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ON A DISCRETE HILBERT TYPE INEQUALITY WITH NON–HOMOGENEOUS KERNEL

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Abstract. New extensions are given for the discrete Hilbert type inequality with non–homogeneous kernel. By this, recently published results by Pogány have been improved by a Hilbert type inequalities in homogeneous kernel case derived by Krnić and Pečarić. Mathematical tools also used are the Dirichlet series Laplace-integral representation and the classical Hölder inequality.

1. Introduction

Let us consider a famous discrete Hilbert inequality (or double series theorem). Let ℓ_p be the space of all complex sequences $\mathbf{x} = (x_n)_{n=1}^\infty$ with the finite norm $\|\mathbf{x}\|_p := \left(\sum_{n=1}^\infty |x_n|^p\right)^{1/p}$ endowed. Let $\mathbf{a} = (a_n)_{n=1}^\infty \in \ell_p,\ \mathbf{b} = (b_n)_{n=1}^\infty \in \ell_q$ be nonnegative sequences and $1/p + 1/q = 1,\ p > 1$. Then

$$\sum_{m,n=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \|\mathbf{a}\|_p \|\mathbf{b}\|_q\,, \tag{1}$$

where the constant $\pi/\sin(\pi/p)$ is the best possible [1, p. 253].

This classical inequality produced a large interest among mathematicians and recently become one of the most frequently investigated research topics. The standard way in deriving Hilbert's inequality is to apply the Hölder inequality to suitably transformed Hilbert type double sum expression, i.e. to the bilinear form

$$\mathfrak{H}_K^{\mathbf{a},\mathbf{b}} := \sum_{m,n=1}^\infty K(m,n)\, a_m b_n \tag{2}$$

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where $\mathbf{a}, \mathbf{b}$ are nonnegative. $K(\cdot,\cdot)$ we call kernel function (of the double series (2)).

So, to obtain discrete Hilbert type inequalities (or double series theorems) one derives sharp upper bounds for $\mathfrak{H}_K^{\mathbf{a},\mathbf{b}}$ in terms of weighted ℓ_p-norms of $\mathbf{a}$, $\mathbf{b}$.

In this article we make use an approach similar to one in [4], but instead of Mathieu–series techniques applied in [4], we use an inequality by Krnić *et al.* [2, §4, Corollary 2]. Namely, in that paper it has been proved that with $p > 1, p^{-1} + q^{-1} = 1$; $\alpha, \beta > 0$ and with a real parameter γ satisfying

$$\Big(\frac{1-\nu}{p}\Big)_+ - \frac{1}{q} < \gamma < \frac{1}{p} + \Big(\frac{1-\nu}{q}\Big)_+ , \tag{3}$$

the following inequality holds

$$\begin{aligned}\sum_{m,n=1}^{\infty} \frac{a_m b_n}{(m^\alpha + n^\beta)^\nu} &\le \alpha^{-1/q}\beta^{-1/p} L^\star \left(\sum_{m=1}^{\infty} m^{(1-\nu)\alpha+(\alpha-1)(1-(1+\gamma)p)} a_m^p\right)^{1/p} \\ &\quad \times \left(\sum_{n=1}^{\infty} n^{(1-\nu)\beta+(\beta-1)(1-(1-\gamma)q)} b_n^q\right)^{1/q}, \end{aligned}\tag{4}$$

where the constant

$$L^\star = \mathrm{B}\Big(\frac{\nu-2+p}{p} + \gamma, \frac{\nu-2+q}{q} - \gamma\Big) \tag{5}$$

is the best possible. The condition (3) has two cases whether γ is less then 1 or greater or equal to 1. In the first case the condition comes to

$$\frac{1}{p} - 1 < \gamma < \frac{1}{p},$$

and in the second case

$$-1 < \gamma - \frac{1}{p} < 0.$$

Here, and in what follows $\mathcal{D}_\lambda(x)$ denotes the Laplace integral of the Dirichlet series [5, §5]

$$\mathcal{D}_\lambda(x) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n x} = x\int_0^\infty e^{-xt}\Big(\sum_{n=1}^{[\lambda^{-1}(t)]} a_n\Big)\,\mathrm{d}t \tag{6}$$

for positive monotone increasing $(\lambda_n)_{n=1}^\infty$ such that

$$\lim_{x\to\infty} \lambda(x) = \lim_{x\to\infty} \rho(x) = \infty\,. \tag{7}$$

The internal sum we find using the suitable form of the Euler–Maclaurin summation formula: Consider the real valued function $x \mapsto a(x)$ and suppose that $a \in \mathrm{C}^1[k,m]$, $k,m \in \mathbb{Z}$, $k < m$. Then by the classical Euler–Maclaurin summation formula we have

$$\sum_{j=k}^{m} a_j = \int_k^m a(x)\mathrm{d}x + \frac{1}{2}(a_k + a_m) + \int_k^m \left(x - [x] - \frac{1}{2}\right) a'(x)\mathrm{d}x.$$

Introducing the operator

$$\mathfrak{d}_x := 1 + \{x\}\frac{\partial}{\partial x},$$

with obvious transformations we get the desired condensed form of the Euler–Maclaurin summation formula

$$\sum_{j=k+1}^{\ell} a_j = \int_k^\ell \mathfrak{d}_x a(x)\,\mathrm{d}x \qquad (k,\ell \in \mathbb{Z}). \tag{8}$$

2. Main result

We are ready now to formulate our main result.

Theorem 1. *Suppose $p, r > 1$, $p^{-1} + q^{-1} = 1$, $r^{-1} + s^{-1} = 1$, $\mu > 0$, $\mathbf{a}, \mathbf{b}$ are nonnegative sequences such that*

$$\big(n^{B_1/r} a_n^p\big)_{n=1}^\infty \in \ell_r, \quad B_1 := \alpha(1-\nu) + (\alpha - 1)\big(1 - (1+\gamma)r\big)\,,$$
$$\big(n^{B_2/s} b_n^p\big)_{n=1}^\infty \in \ell_s, \quad B_2 := \beta(1-\nu) + (\beta - 1)\big(1 - (1-\gamma)s\big)\,;$$

$\alpha, \beta > 0$, γ satisfies (3) and λ, ρ are positive monotone increasing functions satisfying (7), while $\nu > 0$ is such that the series

$$\sum_{m,n=1}^\infty \frac{(a_m b_n)^p}{(m^\alpha + n^\beta)^\nu} \tag{9}$$

converges. Then

$$\sum_{m,n=1}^\infty \frac{a_m b_n}{(\lambda_m + \rho_n)^\mu} < \mathsf{C}_{p,r}^{\mu,\alpha,\beta}(\lambda,\rho)\, \|\mathbf{n}^{B_1/r}\mathbf{a}^p\|_r^{1/p} \|\mathbf{n}^{B_2/s}\mathbf{b}^p\|_s^{1/p}\,, \tag{10}$$

where

$$\mathsf{C}_{\nu,p,r}^{\mu,\alpha,\beta}(\lambda,\rho) := \big(\alpha^{1/s}\beta^{1/r}\big)^{-p}\, \mathrm{B}^{1/p}\Big(\frac{\nu - 2 + r}{r} + \gamma, \frac{\nu - 2 + s}{s} - \gamma\Big)$$
$$\times\, (\mu q(\mu q + 1))^{1/q} \left(\int_{\lambda_1}^\infty \int_{\rho_1}^\infty \frac{\int_0^{[\lambda^{-1}(t)]}\int_0^{[\rho^{-1}(u)]} P(v,w)\,\mathrm{d}v\mathrm{d}w}{(t+u)^{\mu q + 2}}\,\mathrm{d}t\mathrm{d}u \right)^{1/q}$$

and

$$P(v,w) = \sum_{j=0}^{\infty} \binom{\nu(q-1)}{j} \mathfrak{d}_v(v^{\alpha j})\, \mathfrak{d}_w(w^{\beta(\nu(q-1)-j)})\,. \tag{11}$$

Moreover, $\mathsf{C}_{\nu,p,r}^{\mu,\alpha,\beta}(\lambda,\rho)$ *is the best possible.*

Proof. Assume $p > 1$ and rewrite the Hilbert's bilinear double sum into the form:

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{(\lambda_m+\rho_n)^\mu} = \sum_{m,n=0}^{\infty} \frac{a_m b_n}{(m^\alpha+n^\beta)^{\nu/p}} \cdot \frac{(m^\alpha+n^\beta)^{\nu/p}}{(\lambda_m+\rho_n)^\mu}$$

where the positive scaling parameter ν will be ordered so, that the involved sums converge. Then, applying the Hölder inequality with conjugated p, q, $p > 1$ we get

$$\begin{aligned}\sum_{m,n=1}^{\infty} \frac{a_m b_n}{(\lambda_m+\rho_n)^\mu} &< \Big(\sum_{m,n=1}^{\infty} \frac{a_m^p b_n^p}{(m^\alpha+n^\beta)^\nu}\Big)^{1/p} \Big(\sum_{m,n=1}^{\infty} \frac{(m^\alpha+n^\beta)^{\nu(q-1)}}{(\lambda_m+\rho_n)^{\mu q}}\Big)^{1/q} \\ &=: \mathfrak{A}_1^{1/p}\, \mathfrak{A}_2^{1/q}\,. \end{aligned} \tag{12}$$

To estimate $\mathfrak{A}_1$ we apply a Hilbert type inequality with nonhomogeneous kernel function $K(m,n) = (m^\alpha+n^\beta)^{-\nu}$ derived by Krnić *et al.* [2, §4, Corollary 2] taking in (4) certain suitable specifications such us $a_m^p \mapsto a_m$, $b_n^p \mapsto b_n$, $(r,s) \mapsto (p,q)$. Now, making use of the mentioned inequality to $\mathfrak{A}_1$ we arrive at

$$\begin{aligned}\mathfrak{A}_1 &\le \alpha^{-1/s}\beta^{-1/r}\mathrm{B}\Big(\frac{\nu-2+r}{r}+\gamma, \frac{\nu-2+s}{s}-\gamma\Big) \\ &\quad\times \Big(\sum_{n=1}^{\infty} n^{(1-\nu)\alpha+(\alpha-1)(1-(1+\gamma)r)} a_n^{pr}\Big)^{1/r} \\ &\quad\times \Big(\sum_{n=1}^{\infty} n^{(1-\nu)\beta+(\beta-1)(1-(1-\gamma)s)} b_n^{ps}\Big)^{1/s} \\ &= \alpha^{-1/s}\beta^{-1/r}\mathrm{B}\Big(\frac{\nu-2+r}{r}+\gamma, \frac{\nu-2+s}{s}-\gamma\Big) \big\|\mathbf{n}^{B_1/r}\mathbf{a}^p\big\|_r \big\|\mathbf{n}^{B_2/s}\mathbf{b}^p\big\|_s,\end{aligned}$$

where $\big(\frac{1-\nu}{r}\big)_+ - 1/s < \gamma < 1/r + \big(\frac{1-\nu}{s}\big)_+$.

On the other hand in estimating $\mathfrak{A}_2$, we firstly expand the numerator $(m^\alpha + n^\beta)^{\nu(q-1)}$ into a binomial series:

$$\begin{aligned}
\mathfrak{A}_2 &= \sum_{m,n=1}^\infty \sum_{j=0}^\infty \binom{\nu(q-1)}{j} \frac{m^{\alpha j} n^{\beta(\nu(q-1)-j)}}{(\lambda_m+\rho_n)^{\mu q}} \\
&= \sum_{j=0}^\infty \sum_{m,n=1}^\infty \binom{\nu(q-1)}{j} \frac{m^E n^F}{(\lambda_m+\rho_n)^{\mu q}} \\
&= \frac{1}{\Gamma(\mu q)} \sum_{j=0}^\infty \binom{\nu(q-1)}{j} \sum_{m,n=1}^\infty m^E n^F \int_0^\infty x^{\mu q-1} e^{-(\lambda_m+\rho_n)x} \mathrm{d}x \\
&= \frac{1}{\Gamma(\mu q)} \sum_{j=0}^\infty \binom{\nu(q-1)}{j} \int_0^\infty x^{\mu q-1} \Big(\sum_{m=1}^\infty m^E e^{-\lambda_m x} \Big) \Big(\sum_{n=1}^\infty n^F e^{-\rho_n x} \Big) \mathrm{d}x
\end{aligned}$$

for $E = \alpha j$, $F = \beta(\nu(q-1)-j)$. We apply the integral expression result (6) to the Dirichlet series

$$\mathcal{D}_\lambda(x) = \sum_{m=1}^\infty m^E e^{-\lambda_m x} .$$

This results in

$$\mathcal{D}_\lambda(x) = \sum_{m=1}^\infty m^E e^{-\lambda_m x} = x \int_0^\infty e^{-xt} \Big(\sum_{m=1}^{[\lambda^{-1}(t)]} m^E \Big) \mathrm{d}t .$$

We calculate the innner–most *counting sum* by the Euler–Maclaurin summation formula (8). One concludes

$$\begin{aligned}
\mathcal{D}_\lambda(x) &= x \int_0^\infty e^{-xt} \Big(\sum_{m=1}^{[\lambda^{-1}(t)]} m^E \Big) \mathrm{d}t \\
&= x \int_0^\infty e^{-xt} \Big(\int_0^{[\lambda^{-1}(t)]} \mathfrak{d}_v (v^E) \, \mathrm{d}v \Big) \mathrm{d}t \\
&= x \int_{\lambda_1}^\infty \int_0^{[\lambda^{-1}(t)]} e^{-xt} \mathfrak{d}_v(v^E) \, \mathrm{d}t \mathrm{d}v ;
\end{aligned}$$

similarly

$$\mathcal{D}_\rho(x) = \sum_{n=1}^\infty n^F e^{-\rho_n x} = x \int_{\rho_1}^\infty \int_0^{[\rho^{-1}(u)]} e^{-xu} \mathfrak{d}_w(w^F) \, \mathrm{d}u \mathrm{d}w .$$

Therefore we easily deduce

$$\begin{aligned}
\mathfrak{A}_2 &= \frac{1}{\Gamma(\mu q)}\sum_{j=0}^{\infty}\binom{\nu(q-1)}{j}\int_0^\infty x^{\mu q-1}\mathcal{D}_\lambda(x)\,\mathcal{D}_\rho(x)\,\mathrm{d}x \\
&= \frac{1}{\Gamma(\mu q)}\sum_{j=0}^{\infty}\binom{\nu(q-1)}{j}\int_0^\infty x^{\mu q+1}\bigg(\int_{\lambda_1}^\infty\int_0^{[\lambda^{-1}(t)]} e^{-xt}\mathfrak{d}_v(v^E)\,\mathrm{d}t\mathrm{d}v\bigg) \\
&\quad\times\bigg(\int_{\rho_1}^\infty\int_0^{[\rho^{-1}(u)]} e^{-xu}\mathfrak{d}_w(w^F)\,\mathrm{d}u\mathrm{d}w\bigg)\mathrm{d}x \\
&= \frac{1}{\Gamma(\mu q)}\sum_{j=0}^{\infty}\binom{\nu(q-1)}{j}\int_{\lambda_1}^\infty\int_{\rho_1}^\infty\bigg(\int_0^\infty x^{\mu q+1}e^{-x(t+u)}\mathrm{d}x\bigg) \\
&\quad\times\int_0^{[\lambda^{-1}(t)]}\int_0^{[\rho^{-1}(u)]}\mathfrak{d}_v(v^E)\,\mathfrak{d}_w(w^F)\,\mathrm{d}t\mathrm{d}u\mathrm{d}v\mathrm{d}w \\
&= \frac{\Gamma(\mu q+2)}{\Gamma(\mu q)}\sum_{j=0}^{\infty}\binom{\nu(q-1)}{j} \\
&\quad\times\int_{\lambda_1}^\infty\int_{\rho_1}^\infty\frac{\int_0^{[\lambda^{-1}(t)]}\int_0^{[\rho^{-1}(u)]}\mathfrak{d}_v(v^E)\mathfrak{d}_w(w^F)\,\mathrm{d}v\mathrm{d}w}{(t+u)^{\mu q+2}}\,\mathrm{d}t\mathrm{d}u \\
&= \mu q(\mu q+1) \\
&\quad\times\int_{\lambda_1}^\infty\int_{\rho_1}^\infty\frac{\int_0^{[\lambda^{-1}(t)]}\int_0^{[\rho^{-1}(u)]}\sum_{j=0}^\infty\binom{\nu(q-1)}{j}\mathfrak{d}_v(v^E)\mathfrak{d}_w(w^F)\,\mathrm{d}v\mathrm{d}w}{(t+u)^{\mu q+2}}\,\mathrm{d}t\mathrm{d}u\,.
\end{aligned} \tag{13}$$

Since all series in $\mathfrak{A}_2$ are convergent, so are the integral expressions as well. So, all interchanges of the integration order are legitimate. Now, denoting the integrand of numerator term in (13) by $P(v,w)$ and replacing the calculated $\mathfrak{A}_1$ and $\mathfrak{A}_2$ back to the starting inequality (12), we finish the proof of (10).

It remains only to show that $\mathsf{C}_{\nu,p,r}^{\mu,\alpha,\beta}(\lambda,\rho)$ is the best possible constant. Indeed, since the Hölder inequality is sharp, and $\mathfrak{A}_2$ is transformed only by equalities, and the constant $L^\star$ is the best possible [2], the assertion is proved. This finishes the proof of the Theorem. □

Remark 1. The inequality given in [4]

$$\sum_{m,n=1}^{\infty}\frac{a_m b_n}{(\lambda_m+\rho_n)^\mu}\le \mathsf{C}_{\boldsymbol{\lambda},\boldsymbol{\rho}}\,\|\mathbf{n}^{(2-r)/p}\mathbf{a}^r\|_p^{1/r}\,\|\mathbf{n}^{(2-r)/q}\mathbf{b}^r\|_q^{1/r}, \tag{14}$$

where the constant

$$\mathsf{C}_{\boldsymbol{\lambda},\boldsymbol{\rho}} = \mathsf{C}_{\boldsymbol{\lambda},\boldsymbol{\rho}}(p,q,r,s,\mu) := \Big(\frac{\mu s}{2}\,(\mu s+1)\Big)^{1/s}\,\mathrm{B}^{1/r}\Big(1+\frac{r-3}{p}, 1+\frac{r-3}{q}\Big)$$
$$\times\,\bigg(\int_{\lambda_1}^\infty\int_{\rho_1}^\infty \frac{[\lambda^{-1}(x)][\rho^{-1}(y)]\,\big([\lambda^{-1}(x)]+[\rho^{-1}(y)]+2\big)}{(x+y)^{s\mu+2}}\,\mathrm{d}x\,\mathrm{d}y\bigg)^{1/s}$$

is the best possible and

$$\int_{\lambda_1}^\infty \frac{\big(\lambda^{-1}(x)\big)^2}{x^{s\mu/2+1}}\,\mathrm{d}x<\infty, \qquad \int_{\rho_1}^\infty \frac{\big(\rho^{-1}(x)\big)^2}{x^{s\mu/2+1}}\,\mathrm{d}x<\infty, \tag{15}$$

turns out to be a special case of (10). Indeed, let $\alpha=\beta=1, \gamma=0$ and $\nu(q-1)=1$, that is $\nu=p-1$ in our Theorem 1. Thus, we conclude $B_1 = B_2 = 2-p$, consequently the norms become $\|\mathbf{n}^{(2-p)/r}\mathbf{a}^p\|_r$, $\|\mathbf{n}^{(2-p)/s}\mathbf{b}^p\|_s$ in (10). Since the numerator of $\mathfrak{A}_2$ now equal to $m+n$, we can separate the double sums into

$$\mathfrak{A}_2 = \sum_{m,n=1}^\infty \frac{m}{(\lambda_m+\rho_n)^{\mu q}} + \sum_{m,n=1}^\infty \frac{n}{(\lambda_m+\rho_n)^{\mu q}}$$

and calculate each sum exactly like in the proof of the Theorem 1:

$$\mathfrak{A}_2 = \frac{1}{\Gamma(\mu q)}\bigg(\int_0^\infty \Big(\sum_{m=1}^\infty e^{-\lambda_m x}\Big)\Big(\sum_{n=1}^\infty n e^{-\rho_n x}\Big)x^{\mu q-1}\mathrm{d}x$$
$$+\int_0^\infty \Big(\sum_{m=1}^\infty m e^{-\lambda_m x}\Big)\Big(\sum_{n=1}^\infty e^{-\rho_n x}\Big)x^{\mu q-1}\mathrm{d}x\bigg).$$

The inner–most Dirichlet series have the following integral forms

$$\sum_{m=1}^\infty e^{-\lambda_m x} = x\int_0^\infty e^{-xt}[\lambda^{-1}(t)]\,\mathrm{d}t$$
$$\sum_{n=1}^\infty n e^{-\rho_n x} = x\int_0^\infty e^{-ux}\frac{[\rho^{-1}(u)]([\rho^{-1}(u)]+1)}{2}\,\mathrm{d}u$$
$$\sum_{n=1}^\infty e^{-\rho_n x} = x\int_0^\infty e^{-xu}[\rho^{-1}(u)]\,\mathrm{d}u$$
$$\sum_{m=1}^\infty m e^{-\lambda_m x} = x\int_0^\infty e^{-tx}\frac{[\lambda^{-1}(t)]([\lambda^{-1}(t)]+1)}{2}\,\mathrm{d}t\,.$$

Repeating a calculation similar to one in the proof of Theorem 1, we get

$$\mathfrak{A}_2 = \frac{\mu q}{2}(\mu q+1)\int_{\lambda_1}^\infty\int_{\rho_1}^\infty \frac{[\lambda^{-1}(t)][\rho^{-1}(u)]([\lambda^{-1}(t)]+[\rho^{-1}(u)])}{(t+u)^{\mu q+2}}\,\mathrm{d}t\mathrm{d}u\,.$$

By this, the constant turns into

$$\mathsf{C}^{\mu,1,1}_{p-1,p,r}(\lambda,\rho) := \mathrm{B}^{1/p}\Big(1+\frac{p-3}{r}, 1+\frac{p-3}{s}\Big)\Big(\frac{\mu q}{2}(\mu q+1)\Big)^{1/q} \\ \times \bigg(\int_{\lambda_1}^\infty\int_{\rho_1}^\infty \frac{[\lambda^{-1}(t)][\rho^{-1}(u)]([\lambda^{-1}(t)]+[\rho^{-1}(u)])}{(t+u)^{\mu q+2}}\,\mathrm{d}t\mathrm{d}u\bigg)^{1/q}.$$

So, switching $(p,q)\leftrightarrow(r,s)$, we deduce the inequality (14).

3. Another extension of inequality (14)

Krnić and Pečarić obtained a result that we nicely incorporate into our recent considerations. Under $r^{-1}+s^{-1}=1, r>1$, $\nu\in(2-\min\{r,s\}, 2+\min\{r,s\}]$, for nonnegative $\mathbf{a},\mathbf{b}$ there holds true

$$\sum_{m,n=1}^\infty \frac{a_m b_n}{(m+n)^\nu} < L_1\|\mathbf{n}^{(1-\nu)/r}\mathbf{a}\|_r\|\mathbf{n}^{(1-\nu)/s}\mathbf{b}\|_s\,. \tag{16}$$

where the constant

$$L_1 = \mathrm{B}\Big(\frac{r+\nu-2}{r}, \frac{s+\nu-2}{s}\Big)$$

is the best possible, see [3, Corollary 1].

Theorem 2. *Suppose $p,r>1$, $p^{-1}+q^{-1}=1$, $r^{-1}+s^{-1}=1$, $\mu>0$, $\nu\in(2-\min\{r,s\}, 2+\min\{r,s\}]$, $\mathbf{a},\mathbf{b}$ are nonnegative sequences such that*

$$\big(n^{(1-\nu)/r}a_n^p\big)_{n=1}^\infty \in \ell_r,$$
$$\big(n^{(1-\nu)/s}b_n^p\big)_{n=1}^\infty \in \ell_s;$$

and λ,ρ are positive monotone increasing functions satisfying (7). *Then*

$$\sum_{m,n=1}^\infty \frac{a_m b_n}{(\lambda_m+\rho_n)^\mu} < \mathsf{C}^{\mu,\nu}_{p,r}(\lambda,\rho)\,\|\mathbf{n}^{(1-\nu)/r}\mathbf{a}^p\|_r^{1/p}\|\mathbf{n}^{(1-\nu)/s}\mathbf{b}^p\|_s^{1/p}\,, \tag{17}$$

where

$$\mathsf{C}^{\mu,\nu}_{p,r}(\lambda,\rho) := \mathrm{B}^{1/p}\Big(\frac{\nu-2+r}{r}, \frac{\nu-2+s}{s}\Big)\,(\mu q(\mu q+1))^{1/q} \\ \times \bigg(\int_{\lambda_1}^\infty\int_{\rho_1}^\infty \frac{\int_0^{[\lambda^{-1}(t)]}\int_0^{[\rho^{-1}(u)]} P(v,w)\,\mathrm{d}v\mathrm{d}w}{(t+u)^{\mu q+2}}\,\mathrm{d}t\mathrm{d}u\bigg)^{1/q}$$

and

$$P(v,w) = \sum_{j=0}^\infty \binom{\nu(q-1)}{j}\mathfrak{d}_v(v^j)\,\mathfrak{d}_w(w^{\nu(q-1)-j})\,.$$

Moreover, the constant $\mathsf{C}^{\mu,\nu}_{p,r}(\lambda,\rho)$ is the best possible.

Proof. By $\alpha = \beta = 1$ the Hilbert type double sum (12) one restricts to

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{(\lambda_m + \rho_n)^\mu} = \sum_{m,n=0}^{\infty} \frac{a_m b_n}{(m+n)^{\nu/p}} \cdot \frac{(m+n)^{\nu/p}}{(\lambda_m + \rho_n)^\mu}\,. \tag{18}$$

Again, by Hölder inequality with $p^{-1} + q^{-1} = 1$ we conclude

$$\begin{aligned}\sum_{m,n=1}^{\infty} \frac{a_m b_n}{(\lambda_m + \rho_n)^\mu} &< \Big(\sum_{m,n=1}^{\infty} \frac{a_m^p b_n^p}{(m+n)^\nu} \Big)^{1/p} \Big(\sum_{m,n=1}^{\infty} \frac{(m+n)^{\nu(q-1)}}{(\lambda_m + \rho_n)^{\mu q}} \Big)^{1/q} \\ &=: \mathfrak{B}_1^{1/p}\, \mathfrak{B}_2^{1/q}\,.\end{aligned}$$

Applying inequality (16) to evaluate $\mathfrak{B}_1$ with $a_m^p \mapsto a_m$, $b_n^p \mapsto b_n$, we get

$$\mathfrak{B}_1 < \mathrm{B}\Big(1 + \frac{\nu-2}{r}, 1 + \frac{\nu-2}{s}\Big) \|\mathbf{n}^{(1-\nu)/r}\mathbf{a}^p\|_r \|\mathbf{n}^{(1-\nu)/s}\mathbf{b}^p\|_s\,.$$

Now, it remains to calculate $\mathfrak{B}_2$. But, the only change with respect to calculation procedure of $\mathfrak{A}_2$ is the specified $\alpha = \beta = 1$ in (11), where the exponents inside operators $\mathfrak{d}_v, \mathfrak{d}_w$, are just $E = j$ and $F = \nu(q-1) - j$. Therefore, we only mimic the transformation of $\mathfrak{A}_2$ into $\mathfrak{B}_2$ with simplified E, F.

The best constant question is clear because we apply the Hölder inequality, and the sharp constant L_1 appearing in the result (16) by Krnić and Pečarić. This finishes the proof of Theorem 2. $\square$

Remark 2. Let us specify $\nu/p = 1/q$, $p > 1$, accordingly, make use at the Hölder inequality (with respect to conjugated p, q) in the right–hand expression in (18). We conclude that

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{(\lambda_m + \rho_n)^\mu} < \Big(\sum_{m,n=1}^{\infty} \frac{a_m^p b_n^p}{(m+n)^{p-1}} \Big)^{1/p} \Big(\sum_{m,n=1}^{\infty} \frac{m+n}{(\lambda_m + \rho_n)^{\mu q}} \Big)^{1/q},$$

which is the same partial result as the one [4, p. 1487, Eq. (13)], considered by Pogány in course to obtain (14).

On the other side Theorem 2 extends substantially the range of r from $(3-\min\{p,q\}, 3]$ (cf. [4, Theorem 1]) to $(3-\min\{p,q\}, 3+\min\{p,q\}]$. Indeed, putting $\nu = r - 1$ in the preambula of Theorem 2, we show the validity of this claim easily.

4. Discussion, final remarks

There are lots of special cases connected to both of our theorems. In Remarks 1,2 we clearly show that the third authors result (14) is only a corollary, a special case of our Theorems. Now, we will discuss some further aspects of these novel results.

4.1. **Around Theorem 1.** We introduce the independent scaling parameters p, r, α, β, μ and ν in Theorem 1 together with two dependent ones r, s such as are the conjugated Hölder pairs of initial p, r respectively. Therefore, this procedure defined by (10) in a *Hilbert type seven parameters inequality family* $\mathfrak{I}(p, r, \alpha, \beta, \gamma, \mu, \nu)$. Any further specification of the independent parameters gives certain novel inequalities which ones become to complicated for nonninteger $\nu(q-1)$, see the differential form $P(v,w)$ in (11). On the other side $P(v,w)$ is a finite sum with integer $\nu = p-1$. Let us show an example, choosing the nonhomogeneous kernel function $K(m,n) = (m^2+n^3)^{-2}$ in the auxiliary Hilbert type inequality.

Corollary 1. *Assume* $\min\{p,r\} > 1$, $\mu > 0$,

$$\Big(\frac{2}{p}-1\Big)_+ - \frac{1}{q} < \gamma < \frac{1}{p} + \Big(\frac{2-p}{q}\Big)_+ .$$

The four–parameter Hilbert type inequality family $\mathfrak{I}\big(p, r, 2, 3, \gamma, \mu, 2(p-1)\big)$ *reads as follows*

$$\sum_{m,n=1}^{\infty} \frac{a_m b_n}{(\lambda_m+\rho_n)^{\mu}} < \mathsf{C}_{p,r}^{\mu,2,3}(\lambda,\rho)\, \|\mathbf{n}^{(7-4p)/r-1-\gamma}\mathbf{a}^p\|_r^{1/p} \|\mathbf{n}^{(11-6p)/s-2(1-\gamma)}\mathbf{b}^p\|_s^{1/p}\,, \tag{19}$$

where and λ, ρ *are positive monotone increasing functions having property* (7)*, while* $\mathbf{a}, \mathbf{b}$ *are nonnegative real sequences satisfying*

$$\big(n^{(7-4p)/r-1-\gamma} a_n^p\big)_{n=1}^\infty \in \ell_r, \qquad \big(n^{(11-6p)/s-2(1-\gamma)} b_n^p\big)_{n=1}^\infty \in \ell_s,$$

and the double series

$$\sum_{m,n=1}^{\infty} \frac{(a_m b_n)^p}{(m^2+n^3)^{2p-2}}$$

converges. The constant

$$\begin{aligned}\mathsf{C}_{p,r}^{\mu,2,3}(\lambda,\rho) &= \frac{\big(\mu q(\mu q+1)\big)^{1/q}}{\big(2^{1/s}3^{1/r}\big)^p}\, \mathrm{B}^{1/p}\Big(\frac{2p-4}{r}+1+\gamma, \frac{2p-4}{s}+1-\gamma\Big)\\ &\quad\times\Bigg(\int_{\lambda_1}^\infty\!\!\int_{\rho_1}^\infty \frac{L_t R_u}{(t+u)^{\mu q+2}} \Big(\frac{(2L_t+1)(3L_t^2+3L_t-1)}{30}\\ &\qquad + \frac{(L_t+1)(2L_t+1)R_u(R_u+1)^2}{12}\\ &\qquad\quad + \frac{(R_u+1)(2R_u+1)(3R_u^4+6R_u^3-3R_u+1)}{42}\Big)\,\mathrm{d}u\mathrm{d}t\Bigg)^{1/q}\end{aligned} \tag{20}$$

is the best possible; here $L_t := [\lambda^{-1}(t)]$, $R_u := [\rho^{-1}(u)]$.

Obviously, further subsequent specifications of remaining independent parameters will result in less complicated, but not necessarily simpler constants.

4.2. **Around Theorem 2.** The family of the second Hilbert type inequality $\mathfrak{J}(p, r, \mu, \nu)$ defined in the Theorem 2 is of mainly simpler structure. Taking the condition $\nu(q-1) = 2$ as above, we present the related result.

Corollary 2. *Let $r > 1$, $\mu > 0$ and $p \in (2 - \frac{1}{2}\min\{r, s\}, 2 + \frac{1}{2}\min\{r, s\}]$; $\mathbf{a}, \mathbf{b}$ are nonnegative sequences such that*

$$\big(n^{(3-2p)/r} a_n^p\big)_{n=1}^\infty \in \ell_r, \qquad \big(n^{(3-2p)/s} b_n^p\big)_{n=1}^\infty \in \ell_s;$$

and λ, ρ are positive monotone increasing functions satisfying (7)*. Then*

$$\sum_{m,n=1}^\infty \frac{a_m b_n}{(\lambda_m + \rho_n)^\mu} < \mathsf{C}_{p,r}^{\mu, 2p-2}(\lambda, \rho) \, \|\mathbf{n}^{(3-2p)/r}\mathbf{a}^p\|_r^{1/p} \|\mathbf{n}^{(3-2p)/s}\mathbf{b}^p\|_s^{1/p}\,, \tag{21}$$

where

$$\begin{aligned} \mathsf{C}_{p,r}^{\mu, 2p-2}(\lambda, \rho) := \frac{(\mu q(\mu q + 1))^{1/q}}{6}\, \mathrm{B}^{1/p}\Big(\frac{2p-4+r}{r}, \frac{2p-4+s}{s}\Big) \\ \times \bigg(\int_{\lambda_1}^\infty \int_{\rho_1}^\infty \frac{L_t R_u}{(t+u)^{\mu q + 2}} \big((L_t+1)(2L_t+1) \\ + 3(L_t+1)(R_u+1) + (R_u+1)(2R_u+1)\big)\, \mathrm{d}t\mathrm{d}u \bigg)^{1/q}. \end{aligned} \tag{22}$$

Here $L_t := [\lambda^{-1}(t)]$, $R_u := [\rho^{-1}(u)]$. Moreover, the constant $\mathsf{C}_{p,r}^{\mu, 2p-2}(\lambda, \rho)$ is the best possible.

The differential form $P(v, w)$ in the Corollary 2 originates back to $(m + n)^2$, therefore the integrand in (22) becomes symmetric in L_t, R_u.

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