SARAJEVO JOURNAL OF MATHEMATICS
Vol.6 (18) (2010), 3–21

AN INSTRUCTIVE TREATMENT OF A GENERALIZATION OF GĂVRUŢĂ'S STABILITY THEOREM

ESZTER GSELMANN AND ÁRPÁD SZÁZ

Abstract. We prove several useful theorems on Hyers sequences and their pointwise limits in quite natural ways which make a straightforward generalization of Găvruţă's stability theorem rather plausible.

Introduction

As M. Kuczma [28, p. 424] already noted, the first results on approximately additive functions were obtained by Gy. Pólya and G. Szegő [34, Part I, Ch. 3, Problem 99] in 1924 and D. H. Hyers [20] in 1941. The first two authors considered only functions of $\mathbb{N}$ to $\mathbb{R}$. While, the latter author, answering a question of S. M. Ulam, proved a somewhat different form of the following

Theorem 1. *If f is a function of one Banach space X to another Y such that*

$$\| f(x+y) - f(x) - f(y) \| \leq \varepsilon$$

for all $x, y \in X$ and some $\varepsilon \geq 0$, then there exists a unique additive function g of X to Y such that

$$\| f(x) - g(x) \| \leq \varepsilon$$

for all $x \in X$. Moreover, $g(x) = \lim_{n \to \infty} 2^{-n} f\big(2^n x\big)$ for all $x \in X$.

Hyers's theorem has later been generalized by several authors in various ways. First of all, in 1950 T. Aoki [1] and independently in 1978 Th. M. Rassias [37] proved stability theorems for additive and linear mappings,

---

2000 *Mathematics Subject Classification.* 20M15, 39B52, 39B82.

Key words and phrases. Groupoids and normed spaces, approximately additive and homogeneous functions, Găvruţă's type stability theorems.

The work of the authors has been supported by the Hungarian Scientific Research Fund (OTKA) Grant NK-81402.

respectively, by permitting the Cauchy difference to become unbounded. They assumed that

$$\| f(x+y) - f(x) - f(y) \| \le M \big(\| x \|^p + \| y \|^p \big)$$

for all $x, y \in X$ and some $M \ge 0$ and $0 \le p < 1$.

Following the innovative approach of Th. M. Rassias, in 1982 J. M. Rassias [35] also proved a stability theorem for linear mappings by replacing the sum of the above two norms by their product. In [36], he also considered the more general factor $\| x \|^{p_1} \| y \|^{p_2}$ with $0 \le p_1 + p_2 < 1$. While, the corresponding generalization for the case of the sum of the two norms was only considered in [24] by G. Isac and Th. M. Rassias.

The results and problems, and the scientific activity, of Th. M. Rassias motivated several mathematicians to pursue intensive investigations in the stability of various functional equations and inequalities. These have led to an extensive theory. The interested reader can get a rapid overview on the subject by consulting the books of D. H. Hyers, G. Isac and Th. M. Rassias [21], S.-M. Jung [25], and S. Czerwik [6], or the survey papers of D. H. Hyers and Th. M. Rassias [22], G. L. Forti [9], R. Ger [16], and L. Székelyhidi [52].

Curiously enough, in 1951 D. G. Bourgin [5] already remarked that a direct generalization of Hyers's theorem can also be obtained by replacing ε by the more general quantity $\psi(x, y)$. However, such a generalization of Hyers's theorem was only proved in 1994 by P. Găvruţă [12]. (For some more general results, see Forti [10] and Grabiec [19].) As a natural extension of Th. M. Rassias's theorem, Găvruţă proved a somewhat different form of the following

Theorem 2. *If f is a function of a commutative group U to a Banach space X such that*

$$\| f(u+v) - f(u) - f(v) \| \le \Phi(u, v)$$

for all $u, v \in X$ and some function Φ of U^2 to X, with

$$\Psi(u, v) = \sum_{n=0}^{\infty} \frac{1}{2^n} \Phi\big(2^n u, 2^n v\big) < +\infty$$

for all $u, v \in U$, then there exists a unique additive function g of U to $\mathbb{R}$ such that

$$\| f(u) - g(u) \| \le \frac{1}{2} \Psi(u, u)$$

for all $u \in U$. Moreover, g is given by the same formula as in Theorem 1.

In the present paper, we shall prove the following counterpart of a straightforward generalization of Găvruţă's theorem.

Theorem 3. *If f is a function of a semigroup U to a Banach space X such that*

$$\big\| f(2u) - 2f(u) \big\| \leq \varphi(u)$$

for all $u, v \in X$ and some function φ of U to $\mathbb{R}$, with

$$\psi(u) = \sum_{n=0}^{\infty} \frac{1}{2^n} \varphi\big(2^n u\big) < +\infty$$

for all $u \in U$, then there exists a unique 2–homogeneous function g of U to X such that

$$\| f(u) - g(u) \| \leq \frac{1}{2} \psi(u)$$

for all $u \in U$. Moreover, g is given by the same formula as in Theorem 1.

A straightforward generalization of Găvruţă's theorem can be easily derived from this theorem. However, the novelty of our paper lies not in Theorem 3, but in the numerous auxiliary notions and results which make the subject rather plausible.

1. Additive groupoids and functions

Instead of groupoids, it is usually sufficient to consider only semigroups. However, several definitions on semigroups can be naturally extended to groupoids.

Definition 1.1. *If U is a groupoid and $u \in U$, then we define $1u = u$. Moreover, if $n \in \mathbb{N}$ such that nu is already defined, then we define*

$$(n+1)u = nu + u.$$

Definition 1.2. *A function f of one groupoid U to another V is called additive if*

$$f(u+v) = f(u) + f(v)$$

for all $u, v \in U$.

Theorem 1.3. *If f is a function of $\mathbb{N}$ to a semigroup U such that*

$$f(n+1) = f(n) + f(1)$$

for all $n \in \mathbb{N}$, then f is already additive.

Proof. If $m \in \mathbb{N}$, then $f(m+1) = f(m) + f(1)$. Moreover, if $n \in \mathbb{N}$ such that $f(m+n) = f(m) + f(n)$, then we also have

$$f\big(m + (n+1)\big) = f\big((m+n)+1\big) = f(m+n) + f(1) =$$
$$= \big(f(m)+f(n)\big)+f(1) = f(m)+\big(f(n)+f(1)\big) = f(m)+f(n+1).$$

□

Corollary 1.4. *If U is a semigroup, then for any $u \in U$ and $m, n \in \mathbb{N}$ we have*

$$(m+n)u = mu + nu .$$

Proof. Let $u \in U$, and define $f(n) = nu$ for all $n \in \mathbb{N}$. Then, the hypothesis of Theorem 1.3 is satisfied. Therefore, f is additive, and thus the required assertion is also true. □

Definition 1.5. *A function f of one groupoid U to another V is called n–homogeneous, for some $n \in \mathbb{N}$, if*

$$f(nu) = nf(u)$$

for all $u \in U$. Moreover, f is called $\mathbb{N}$-homogeneous if it is n–homogeneous for all $n \in \mathbb{N}$.

Theorem 1.6. *If f is an additive function of U to V, then f is $\mathbb{N}$–homogeneous.*

Proof. If $u \in U$, then $f(1u) = f(u) = 1f(u)$. Moreover, if $n \in \mathbb{N}$ such that $f(nu) = nf(u)$, then we also have

$$f\big((n+1)u\big) = f(nu+u) = f(nu)+f(u) = nf(u)+f(u) = (n+1)f(u).$$

□

Corollary 1.7. *If U is a semigroup, then for any $u \in U$ and $m, n \in \mathbb{N}$ we have*

$$(nm)u = n(mu).$$

Proof. If $u \in U$ and $f(u) = nu$ for all $n \in \mathbb{N}$, then by Corollary 1.4 f is additive. Therefore, by Theorem 1.6, f is $\mathbb{N}$–homogeneous. Thus, the required assertion is also true. □

Theorem 1.8. *If $(f_n)_{n=1}^{\infty}$ is a sequence of functions of a groupoid U to a commutative semigroup V such that f_1 is additive and*

$$f_{n+1} = f_n + f_1$$

for all $n \in \mathbb{N}$, then f_n is additive for all $n \in \mathbb{N}$.

Proof. If $n \in \mathbb{N}$ such that f_n is additive, then

$$\begin{aligned} f_{n+1}(u+v) &= f_n(u+v)+f_1(u+v) = \big(f_n(u)+f_n(v)\big)+\big(f_1(u)+f_1(v)\big) \\ &= \big(f_n(u)+f_1(u)\big)+\big(f_n(v)+f_1(v)\big) = f_{n+1}(u)+f_{n+1}(v) \end{aligned}$$

for all $u, v \in U$. Therefore, f_{n+1} is also additive. □

Corollary 1.9. *If U is a commutative semigroup, then for any $u, v \in U$ and $n \in \mathbb{N}$ we have*

$$n(u+v) = nu + nv .$$

Proof. Define $f_n(u) = n\,u$ for all $u \in U$ and $n \in \mathbb{N}$. Then, the hypotheses of Theorem 1.8 are satisfied. Therefore, f_n is additive for all $n \in \mathbb{N}$. Thus, the required assertion is also true. □

Remark 1.10. A commutative group U can be made a module over the ring $\mathbb{Z}$ of integers by using the definitions $0\,u = 0$ and $(-n)\,u = -(n\,u)$ for all $u \in U$ and $n \in \mathbb{N}$.

Moreover, U can be sometimes extended to a vector space V over the field $\mathbb{Q}$ of rationals by using the quotients $u/k = \{ (l, v) \in \mathbb{Z}\times U : \quad l\,u = k\,v \}$ with $u \in U$ and $0 \neq k \in \mathbb{Z}$.

Remark 1.11. In the sequel, $\mathbb{K}$ will denote any one of the number fields $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$. Moreover, we shall only consider vector spaces over $\mathbb{K}$.

Note that if X is a vector space then $1\,x = x$ and $(n+1)\,x = n\,x + x$, and moreover $0\,x = 0$ and $(-n)\,x = -n\,x$ for all $x \in X$ and $n \in \mathbb{N}$. Therefore, the two possible definitions for $k\,x$, with $k \in \mathbb{Z}$ and $x \in X$, coincide.

2. The associated Hyers sequences

According to Hyers's paper [20], we may naturally introduce the following

Definition 2.1. *If f is a function of a groupoid U to a vector space X, then we define*

$$f_n(u) = \frac{1}{2^n} f\left(2^n u\right)$$

for all $u \in U$ and $n \in \mathbb{N}$. The sequence $(f_n)_{n=1}^{\infty}$ is called the Hyers sequence associated with f.

Remark 2.2. In accordance with this definition, sometimes we shall also use the notation $f_0 = f$.

Hyers's sequences were generalized in 1991 by Th. M. Rassias [38] who replaced the number 2 by an integer $k > 2$.

By using the corresponding definitions, we can easily prove the following theorems.

Theorem 2.3. *If f and g are functions of a groupoid U to a vector space X over $\mathbb{K}$ and $\lambda \in \mathbb{K}$, then for any $n \in \mathbb{N}$ we have*

(1) $(f+g)_n = f_n + g_n$; (2) $(\lambda f)_n = \lambda f_n$.

Theorem 2.4. *If f and g are functions of a groupoid U to $\mathbb{R}$, then*

$$f \leq g \qquad \text{implies} \qquad f_n \leq g_n$$

for all $n \in \mathbb{N}$.

Theorem 2.5. *If f is a function and $(f_{(\nu)})_{\nu=1}^{\infty}$ is a sequence of functions of a groupoid U to a normed space X, then*

$$f = \lim_{\nu\to\infty} f_{(\nu)} \quad \textit{implies} \quad f_n = \lim_{\nu\to\infty} \big(f_{(\nu)}\big)_n$$

for all $n \in \mathbb{N}$.

Proof. Namely, if $\lim_{\nu\to\infty} f_{(\nu)}(u) = f(u)$ for all $u \in U$, then we also have

$$\lim_{\nu\to\infty} \big(f_{(\nu)}\big)_n(u) = \lim_{\nu\to\infty} \frac{1}{2^n} f_{(\nu)}\big(2^n u\big) = \frac{1}{2^n} f\big(2^n u\big) = f_n(u)$$

for all $u \in U$ and $n \in \mathbb{N}$. □

Now, as an immediate consequence of Theorems 2.3 and 2.5, we can also state

Corollary 2.6. *If f is a function and $(f_{(\nu)})_{\nu=1}^{\infty}$ is a sequence of functions of a groupoid U to a normed space X, then*

$$f = \sum_{\nu=1}^{\infty} f_{(\nu)} \quad \textit{implies} \quad f_n = \sum_{\nu=1}^{\infty} \big(f_{(\nu)}\big)_n$$

for all $n \in \mathbb{N}$.

Concerning Hyers's sequences, we can also easily prove the following theorems.

Theorem 2.7. *If f is a function of a groupoid U to a vector space X and g is a function of X to another vector space Y, then for any $n \in \mathbb{N}$ we have*

$$(g \circ f)_n = g_n \circ f_n .$$

Theorem 2.8. *If f is a function of a semigroup U to a vector space X, then for any $n, m \in \mathbb{N}$ we have*

$$\big(f_n\big)_m = f_{n+m} .$$

Proof. By Definition 2.1 and Corollary 1.7, it is clear that

$$\begin{aligned}\big(f_n\big)_m(u) &= \frac{1}{2^m} f_n\big(2^m u\big) = \frac{1}{2^m}\left(\frac{1}{2^n} f\Big(2^n\big(2^m u\big)\Big)\right) \\ &= \left(\frac{1}{2^m}\frac{1}{2^n}\right) f\Big(\big(2^n 2^m\big) u\Big) = \frac{1}{2^{m+n}} f\big(2^{n+m} u\big) = f_{n+m}(u)\end{aligned}$$

for all $u \in U$. □

In particular, it is also worth noticing that we also have

Corollary 2.9. *If f is a function of a semigroup U to a vector space X, then for any $u \in U$ and $n \in \mathbb{N}$ we have*

$$f_n(2u) = 2 f_{n+1}(u).$$

Proof. By Definition 2.1 and Theorem 2.8, it is clear that

$$f_n(2u) = 2\,\frac{1}{2} f_n(2u) = 2\big(f_n\big)_1(u) = 2 f_{n+1}(u).$$

□

Remark 2.10. If f is a function of a groupoid U to a vector space X, then by the corresponding definitions we also have $f_0(2u) = 2 f_1(u)$ for all $u \in U$.

As a useful consequence of the corresponding definitions, we also have

Theorem 2.11. *If f is a function of a groupoid U to a vector space X, then the following assertions are equivalent:*

(1) $f_1 = f$; (2) f *is* 2*–homogeneous*.

Proof. For any $u \in U$, we have

$$f_1(u) = f(u) \iff \frac{1}{2} f(2u) = f(u) \iff f(2u) = 2 f(u).$$

□

Hence, by using Theorem 2.8, we can easily get the following

Theorem 2.12. *If f is a 2-homogeneous function of a semigroup U to a vector space X, then $f_n = f$ for all $n \in \mathbb{N}$.*

Proof. By Theorems 2.8 and 2.11, for any $n \in \mathbb{N}$, we have

$$f_{n+1} = f_{1+n} = \big(f_1\big)_n = f_n.$$

Hence, by induction, it is clear that $f_n = f_1 = f$ also holds. □

Remark 2.13. Note that this theorem can be applied to additive functions since they are in particular 2–homogeneous.

Moreover, it is also worth noticing that if X is a normed space and $p(x) = \| x \|$ for all $x \in X$, then by the above theorem $p_n = p$ for all $n \in \mathbb{N}$.

3. Regular and normal functions

Definition 3.1. *A function f of a groupoid U to a normed space X is called regular if the limit*

$$g_f(u) = \lim_{n\to\infty} f_n(u)$$

exists in X for all $u \in U$. In particular, f is called null-regular if $g_f(u) = 0$ for all $u \in U$.

Remark 3.2. In addition, the function f may be naturally called uniformly regular if the sequence (f_n) is uniformly convergent.

By using the above definition, from the corresponding results of Section 3 we can easily derive the following theorems.

Theorem 3.3. *If f and h are regular functions of a groupoid U to a normed space X over $\mathbb{K}$ and $\lambda \in \mathbb{K}$, then $f+h$ and λf are also regular functions of U to X, and*

(1) $g_{f+h} = g_f + g_h$; (2) $g_{\lambda f} = \lambda g_f$.

Theorem 3.4. *If f and h are regular functions of a groupoid U to $\mathbb{R}$, then*

$$f \le h \quad \textit{implies} \quad g_f \le g_h .$$

Theorem 3.5. *If f is a regular function of a semigroup U to a normed space X, then for any $n \in \mathbb{N}$ the function f_n is also regular and*

$$g_{f_n} = g_f .$$

Proof. If $u \in U$, then by Definition 3.1 and Theorem 2.8 we have

$$g_{f_n}(u) = \lim_{m\to\infty} \big(f_n\big)_m(u) = \lim_{m\to\infty} f_{n+m}(u) = \lim_{k\to\infty} f_k(u) = g_f(u) .$$

□

Theorem 3.6. *If f is a regular function of a semigroup U to a normed space X, then g_f is 2–homogeneous.*

Proof. If $u \in U$, then by Definition 3.1 and Corollary 2.9 we have

$$g_f(2u) = \lim_{n\to\infty} f_n(2u) = \lim_{n\to\infty} 2 f_{n+1}(u) = 2 \lim_{k\to\infty} f_k(u) = 2 g_f(u) .$$

□

Theorem 3.7. *If f and g are functions of a semigroup U to a normed space X and φ is a function of U to $\mathbb{R}$ such that*

(1) *$\| f(u) - g(u) \| \le \varphi(u)$ for all $u \in U$;*

(2) *g is 2–homogeneous and φ is null-regular ;*

then f *is regular and* $g_f = g$ *.*

Proof. Define $p(x) = \| x \|$ for all $x \in X$. Then, for any $u \in U$, we have

$$\big(p \circ (f - g)\big)(u) = \big\| f(u) - g(u) \big\| \leq \varphi(u).$$

Hence, by Theorems 2.12, 2.3, 2.7 and 2.4, it is clear that

$$\big\| f_n(u) - g(u) \big\| = \big(p \circ (f_n - g)\big)(u) = \big(p_n \circ \big(f_n - g_n\big)\big)(u) =$$
$$= \big(p_n \circ (f - g)_n\big)(u) = \big(p \circ (f - g)\big)_n(u) \leq \varphi_n(u)$$

for all $n \in \mathbb{N}$. This implies that $\lim_{n\to\infty} \big\| f_n(u) - g(u) \big\| = 0$, and thus

$$g(u) = \lim_{n\to\infty} f_n(u) = g_f(u).$$

□

Corollary 3.8. *If* f *is a function of a semigroup* U *to a normed space* X *and* φ *is a mull-regular function of* U *to* $\mathbb{R}$, *then there exists at most one* 2*–homogeneous function* g *of* U *to* X *such that*

$$\| f(u) - g(u) \| \leq \varphi(u)$$

for all $u \in U$ *.*

Definition 3.9. *A regular function* f *of a groupoid* U *to a normed space* X *is called normal if the sum*

$$S_f(u) = \sum_{n=0}^{\infty} \big(f_n(u) - g_f(u) \big)$$

exists in X *for all* $u \in U$ *. In particular, a normal function is called null-normal if it is null-regular.*

Remark 3.10. In addition, the function f may be naturally called uniformly normal if the if the series $\sum (f_n - g_f)$ is uniformly convergent.

Theorem 3.11. *If* f *is a normal function of a semigroup* U *to a normed space* X, *then* S_f *is a null-regular function of* U *to* X *.*

Proof. By Definition 3.9, we have

$$S_f = \sum_{i=0}^{\infty} \big(f_i - g_f \big).$$

Hence, by using Corollary 2.6 and Theorems 2.3, 2.8, 3.6 and 2.12, we can infer that

$$\big(S_f\big)_n = \sum_{i=0}^{\infty} \big(f_i - g_f \big)_n = \sum_{i=0}^{\infty} \big((f_i)_n - (g_f)_n \big)$$

$$= \sum_{i=0}^{\infty} \big(f_{i+n} - g_f \big) = \sum_{k=n}^{\infty} \big(f_k - g_f \big) .$$

This implies that

$$\lim_{n\to\infty} \big(S_f \big)_n = \lim_{n\to\infty} \sum_{k=n}^{\infty} \big(f_k - g_f \big) = 0 .$$

Therefore, the required assertion is also true. □

Remark 3.12. If in particular f is uniformly normal, then the above proof also shows that S_f is uniformly null-regular.

4. Approximately homogeneous functions

Definition 4.1. *A function f of a groupoid U to a normed group X is called φ–approximately n–homogeneous, for some $n \in \mathbb{N}$ and $\varphi \in \mathbb{R}^U$, if*

$$\big\| f(n\,u) - n\,f(u) \big\| \leq \varphi(u)$$

for all $u \in U$.

Remark 4.2. Now, f may be called ε–approximately n–homogeneous, for some $\varepsilon \geq 0$, if it is φ–approximately n–homogeneous with $\varphi = U \times \{\varepsilon\}$.

A simple reformulation of the $n = 2$ particular case of above definition yields the following

Theorem 4.3. *If f is a function of a groupoid U to a normed space X and $\varphi \in \mathbb{R}^U$, then the following assertions are equivalent:*

(1) *f is φ-approximately 2–homogeneous;*
(2) $\big\| f_1(u) - f(u) \big\| \leq \frac{1}{2}\,\varphi(u)$ *for all* $u \in U$.

Proof. For any $u \in U$, we have

$$\big\| f(2\,u) - 2\,f(u) \big\| \leq \varphi(u)$$
$$\Longleftrightarrow \ \left\| \frac{1}{2} f(2\,u) - f(u) \right\| \leq \frac{1}{2}\,\varphi(u) \ \Longleftrightarrow \ \big\| f_1(u) - f(u) \big\| \leq \frac{1}{2}\,\varphi(u) .$$

□

As an extension of the implication (1) $\Longrightarrow$ (2), we can prove the following

Theorem 4.4. *If f is a φ–approximately 2–homogeneous function of a semigroup U to a normed space X, then for any $u \in U$ and $n \in \{0\} \cup \mathbb{N}$ we have*

$$\big\| f_{n+1}(u) - f_n(u) \big\| \leq \frac{1}{2}\,\varphi_n(u) .$$

Proof. Define $p(x) = \| x \|$ for all $x \in X$. Then, by Theorem 2.12, we have $p_n = p$. Moreover, by Theorem 4.3, we also have

$$\big(p \circ (f_1 - f) \big)(u) = \big\| f_1(u) - f(u) \big\| \leq \frac{1}{2} \varphi(u).$$

Now, by Theorems 2.8, 2.3, 2.7 and 2.4, it is clear that

$$\big\| f_{n+1}(u) - f_n(u) \big\| = \big(p \circ (f_{n+1} - f_n) \big)(u) = \big(p \circ \big((f_1)_n - f_n \big) \big)(u)$$
$$= \big(p_n \circ (f_1 - f)_n \big)(u) = \big(p \circ (f_1 - f) \big)_n (u) \leq \Big(\frac{1}{2} \varphi \Big)_n (u) = \frac{1}{2} \varphi_n(u).$$

□

Corollary 4.5. *If f is a φ–approximately 2–homogeneous function of a semigroup U to a normed space X, then f_n is φ_n–approximately 2–homogeneous for all $n \in \mathbb{N}$.*

Proof. By Theorems 2.8 and 4.4, we have

$$\big\| (f_n)_1(u) - f_n(u) \big\| = \big\| f_{n+1}(u) - f_n(u) \big\| \leq \frac{1}{2} \varphi_n(u)$$

for all $u \in U$ and $n \in \mathbb{N}$. Therefore, by Theorem 4.3, the required assertion is also true. □

By using Theorem 4.4, we can also easily prove the following more general

Theorem 4.6. *If f is a φ-approximately 2–homogeneous function of a semigroup U to a normed space X, then for any $u \in U$, $n \in \mathbb{N}$ and $k \in \{0\} \cup \mathbb{N}$ we have*

$$\big\| f_{n+k}(u) - f_k(u) \big\| \leq \frac{1}{2} \sum_{i=k}^{n+k-1} \varphi_i(u).$$

Proof. By using Theorem 4.4, we can easily see that

$$\big\| f_{n+k}(u) - f_k(u) \big\| = \Big\| \sum_{j=1}^{n} \big(f_{j+k}(u) - f_{j+k-1}(u) \big) \Big\|$$
$$\leq \sum_{j=1}^{n} \| f_{j+k}(u) - f_{j+k-1}(u) \| \leq \sum_{j=1}^{n} \frac{1}{2} \varphi_{j+k-1}(u) = \frac{1}{2} \sum_{i=k}^{n+k-1} \varphi_i(u).$$

□

The $k = 0$ particular case of this theorem immediately yields the following

Corollary 4.7. *If f is a φ–approximately 2–homogeneous function of a semigroup U to a normed space X, then for any $u \in U$ and $n \in \mathbb{N}$ we have*

$$\| f_n(u) - f(u) \| \leq \frac{1}{2} \sum_{i=0}^{n-1} \varphi_i(u) .$$

5. The regularity of approximately homogeneous functions

Theorem 5.1. *If f is a φ–approximately 2–homogeneous function of a semigroup U to a normed space X and φ is null-normal, then for any $u \in U$ and $n \in \mathbb{N}$ we have*

$$\overline{\lim_{n \to \infty}} \| f_n(u) - f(u) \| \leq \frac{1}{2} S_\varphi(u) .$$

Proof. From Corollary 4.7, we can easily see that

$$\overline{\lim_{n \to \infty}} \| f_n(u) - f(u) \| \leq \overline{\lim_{n \to \infty}} \frac{1}{2} \sum_{i=0}^{n-1} \varphi_i(u)$$

$$= \frac{1}{2} \lim_{n \to \infty} \sum_{i=0}^{n-1} \varphi_i(u) = \frac{1}{2} \sum_{n=0}^{\infty} \varphi_n(u) = \frac{1}{2} S_\varphi(u) .$$

□

Corollary 5.2. *If f is a regular φ-approximately 2–homogeneous function of a semigroup U to a normed space X and φ is null-normal, then for any $u \in U$ we have*

$$\| f(u) - g_f(u) \| \leq \frac{1}{2} S_\varphi(u)$$

Proof. By the regularity of f and Theorem 5.1, it is clear that

$$\|f(u) - g_f(u)\| = \lim_{n \to \infty} \| f(u) - f_n(u) \| = \overline{\lim_{n \to \infty}} \| f_n(u) - f(u) \| \leq \frac{1}{2} S_\varphi(u) .$$

□

Theorem 5.3. *If f is a φ-approximately 2–homogeneous function of a semigroup U to a Banach space X and φ is null-normal, then f is regular.*

Proof. If $u \in U$, then by Theorem 4.6 for any $k, l \in \mathbb{N}$, with $k < l$, we have

$$\| f_l(u) - f_k(u) \| \leq \frac{1}{2} \sum_{i=k}^{l-1} \varphi_i(u) \leq \frac{1}{2} \sum_{i=k}^{\infty} \varphi_i(u) .$$

Moreover, by null-normality of φ, we also have

$$\sum_{i=0}^{\infty} \varphi_i(u) < +\infty , \qquad \text{and hence} \qquad \lim_{k \to \infty} \sum_{i=k}^{\infty} \varphi_i(u) = 0 .$$

Therefore, for each $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for any $k \in \mathbb{N}$, with $k_0 \leq k$, we have

$$\sum_{i=k}^{\infty} \varphi_i(u) < \varepsilon .$$

Hence, it is clear that for any $k, l \in \mathbb{N}$, with $k_0 \leq k, l$ and $k \leq l$, we also have

$$\| f_l(u) - f_k(u) \| \leq \frac{1}{2} \sum_{i=k}^{\infty} \varphi_i(u) < \frac{1}{2} \varepsilon < \varepsilon .$$

Therefore, $\big(f_n(u) \big)_{n=1}^{\infty}$ is a Cauchy sequence in X. Thus, by the completeness of X, the limit

$$g_f(u) = \lim_{n\to\infty} f_n(u)$$

exists in X. This shows that f is regular. □

Remark 5.4. If in particular φ is uniformly null-normal, then we can also prove that f is uniformly regular.

The above results allow us to easily establish the following counterpart of a straightforward generalization of Găvruţă's theorem [12].

Theorem 5.5. *If f is a φ-approximately 2–homogeneous function of a semigroup U to a Banach space X and φ is null-normal, then there exists a unique 2–homogeneous function g of U to X such that*

$$\| f(u) - g(u) \| \leq \frac{1}{2} S_\varphi(u)$$

for all $u \in U$. Moreover, g is the pointwise limit of the Hyers sequence associated with f.

Proof. By Theorem 5.3, f is regular. Thus, by Definition 3.1, the limit

$$g_f(u) = \lim_{n\to\infty} f_n(u)$$

exists for all $u \in U$.

Moreover, by Corollary 5.2, we have

$$\| f(u) - g_f(u) \| \leq \frac{1}{2} S_\varphi(u)$$

for all $u \in U$. Furthermore, by Theorem 3.6, g_f is 2–homogeneous.

Finally, by Theorem 3.11, S_φ is null–regular. Hence, $(1/2)\, S_\varphi$ is also null-regular. Therefore, by Corollary 3.8, there exists at most one function g of U to X such that

$$\| f(u) - g(u) \| \leq \frac{1}{2} S_\varphi(u)$$

for all $u \in U$. Thus, the proof is complete. □

Remark 5.6. If in particular φ is uniformly null-normal, then by Remark 6.4 we can also state that g is the uniform limit of the associated Hyers sequence.

6. Approximately additive functions

Definition 6.1. *A function f of a groupoid U to a normed group X is called Φ–approximately additive, for some $\Phi \in \mathbb{R}^{U^2}$, if*

$$\| f(u+v) - \big(f(u) + f(v) \big) \| \leq \Phi(u, v)$$

for all $u, v \in U$.

Remark 6.2. Now, f may be called ε–approximately additive, for some $\varepsilon \geq 0$, if it is Φ–approximately additive with $\Phi = U^2 \times \{ \varepsilon \}$.

Theorem 6.3. *If f is an Φ–approximately additive function of a groupoid U to a normed group X, $1 < n \in \mathbb{N}$ and*

$$\varphi(u) = \sum_{k=1}^{n-1} \Phi(ku, u)$$

for all $u \in U$, then f is φ–approximately n–homogeneous.

Proof. By Definition 6.1, we evidently have

$$\| f(2u) - 2f(u) \| = \| f(u+u) - \big(f(u) + f(u) \big) \| \leq \Phi(u, u)$$

for all $u \in U$. Therefore, the required assertion is true for $n = 2$.

Moreover, if the required assertion is true for some $2 \geq n$, then we can easily see that

$$\begin{aligned}
&\| f\big((n+1)u\big) - (n+1) f(u) \| \\
&= \| f(nu+u) - \big(n f(u) + f(u) \big) \| = \| f(nu+u) - f(u) - n f(u) \| \\
&= \| f(nu+u) - f(u) - f(nu) + f(nu) - n f(u) \| \\
&= \| f(nu+u) - \big(f(nu) + f(u) \big) + f(nu) - n f(u) \| \\
&\leq \| f(nu+u) - \big(f(nu) + f(u) \big) \| + \| f(nu) - n f(u) \| \\
&\leq \Phi(nu, u) + \sum_{k=1}^{n-1} \Phi(ku, u) = \sum_{k=1}^{n} \Phi(ku, u)
\end{aligned}$$

for all $u \in U$. Therefore, the required assertion is also true for $n+1$. □

The $n = 2$ particular case of the above theorem gives the following

Corollary 6.4. *If* f *is an* Φ*–approximately additive function of a groupoid* U *to a normed group* X*, and*

$$\varphi(u) = \Phi(u, u)$$

for all $u \in U$*, then* f *is* φ*–approximately* 2*–homogeneous.*

Remark 6.5. Therefore, the results of Sections 4 and 5, can be immediately applied to Φ–approximately additive functions.

Concerning approximately additive functions, we can also easily prove the following

Theorem 6.6. *If* f *is an* Φ*–approximately additive function of a commutative semigroup* U *to a normed space* X*, then* f_n *is* Φ_n*–approximately additive for all* $n \in \mathbb{N}$*.*

Proof. If $n \in \mathbb{N}$ and $u, v \in U$, then by the corresponding definitions and Corollary 1.9 it is clear that

$$\begin{aligned}
&\| f_n(u+v) - \big(f_n(u) + f_n(v) \big) \| \\
&\quad = \left\| \frac{1}{2^n} f\big(2^n(u+v)\big) - \Big(\frac{1}{2^n} f\big(2^n u\big) + \frac{1}{2^n} f\big(2^n v\big) \Big) \right\| \\
&\quad\quad = \frac{1}{2^n} \| f\big(2^n u + 2^n v\big) - \big(f\big(2^n u\big) + f\big(2^n v\big)\big) \| \\
&\quad\quad\quad \le \frac{1}{2^n} \Phi\big(2^n u, 2^n v\big) = \frac{1}{2^n} \Phi\big(2^n (u, v)\big) = \Phi_n(u, v).
\end{aligned}$$

□

From the above theorem, we can easily derive the following counterpart of Theorem 3.6.

Theorem 6.7. *If* f *is a regular* Φ*–approximately additive function of a commutative semigroup* U *to a normed space* X *and* Φ *is null-regular, then* g_f *is additive.*

Proof. If $u, v \in U$, then by the regularity of f, Theorem 6.6 and the null-regularity of Φ we have

$$\begin{aligned}
&\| g_f(u+v) - \big(g_f(u) + g_f(v) \big) \| \\
&\quad = \lim_{n\to\infty} \| f_n(u+v) - \big(f_n(u) + f_n(v) \big) \| \le \lim_{n\to\infty} \Phi_n(u, v) = 0,
\end{aligned}$$

and thus $g_f(u+v) = g_f(u) + g_f(v)$ is also true. □

Now, as an immediate consequence of our former results, we can also state the following straightforward extension of Găvruţă's theorem [12].

Theorem 6.8. *If f is a Φ-approximately additive function of a commutative semigroup U to a Banach space X, Φ is null-regular and the function φ, defined by*

$$\varphi(u) = \Phi(u, u)$$

for all $u \in U$, is null-normal, then there exists a unique additive function g of U to X such that

$$\| f(u) - g(u) \| \leq \frac{1}{2} S_{\varphi}(u)$$

for all $u \in U$. Moreover, g is the pointwise limit of the Hyers sequence associated with f.

Proof. By Corollary 6.4, f is, in particular, φ–approximately 2–homogeneous. Therefore, by Theorem 5.5, there exists a unique 2–homogeneous function g of U to X such that

$$\| f(u) - g(u) \| \leq \frac{1}{2} S_{\varphi}(u)$$

for all $u \in U$. Moreover, g is the pointwise limit of the Hyers sequence associated with f, and thus f is regular and $g = g_f$. Furthermore, by Theorem 6.7, g_f is additive. Hence, it is clear that the required assertions are also true. □

Remark 6.9. If in particular φ is uniformly null-normal, then by Remark 5.6 we can also state that g is the uniform limit of the associated Hyers sequence.

Acknowledgment. The second author is greatly indebted to Th. M. Rassias for his valuable suggestions and continuous encouragement, and also for sending reprints of several papers which were not available in our library.

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan, 2 (1950), 64–66.

[2] R. Badora, *Note on the superstability of the Cauchy functional equation*, Publ. Math. Debrecen, 57 (2000), 421–424.

[3] R. Badora, *On the Hahn–Banach theorem for groups*, Arch. Math., 86 (2006), 517–528.

[4] K. Baron and P. Volkmann, *On functions close to homomorphisms between square symmetric structures*, Seminar LV, 14 (2002), 1–12 (electronic).

[5] D. G. Bourgin, *Classes of transformations and bordering transformations*, Bull. Amer. Math. Soc., 57 (1951), 223–237.

[6] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, London, 2002.

[7] V. A. Faĭziev, Th. M. Rassias and P. K. Sahoo, *The space of* (ψ, γ)*–additive mappings on semigroups*, Trans. Amer. Math. Soc., 354 (1985), 4455-4472.
[8] G. L. Forti, *The stability of homomorphisms and amenability, with applications to functional equations*, Abh. Math. Sem. Uiv. Hamburg, 57 (1987), 215–226.
[9] G. L. Forti, *Hyers–Ulam stability of functional equations in several variables*, Aequationes Math., 50 (1995), 143–190.
[10] G. L. Forti *An existence and stability theorem for a class of functional equations*, Stochastica, 4 (1980), 23–30.
[11] Z. Gajda, *On stability of additive mappings*, Internat J. Math. Sci., 14 (1991), 431–434.
[12] P. Găvruţă, *A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings*, J. Math. Anal. Appl., 184 (1994), 431–436.
[13] P. Găvruţă, *On a problem of G. Isac and Th. M. Rassias concerning the stability of mappings*, J. Math. Anal. Appl., 261 (2001), 543–553.
[14] P. Găvruţă, M. Hosszu, D. Popescu and C. Căprău, *On the stability of Mappings and an answer to a problem of Th. M. Rassias*, Ann. Math. Blaise Pascal, 2 (1995), 55–60.
[15] R. Ger, *The singular case in the stability behaviour of linear mappings*, Grazer Math. Ber., 316 (1992), 59–70.
[16] R. Ger, *A survey of recent results on stability of functional equations*, Proceedings of the 4th International Conference on Functional Equations and Inequalities Pedagogical University of Cracow, 1994, 5–36.
[17] R. Ger and P. Volkmann, *On sums of linear and bounded mappings*, Abh. Math. Sem. Univ. Hamburg, 68 (1998), 103–108.
[18] A. Gilányi, Z. Kaiser and Zs. Páles, *Estimates to the stability of functional equations*, Aequationes Math., 73 (2007), 125–143.
[19] A. Grabiec *The generalized Hyers–Ulam stability of a class of functional equations*, Publ. Math. Debrecen, 48 (1996), 217–235.
[20] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A, 27 (1941), 222–224.
[21] D. H. Hyers, G. Isac and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Boston, 1998
[22] D. H. Hyers and Th. M. Rassias, *Approximate homomorphisms*, Aequationes Math., 44 (1992), 125–153.
[23] G. Isac and Th. M. Rassias, *On the Hyers-Ulam stability of* ψ *additive mappings*, J. Approx. Theory, 72 (1993), 131–137.
[24] G. Isac and Th. M. Rassias,*Functional inequalities for approximately additive mappings*, In: Th. M. Rassias and Jo. Tabor (Eds.), Stability of mappings of Hyers–Ulam type, Hadronic Press Collect. Orig. Art., Palm Harbor, FL, 1994, 117–125.
[25] S.-M. Jung, *Hyers–Ulam–Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, 2001.
[26] Z. Kaiser and Zs. Páles, *An example of a stable functional equation when the Hyers method does not work*, J. Ineq. Pure Appl. Math., 6 (2005), 1–11.
[27] G. H. Kim, *On the stability of functional equations with square-symmetric operation*, Math. Ineq. Appl., 4 (2001), 257–266.
[28] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*, Polish Sci. Publ. and Univ. Ślaski, Warszawa, 1985.
[29] Y.-H. Lee and K.-W. Jun, *On the stability of approximately additive mappings*, Proc. Amer. Math. Soc., 128 (1999), 1361–1369.

[30] Zs. Páles, *Generalized stability of the Cauchy functional equations*, Aequationes Math., 56 (1998), 222–232.
[31] Zs. Páles, *Hyers–Ulam stability of the Cauchy functional equation on square-symmetric groupoids*, Publ. Math. Debrecen, 58 (2001), 651–666.
[32] Zs. Páles, P. Volkmann and R. D. Luce, *Hyers–Ulam stability of functional equations with a square-symmetric operation*, Proc. Natl. Acad. Sci. USA, 95 (1998), 12772–12775.
[33] C.-G. Park, *On the stability of the linear mapping in Banach modules*, J. Math. Anal. Appl., 275 (2002), 711–720.
[34] Gy. Pólya and G. Szegő, *Problems and Theorems in Analysis I*, Springer Verlag, Berlin, 1972.
[35] J. M. Rassias, *On approximation of approximately linear mappings by linear mappings*, J. Funct. Anal., 46 (1982), 126–130.
[36] J. M. Rassias, *Solution of a problem of Ulam*, J. Approximation Theory, 57 (1989), 268–273.
[37] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc., 72 (1978), 297–300.
[38] Th. M. Rassias, *On a modified Hyers-Ulam sequence*, J. Math. Anal. Appl., 158 (1991), 106–113.
[39] Th. M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Applicandae Math., 62 (2000), 23–130.
[40] Th. M. Rassias, *The problem of S. M. Ulam for approximately multiplicative mappings*, J. Math. Anal. Appl., 246 (2000), 353–378.
[41] Th. M. Rassias *On the stability of functional equations originated by a problem of Ulam*, Mathematica (Cluj), 44 (2002), 39–75.
[42] Th. M. Rassias and P. Semrl, *On the behavior of mappings which do not satisfy Hyers–Ulam stability*, Proc. Amer. Math. Soc., 114 (1992), 989–993.
[43] Th. M. Rassias and P. Semrl, *On the Hyers–Ulam stability of linear mappings*, J. Math. Anal. Appl., 173 (1993), 325–338.
[44] Th. M. Rassias and J. Tabor, *On approximately additive mappings in Banach spaces*, In: Th. M. Rassias and Jo. Tabor (Eds.), Stability of mappings of Hyers–Ulam type, Hadronic Press Collect. Orig. Art., Palm Harbor, FL, 1994, 123–137.
[45] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory, 4 (2003), 91–96.
[46] J. Rätz, *On approximately additive mappings*, In: E. F. Beckenbach (Ed.), General Inequalities 2 (Oberwolfach, 1978), Int. Ser. Num. Math. (Birkhäuser, Basel-Boston), 47 (1980), 233–251.
[47] J. Špakula and P. Zlatoš *Almost homomorphisms of compact groups*, Illinois J. Math., 48 (2004), 1183–1189.
[48] Á. Száz *An extension of an additive selection theorem of Z. Gajda and R. Ger*, Tech. Rep., Inst. Math., Univ. Debrecen, 8 (2006), 1–24.
[49] Á. Száz, *A instructive treatment of a generalization of Hyers's stability theorem*, In: Th. M. Rassias and D. Andrica (Eds.), Inequalities and Applications, Cluj University Press, Romania, 2008, 245–271.
[50] Á. Száz *Applications of relations and relators in the extensions of stability theorems for homogeneous and additive functions*, Aust. J. Math. Anal. Appl., 6 (2009), 1-66.
[51] L. Székelyhidi, *Remark 17*, Aequationes Math., 29 (1985), 95–96.

[52] L. Székelyhidi, *Ulam's problem, Hyers's solution – and to where they led*, In: Th. M. Rassias and Jo. Tabor (Eds.), Functional Equations and Inequalities, Math. Appl., 518, Kluwer Acad. Publ., Dordrecht, 2000, 259–285.

[53] J. Tabor and J. Tabor, *Homogeneity is superstable*, Publ. Math. Debrecen, 45 (1994), 123–130.

[54] J. Tabor and J. Tabor, *Restricted stability and shadowing*, Publ. Math. Debrecen, 73 (2008), 49–58.

(Received: June 24, 2009)
(Revised: July 19, 2009)

Institute of Mathematics
University of Debrecen
H-4010 Debrecen, Pf. 12, Hungary
E–mail: gselmann@math.klte.hu
szaz@math.klte.hu