u) > 0$ *on* $(0, \infty)$. *If*

$$u(t, x) \le c + \int_0^t \int_B k(t, x, s, y)\, g(u(s, y))\, dy ds, \tag{2.4}$$

for $(t,x) \in E$, *then for* $0 \le t \le t_1; t, t_1 \in R_+, x \in B$,

$$u(t,x) \le W^{-1}\left[W(c) + \int_0^t A(\sigma,x)\,d\sigma\right], \tag{2.5}$$

where

$$W(r) = \int_{r_0}^{r} \frac{ds}{g(s)}, r > 0, \tag{2.6}$$

$r_0 > 0$ *is arbitrary and* W^{-1} *is the inverse of* W *and* $A(t,x)$ *is given by* (2.3) *and* $t_1 \in R_+$ *is chosen so that*

$$W(c) + \int_0^t A(\sigma,x)\,d\sigma \in \mathrm{Dom}\left(W^{-1}\right),$$

for all $t \in R_+$ *lying in the interval* $0 \le t \le t_1$ *and* $x \in B$.

$(\boldsymbol{a_3})$ *If*

$$u^2(t,x) \le c + \int_0^t \int_B k(t,x,s,y)\,u(s,y)\,dyds, \tag{2.7}$$

for $(t,x) \in E$, *then*

$$u(t,x) \le \sqrt{c} + \frac{1}{2}\int_0^t A(\sigma,x)\,d\sigma, \tag{2.8}$$

for $(t,x) \in E$, *where* $A(t,x)$ *is given by* (2.3).

$(\boldsymbol{a_4})$ *Let* $g(u)$ *be as in part* (a_2). *If*

$$u^2(t,x) \le c + \int_0^t \int_B k(t,x,s,y)\,u(s,y)\,g(u(s,y))\,dyds, \tag{2.9}$$

for $(t,x) \in E$, *then for* $0 \le t \le t_2; t, t_2 \in R_+, x \in B$,

$$u(t,x) \le W^{-1}\left[W\left(\sqrt{c}\right) + \frac{1}{2}\int_0^t A(\sigma,x)\,d\sigma\right], \tag{2.10}$$

where W, W^{-1}, $A(t,x)$ *are as in part* (a_2) *and* $t_2 \in R_+$ *is chosen so that*

$$W\left(\sqrt{c}\right) + \frac{1}{2}\int_0^t A(\sigma,x)\,d\sigma \in \mathrm{Dom}\left(W^{-1}\right),$$

for all $t \in R_+$ *lying in the interval* $0 \le t \le t_2$ *and* $x \in B$.

$(\boldsymbol{a_5})$ *Suppose that* $u(t,x) \in C(E, R_1)$ *and* $c \ge 1$. *If*

$$u(t,x) \le c + \int_0^t \int_B k(t,x,s,y)\,u(s,y)\log u(s,y)\,dyds, \tag{2.11}$$

for $(t,x) \in E$, *then*

$$u(t,x) \le c^{\exp\left(\int_0^t A(\sigma,x)d\sigma\right)}, \tag{2.12}$$

for $(t,x)\in E$, *where* $A(t,x)$ *is given by* (2.3).

$(\boldsymbol{a_6})$ *Let* $u(t,x)\in C(E,R_1)$, $c\geq 1$ *and* $g(u)$ *be as in part* (a_2). *If*

$$u(t,x)\leq c+\int_0^t\int_B k(t,x,s,y)\,u(s,y)\,g(\log u(s,y))\,dyds, \tag{2.13}$$

for $(t,x)\in E$, *then for* $0\leq t\leq t_3; t,t_3\in R_+, x\in B$,

$$u(t,x)\leq W^{-1}\left[W(\log c)+\int_0^t A(\sigma,x)\,d\sigma\right], \tag{2.14}$$

where W, W^{-1}, $A(t,x)$ *are as in part* (a_2) *and* $t_3\in R_+$ *is chosen so that*

$$W(\log c)+\int_0^t A(\sigma,x)\,d\sigma\in \mathrm{Dom}\left(W^{-1}\right),$$

for all $t\in R_+$ *lying in the interval* $0\leq t\leq t_3$ *and* $x\in B$.

The discrete analogues of the inequalities in Theorem 1 are given as follows.

Theorem 2. *Let* $u(t,x)\in D(H,R_+), h(n,x,s,y), \Delta_1 h(n,x,s,y)\in D(\Omega,R_+)$ *and* $c\geq 0$ *is a real constant.*

$(\boldsymbol{b_1})$ *If*

$$u(n,x)\leq c+\sum_{s=0}^{n-1}\sum_G h(n,x,s,y)\,u(s,y), \tag{2.15}$$

for $(n,x)\in H$, *then*

$$u(n,x)\leq c\prod_{\sigma=0}^{n-1}\left[1+\bar{A}(\sigma,x)\right], \tag{2.16}$$

for $(n,x)\in H$, *where*

$$\bar{A}(n,x)=\sum_G h(n+1,x,n,y)+\sum_{s=0}^{n-1}\sum_G \Delta_1 h(n,x,s,y), \tag{2.17}$$

for $(n,x)\in H$.

$(\boldsymbol{b_2})$ *Let* g *be as in Theorem* 1 *part* (a_2). *If*

$$u(n,x)\leq c+\sum_{s=0}^{n-1}\sum_G h(n,x,s,y)\,g(u(s,y)), \tag{2.18}$$

for $(n,x) \in H,$ *then for* $0 \le n \le n_1; n, n_1 \in N_0, x \in G,$

$$u(n,x) \le W^{-1}\left[W(c) + \sum_{\sigma=0}^{n-1} \bar{A}(\sigma,x)\right], \tag{2.19}$$

where $W,\ W^{-1}$ *are as in Theorem* 1 *part* (a_2), $\bar{A}(n,x)$ *is given by* (2.17) *and* $n_1 \in N_0$ *be chosen so that*

$$W(c) + \sum_{\sigma=0}^{n-1} \bar{A}(\sigma,x) \in \operatorname{Dom}\left(W^{-1}\right),$$

for all $n \in N_0$ *lying in* $0 \le n \le n_1$ *and* $x \in G.$

(b_3) *If*

$$u^2(n,x) \le c + \sum_{s=0}^{n-1} \sum_{G} h(n,x,s,y)\, u(s,y), \tag{2.20}$$

for $(n,x) \in H,$ *then*

$$u(n,x) \le \sqrt{c} + \frac{1}{2} \sum_{\sigma=0}^{n-1} \bar{A}(\sigma,x), \tag{2.21}$$

for $(n,x) \in H,$ *where* $\bar{A}(n,x)$ *is given by* (2.17).

(b_4) *Let* g *be as in Theorem* 1 *part* (a_2). *If*

$$u^2(n,x) \le c + \sum_{s=0}^{n-1} \sum_{G} h(n,x,s,y)\, u(s,y)\, g(u(s,y)), \tag{2.22}$$

for $(n,x) \in H,$ *then for* $0 \le n \le n_2,\ n_1, n_2 \in N_0,\ x \in G,$

$$u(n,x) \le W^{-1}\left[W\left(\sqrt{c}\right) + \frac{1}{2} \sum_{\sigma=0}^{n-1} \bar{A}(\sigma,x)\right], \tag{2.23}$$

where $W,\ W^{-1},\ \bar{A}(n,x)$ *are as in part* (b_2) *and* $n_2 \in N_0$ *be chosen so that*

$$W\left(\sqrt{c}\right) + \frac{1}{2} \sum_{\sigma=0}^{n-1} \bar{A}(\sigma,x) \in \operatorname{Dom}\left(W^{-1}\right),$$

for all $n \in N_0$ *lying in* $0 \le n \le n_2$ *and* $x \in G.$

(b_5) *Suppose that* $u(n,x) \in D(H, R_1)$ *and* $c \ge 1.$ *If*

$$u(n,x) \le c + \sum_{s=0}^{n-1} \sum_{G} h(n,x,s,y)\, u(s,y) \log u(s,y), \tag{2.24}$$

for $(n,x)\in H,$ *then*

$$u(n,x)\leq c^{\prod_{\sigma=0}^{n-1}\left[1+\bar{A}(\sigma,x)\right]}, \tag{2.25}$$

for $(n,x)\in H,$ *where* $\bar{A}(n,x)$ *is given by* (2.17).

(b_6) *Let* $u(n,x)\in D(H,R_1),$ $c\geq 1$ *and* $g(u)$ *be as in Theorem* 1 *part* (b_2). *If*

$$u(n,x)\leq c+\sum_{s=0}^{n-1}\sum_{G}h(n,x,s,y)\,u(s,y)\,g(\log u(s,y)), \tag{2.26}$$

for $(n,x)\in H,$ *then for* $0\leq n\leq n_3$*;* $n,n_3\in N_0,$ $x\in G,$

$$u(n,x)\leq \exp\left(W^{-1}\left[W(\log c)+\sum_{\sigma=0}^{n-1}\bar{A}(\sigma,x)\right]\right), \tag{2.27}$$

where $W,$ $W^{-1},$ $\bar{A}(n,x)$ *are as in part* (b_2) *and* $n_3\in N_0$ *is chosen so that*

$$W(\log c)+\sum_{\sigma=0}^{n-1}\bar{A}(\sigma,x)\in \operatorname{Dom}\left(W^{-1}\right),$$

for all $n\in N_0$ *lying in* $0\leq n\leq n_3$ *and* $x\in G.$

3. Proofs of Theorems 1 and 2

The proofs resemble one another, we give the details for $(a_1)-(a_4)$ and $(b_5),(b_6)$ only; the proofs of $(a_5),(a_6)$ and $(b_1)-(b_4)$ can be complated by following the proofs of the above mentioned inequalities, see also [6-8]. To prove $(a_1)-(a_4)$, it is sufficient to assume that $c>0$, since the standard limiting argument can be used to treat the remaing case, see [6, p. 108].

(a_1) For an arbitrary $X\in B$ from (2.1), we have

$$u(t,X)\leq c+\int_0^t\int_B k(t,X,s,y)\,u(s,y)\,dyds. \tag{3.1}$$

Setting

$$e(t,s)=\int_B k(t,X,s,y)\,u(s,y)\,dy, \tag{3.2}$$

the inequality (3.1) can be restated as

$$u(t,X)\leq c+\int_0^t e(t,s)\,ds. \tag{3.3}$$

Define

$$z(t)=c+\int_0^t e(t,s)\,ds, \tag{3.4}$$

then $z(0) = c$ and

$$u\,(t, X) \le z\,(t)\,. \tag{3.5}$$

From (3.4), (3.2), (3.5) and the fact that $z(t)$ is nondecreasing in $t \in R_+$ and $u\,(t, x) \le z\,(t)$ (since $X \in B$ is arbitrary), we observe that

$$\begin{aligned} z'\,(t) &= e\,(t, t) + \int_0^t D_1 e\,(t, s)\, ds \\ &= \int_B k\,(t, X, t, y)\, u\,(t, y)\, dy + \int_0^t D_1 \left\{ \int_B k\,(t, X, s, y)\, u\,(s, y)\, dy \right\} ds \\ &\le \int_B k\,(t, X, t, y)\, z\,(t)\, dy + \int_0^t \int_B D_1 k\,(t, X, s, y)\, z\,(s)\, dy ds \\ &\le A\,(t, X)\, z\,(t)\,. \end{aligned} \tag{3.6}$$

The inequality (3.6) implies

$$z\,(t) \le c \,\exp\left(\int_0^t A\,(\sigma, X)\, d\sigma \right). \tag{3.7}$$

Using (3.7) in (3.5) and the fact that $X \in B$ is arbitrary, we get the required inequality in (2.2).

(a_2) For an arbitrary $X \in B$ from (2.4), we have

$$u\,(t, X) \le c + \int_0^t \int_B k\,(t, X, s, y)\, g\,(u\,(s, y))\, dy ds. \tag{3.8}$$

Setting

$$r\,(t, s) = \int_B k\,(t, X, s, y)\, g\,(u\,(s, y))\, dy, \tag{3.9}$$

the inequality (3.8) can be restated as

$$u\,(t, X) \le c + \int_0^t r\,(t, s)\, ds. \tag{3.10}$$

Defining by $z(t)$ the right hand side of (3.10) and following the proof of part (a_1) given above, we get

$$z'\,(t) \le A\,(t, X)\, g\,(z\,(t))\,. \tag{3.11}$$

Now by following the proof of Theorem 2.3.1 given in [6, p. 107] and in view of the proof of part (a_1), we get the desired inequality in (2.5).

(a_3) For an arbitrary $X \in B$ from (2.7), we have

$$u^2\,(t, X) \le c + \int_0^t \int_B k\,(t, X, s, y)\, u\,(s, y)\, dy ds. \tag{3.12}$$

Let $e(t,s)$ be given by (3.2). Then (3.12) can be restated as

$$u^2(t,X) \le c + \int_0^t e(t,s)\,ds. \tag{3.13}$$

Define by $z(t)$ the right hand side of (3.13), then $z(0) = c$, $z(t)$ is nondecreasing in $t \in R_+$ and $u(t,X) \le \sqrt{z(t)}$. Following the proof of part (a_1), we get

$$z'(t) \le A(t,X)\sqrt{z(t)}. \tag{3.14}$$

The inequality (3.14) implies

$$\sqrt{z(t)} \le \sqrt{c} + \frac{1}{2}\int_0^t A(\sigma,X)\,d\sigma. \tag{3.15}$$

The required inequality in (2.8) follows by using (3.15) in $u(t,X) \le \sqrt{z(t)}$ and the fact that $X \in B$ is arbitrary.

(a_4) For an arbitrary $X \in B$ from (2.9), we have

$$u^2(t,X) \le c + \int_0^t \int_B k(t,X,s,y)\,u(s,y)\,g(u(s,y))\,dyds. \tag{3.16}$$

Setting

$$p(t,s) = \int_B k(t,X,s,y)\,u(s,y)\,g(u(s,y))\,dy, \tag{3.17}$$

the inequality (3.16) can be restated as

$$u^2(t,X) \le c + \int_0^t p(t,s)\,ds. \tag{3.18}$$

Defining by $z(t)$ the right hand side of (3.18) and following the proof of part (a_1), we get

$$z'(t) \le A(t,X)\sqrt{z(t)}g\left(\sqrt{z(t)}\right). \tag{3.19}$$

The inequality (3.19) implies

$$\sqrt{z(t)} \le \sqrt{c} + \frac{1}{2}\int_0^t A(\sigma,X)\,g\left(\sqrt{z(\sigma)}\right)d\sigma. \tag{3.20}$$

Now an application of Bihari's inequality given in Theorem 2.3.1 in [6, p. 107] to (3.20), we get

$$\sqrt{z(t)} \le W^{-1}\left[W\left(\sqrt{c}\right) + \frac{1}{2}\int_0^t A(\sigma,X)\,d\sigma\right]. \tag{3.21}$$

Using (3.21) in $u(t,X) \le \sqrt{z(t)}$ and the fact that $X \in B$ is arbitrary, gives the required inequality in (2.10).

(b_5) For an arbitrary $X \in G$ from (2.24), we have

$$u(n,X) \le c + \sum_{s=0}^{n-1} \sum_{G} h(n,X,s,y)\, u(s,y) \log u(s,y). \tag{3.22}$$

Setting

$$q(n,s) = \sum_{G} h(n,X,s,y)\, u(s,y) \log u(s,y), \tag{3.23}$$

the inequality (3.22) can be restated as

$$u(n,X) \le c + \sum_{s=0}^{n-1} q(n,s). \tag{3.24}$$

Define

$$z(n) = c + \sum_{s=0}^{n-1} q(n,s), \tag{3.25}$$

then $z(0) = c$ and

$$u(n,X) \le z(n). \tag{3.26}$$

From (3.25), (3.23), (3.26) and the fact that $z(n)$ is nondecreasing in $n \in N_0$ and $u(n,x) \le z(n)$ (since $X \in G$ is arbitrary), we observe that

$$\begin{aligned}
\Delta z(n) &= q(n+1,n) + \sum_{s=0}^{n-1} \Delta_1 q(n,s) \\
&= \sum_{G} h(n+1,X,n,y)\, u(n,y) \log u(n,y) \\
&\qquad + \sum_{s=0}^{n-1} \Delta_1 \left\{ \sum_{G} h(n,X,s,y)\, u(s,y) \log u(s,y) \right\} \\
&\le \sum_{G} h(n+1,X,n,y)\, z(n) \log z(n) \\
&\qquad + \sum_{s=0}^{n-1} \sum_{G} \Delta_1 h(n,X,s,y)\, z(s) \log z(s) \\
&\le \left[\bar{A}(n,X) \log z(n) \right] z(n).
\end{aligned} \tag{3.27}$$

Now a suitable application of Theorem 1.2.1 given in [7, p. 11] to (3.27) yields

$$z(n) \le c \prod_{\sigma=0}^{n-1} \left[1 + \bar{A}(\sigma, X) \log z(\sigma)\right]$$
$$\le c \exp\left(\sum_{\sigma=0}^{n-1} \bar{A}(\sigma, X) \log z(\sigma)\right). \qquad (3.28)$$

From (3.28), we observe that

$$\log z(n) \le \log c + \sum_{\sigma=0}^{n-1} \bar{A}(\sigma, X) \log z(\sigma). \qquad (3.29)$$

Now a suitable application of Theorem 1.2.2 given in [7, p.12] to (3.29) yields

$$\log z(n) \le (\log c) \prod_{\sigma=0}^{n-1} \left[1 + \bar{A}(\sigma, X)\right]$$
$$= \log c^{\prod_{\sigma=0}^{n-1} \left[1+\bar{A}(\sigma,X)\right]}. \qquad (3.30)$$

From (3.30), we observe that

$$z(n) \le c^{\prod_{\sigma=0}^{n-1} \left[1+\bar{A}(\sigma,X)\right]}. \qquad (3.31)$$

Using (3.31) in (3.26) and the fact that $X \in G$ is arbitrary, we get the required inequality in (2.25).

(b_6) For an arbitrary $X \in G$ from (2.26), we have

$$u(n, X) \le c + \sum_{s=0}^{n-1} \sum_{G} h(n, X, s, y) u(s, y) g(\log u(s, y)). \qquad (3.32)$$

The proof can be completed by setting

$$\phi(n, s) = \sum_{G} h(n, X, s, y) u(s, y) g(\log u(s, y)), \qquad (3.33)$$

and following the proof of (b_5) and closely looking at the proof of Theorem 3.5.3 given in [7, p. 245]. Here, we omit the details.

4. Some applications

In [9], Pachpatte studied the existence and uniqueness of solutions of the nonlinear mixed Volterra-Fredholm integral equation of the form

$$u(t, x) = f(t, x) + \int_0^t \int_B F(t, x, s, y, u(s, y))\, dy ds, \qquad (4.1)$$

by using the well known Banach fixed point theorem coupled with Bielecki type norm. In equation (4.1) f, F are given functions, u is the unknown function and B is as defined in section 2. We assume that $f \in C(E, R)$, $F \in C(S \times R, R)$. In this section we apply the inequality in Theorem 1 part (a_1) to study some fundamental qualitative properties of solutions of equation (4.1) under some suitable conditions on the functions involved therein. The generality of equation (4.1) allow us to obtain similar results concerning the equation (1.1). The detailed analysis related to equations (1.1) and (4.1) may be found in [4,11] and [3,5].

The following result concerning the estimate on the solution of equation (4.1) holds.

Theorem 3. *Suppose that the function F in equation* (4.1) *satisfies the condition*

$$|F(t,x,s,y,u) - F(t,x,s,y,v)| \leq k(t,x,s,y)\,|u-v|, \tag{4.2}$$

where $k(t,x,s,y), D_1 k(t,x,s,y) \in C(S, R_+)$. Let

$$c = \sup_{(t,x)\in E} \left| f(t,x) + \int_0^t \int_B F(t,x,s,y,0)\,dyds \right| < \infty, \tag{4.3}$$

where f, F are the functions in equation (4.1)*. If $u(t,x)$ is any solution of equation* (4.1) *on E, then*

$$|u(t,x)| \leq c \exp\left(\int_0^t A(\sigma, x)\,d\sigma \right), \tag{4.4}$$

for $(t,x) \in E$, where $A(t,x)$ is given by (2.3).

Proof. Using the fact that $u(t,x)$, $(t,x) \in E$ is a solution of equation (4.1) and hypotheses, we have

$$\begin{aligned} |u(t,x)| &\leq \left| f(t,x) + \int_0^t \int_B F(t,x,s,y,0)\,dyds \right| \\ &\quad + \int_0^t \int_B |F(t,x,s,y,u(s,y)) - F(t,x,s,y,0)|\,dyds \\ &\leq c + \int_0^t \int_B k(t,x,s,y)\,|u(s,y)|\,dyds. \end{aligned} \tag{4.5}$$

Now an application of the inequality in Theorem 1 part (a_1) to (4.5) yields (4.4). □

A slight variant of Theorem 3 is embodied in the following theorem.

Theorem 4. *Suppose that the function F in equation* (4.1) *satisfies the condition* (4.2)*. Let*

$$d = \sup_{(t,x)\in E} \int_0^t \int_B |F(t,x,s,y,f(s,y))|\, dyds < \infty, \tag{4.6}$$

where f, F are the functions in equation (4.1)*. If $u(t,x)$ is any solution of equation* (4.1) *on E, then*

$$|u(t,x) - f(t,x)| \le d \exp\left(\int_0^t A(\sigma,x)\, d\sigma\right), \tag{4.7}$$

for $(t,x) \in E$, where $A(t,x)$ is given by (2.3).

Proof. Let $e(t,x) = |u(t,x) - f(t,x)|$ for $(t,x) \in E$. Using the fact that $u(t,x)$ is a solution of equation (4.1) and hypotheses, we have

$$\begin{aligned} e(t,x) &\le \int_0^t \int_B |F(t,x,s,y,u(s,y)) - F(t,x,s,y,f(s,y)) \\ &\qquad + F(t,x,s,y,f(s,y))|\, dyds \\ &\le \int_0^t \int_B |F(t,x,s,y,u(s,y)) - F(t,x,s,y,f(s,y))|\, dy\, ds \\ &\qquad + \int_0^t \int_B |F(t,x,s,y,f(s,y))|\, dy\, ds \\ &\le d + \int_0^t \int_B k(t,x,s,y)\, e(s,y)\, dy\, ds. \end{aligned} \tag{4.8}$$

Now an application of the inequality in Theorem 1 part (a_1) to (4.8) yields (4.7).

We call the function $u \in C(E,R)$ an ε-approximate solution to equation (4.1) if there exists a constant $\varepsilon \ge 0$ such that

$$\left| u(t,x) - \left\{ f(t,x) + \int_0^t \int_B F(t,x,s,y,u(s,y))\, dyds \right\} \right| \le \varepsilon, \tag{4.9}$$

for all $(t,x) \in E$. □

The next theorem deals with the estimate on the difference between the two approximate solutions of equation (4.1).

Theorem 5. *Let $u_1(t,x)$ and $u_2(t,x)$ be respectively, ε_1- and ε_2- approximate solutions of equation* (4.1) *on E. Suppose that the function F in equation* (4.1) *satisfies the condition* (4.2)*. Then*

$$|u_1(t,x) - u_2(t,x)| \le (\varepsilon_1 + \varepsilon_2) \exp\left(\int_0^t A(\sigma,x)\, d\sigma\right), \tag{4.10}$$

for $(t,x) \in E$, where $A(t,x)$ is given by (2.3).

Proof. Since $u_1(t,x)$ and $u_2(t,x)$ for $(t,x) \in E$ are respectively, ε_1- and ε_2-approximate solutions of equation (4.1), we have

$$\left|u_i(t,x) - \left\{f(t,x) + \int_0^t \int_B F(t,x,s,y,u_i(s,y))\,dyds\right\}\right| \le \varepsilon_i, \tag{4.11}$$

for $i = 1,2$. From (4.11) and using the elementary inequalities $|v - z| \le |v| + |z|$ and $|v| - |z| \le |v - z|$, we observe that

$$\begin{aligned}
\varepsilon_1 + \varepsilon_2 \ge & \left|u_1(t,x) - \left\{f(t,x) + \int_0^t \int_B F(t,x,s,y,u_1(s,y))\,dy\,ds\right\}\right| \\
& + \left|u_2(t,x) - \left\{f(t,x) + \int_0^t \int_B F(t,x,s,y,u_2(s,y))\,dy\,ds\right\}\right| \\
\ge & \left|[u_1(t,x) - u_2(t,x)] - \left[\left\{f(t,x) + \int_0^t \int_B F(t,x,s,y,u_1(s,y))\,dy\,ds\right\}\right.\right. \\
& \left.\left. - \left\{f(t,x) + \int_0^t \int_B F(t,x,s,y,u_2(s,y))\,dy\,ds\right\}\right]\right| \\
\ge & |u_1(t,x) - u_2(t,x)| - \left|\int_0^t \int_B \Big\{F(t,x,s,y,u_1(s,y))\right. \\
& \left. - F(t,x,s,y,u_2(s,y))\Big\}dy\,ds\right|.
\end{aligned} \tag{4.12}$$

Let $w(t,x) = |u_1(t,x) - u_2(t,x)|$, $(t,x) \in E$. From (4.12) and using the condition (4.2), we have

$$\begin{aligned}
w(t,x) &\le (\varepsilon_1 + \varepsilon_2) + \int_0^t \int_B |F(t,x,s,y,u_1(s,y)) \\
&\qquad - F(t,x,s,y,u_2(s,y))|dy\,ds \\
&\le (\varepsilon_1 + \varepsilon_2) + \int_0^t \int_B k(t,x,s,y)\,w(s,y)\,dy\,ds.
\end{aligned} \tag{4.13}$$

Now an application of the inequality in Theorem 1 part (a_1) to (4.13) yields (4.10). □

Consider the equation (4.1) and the following mixed Volterra-Fredholm integral equation

$$v(t,x) = \bar{f}(t,x) + \int_0^t \int_B \bar{F}(t,x,s,y,v(s,y))\,dyds, \tag{4.14}$$

where $\bar{f} \in C(E,R)$, $\bar{F} \in C(S \times R, R)$.

The following result that relates the solutions of equations (4.1) and (4.14) holds.

Theorem 6. *Suppose that the function F in equation (4.1) satisfies the condition (4.2) and there exist constants $\delta_i \geq 0$ $(i = 1, 2)$ such that*

$$\left|f(t,x) - \bar{f}(t,x)\right| \leq \delta_1, \tag{4.15}$$

$$\int_0^t \int_B \left|F(t,x,s,y,p) - \bar{F}(t,x,s,y,p)\right| dy\, ds \leq \delta_2, \tag{4.16}$$

where f, F and $\bar{f}, \bar{F}$ are as given in (4.1) and (4.14). Let $u(t,x)$ and $v(t,x)$, $(t,x) \in E$ be respectively, solutions to (4.1) and (4.14), then

$$|u(t,x) - v(t,x)| \leq (\delta_1 + \delta_2) \exp\left(\int_0^t A(\sigma, x)\, d\sigma\right), \tag{4.17}$$

for $(t,x) \in E$, where $A(t,x)$ is given by (2.3).

Proof. Let $r(t,x) = |u(t,x) - v(t,x)|$, $(t,x) \in E$. Using the facts that $u(t,x), v(t,x)$ are the solutions of equations (4.1), (4.14) and the hypotheses, we have

$$\begin{aligned} r(t,x) \leq |f(t,x) - \bar{f}(t,x)| &+ \int_0^t \int_B |F(t,x,s,y,u(s,y)) \\ &- F(t,x,s,y,v(s,y))| dy\, ds \\ &+ \int_0^t \int_B \left|F(t,x,s,y,v(s,y)) - \bar{F}(t,x,s,y,v(s,y))\right| dy\, ds \\ &\leq (\delta_1 + \delta_2) + \int_0^t \int_B k(t,x,s,y)\, r(s,y)\, dy\, ds. \end{aligned} \tag{4.18}$$

Now an application of the inequality in Theorem 1 part (a_1) to (4.18) yields (4.17). □

In concluding, we note that the inequality in Theorem 1 part (a_1) can be used to formulate the results on the uniqueness and continuous dependence of solutions of equation (4.1) by closely looking at the corresponding results recently given in [10]. We also note that one can use the inequality in Theorem 2 part (b_1) to establish results similar to those of given in Theorems 3–6 to the solutions of sum-difference equation of the form

$$u(n,x) = h(n,x) + \sum_{s=0}^{n-1} \sum_G L(n,x,s,y,u(s,y)), \tag{4.19}$$

where $h \in D(H,R)$, $L \in D(\Omega \times R, R)$. Moreover, many generalizations, extensions, variants and applications of the inequalities given above are possible. We hope that the results given here will serve as rooted trees for future investigations.

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