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NON-ADDITIVE MEASURES, ENVELOPES AND EXTENSIONS TO QUASI-MEASURES

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ABSTRACT. In the present paper, we introduce the notions of lower envelope and upper envelope for a $[0, \infty]$-valued function μ defined on a proper sublattice M of a locally complete σ-continuous lattice L, and we extend a finite-stable, supermodular *usc*-measure μ on a proper sublattice M of L to a quasi*-measure (i.e., a supermodular *usc*-measure) on L, which is M_δ-inner regular. Analogously, we extend a submodular *lsc*-measure on M to a quasi$_*$-measure (i.e., a submodular *lsc*-measure) on L, which is M_σ-outer regular. Furthermore, we have studied notions of measuring envelopes in D-lattices in the context of null-additive, converse null-additive, superadditive and weak converse null-additive functions.

1. INTRODUCTION

In measure theory, a basic procedure is that of extending the notion of a measure on a given class of sets to a larger class of sets. The possibility of extension in measure theory on logics (orthomodular lattices or posets) was presented as an open problem in [12]. Volauf in [28] proved an extension theorem for orthocomplemented lattices and probability measures using Carathéodory measurability. In [25], Riečan studied an extension theorem for subadditive probability measures defined on a suborthomodular lattice of a σ-continuous, σ-complete orthomodular lattice. Later on, in 2001, Avallone and De Simone gave an extension theorem for modular functions that contains Riečan's result as a particular case. They together with Vitolo further extended the theory in context of lattice ordered effect algebras in [5]. An extension theorem has been proved for measures on MV-algebras in fuzzy measure theory ([6, 26]; see also [18, 21, 27]). Adamski [2] obtained that every non-negative, semifinite, continuous at ϕ and tight function defined on a lattice of sets can be extended to an inner regular measure. The

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concept of an effect algebra has been introduced by Bennett and Foulis [7], as a generalization of Hilbert space effects interpreted as “unsharp” quantum events. Different from the “sharp” events the effects do not satisfy the noncontradiction principle, i.e. the conjunction of a and non a may be different from zero. These new logical structures generalize orthomodular lattices (including Boolean algebras) as well as MV-algebras employed by Chang in the analysis of many valued logics [8]. The categorical equivalence of D-posets and effect algebras is discussed in [10].

Non-additive set functions, as for example outer measures, semivariations of vector measures, naturally appeared earlier in classical measure theory concerning countable additive set functions or more general finite additive set functions [10]. Non-additive measures appear today in many branches of pure mathematics with many important applications ([23, 29]; see also [15, 16, 19, 20]). Wang [29] gave the concept of null-additive set functions. Wang and Klir [30] introduced the concept of converse null-additive fuzzy measures. The notion of weak converse null-additive function is studied in context of fuzzy measures in [21].

The aim of the present paper is to study an extension problem for non-additive measures defined on a proper sublattice M of a locally complete σ-continuous lattice L. Some basic definitions are collected in Section 2. In Section 3, we introduce notions of lower envelope μ_* and upper envelope μ^* of a $[0, \infty]$-valued function μ defined on M, and we extend a finite-stable, supermodular *usc*-measure μ on a proper sublattice M of L to a quasi*-measure (i.e., a supermodular *usc*-measure) on L, which is M_δ-inner regular. Analogously, we extend a submodular *lsc*-measure to a quasi$_*$-measure (i.e., a submodular *lsc*-measure) on L, which is M_σ-outer regular. In Section 4, we have studied the notions of null-additive, converse null-additive, superadditive and weak converse null-additive functions on a proper D-sublattice M of L, where L is a σ-complete σ-continuous D-lattice. We have observed that every superadditive function is converse null-additive, and also weak converse null-additive; the converse need not be true, which is established by a counterexample. We proved that if μ is superadditive, then μ_* is also superadditive. It is also proved that for a monotone null-additive function defined on a proper D-sublattice M, if a is μ-measurable, then $\mu_*(a) = \mu^*(a)$ and, if $d \in L$ such that $\mu_*(d) = \mu^*(d) < \infty$, d is μ-measurable, then μ is weak converse null-additive.

2. Preliminaries and basic facts

Let $L = (L, \leq)$ be a lattice. Let $\{a_n\}$ be a sequence in L. We call $a_n \uparrow a, (a \in L)$ if and only if $a_1 \leq a_2 \leq \cdots \leq a_n \leq \dots$, $\bigvee a_n$ exists and $\bigvee a_n = a$. Similarly, call $a_n \downarrow a, (a \in L)$ if and only if $a_1 \geq a_2 \geq \cdots \geq a_n \geq \dots$, $\bigwedge a_n$

exists and $\bigwedge a_n = a$. In these cases we also write $a = \lim_{n\to\infty} a_n$. If $a_n \uparrow a$, $b_n \uparrow b$ and $a_n \leq b_n$, for all n, then we may deduce that $a \leq b$. The symbol 1 denotes the top element (or supremum) of L and 0 denotes the bottom element (or infimum) of L (see [22]). A lattice L is said to be *σ-continuous* (see [25]) if $a_n \uparrow a$ implies $a_n \wedge b \uparrow a \wedge b$ (or equivalently, $a_n \downarrow a$ implies $a_n \vee b \downarrow a \vee b$) for every $b \in L$. If L is σ-continuous then, for sequences $\{a_n\}$ and $\{b_n\}$ in L such that $a_n \uparrow a$ and $b_n \uparrow b$, we have $a_n \wedge b_n \uparrow a \wedge b$ (or equivalently, $a_n \downarrow a, b_n \downarrow b$ implies $a_n \vee b_n \downarrow a \vee b$). Every infinitely distributive lattice [22] is σ-continuous.

2.1 [1]. A lattice L is *locally complete* if it satisfies one of the following equivalent conditions:

(i) Every non-empty lower bounded subset of L admits an infimum.
(ii) Every non-empty upper bounded subset of L admits a supremum.
(iii) There exists a complete lattice, denoted by $\overline{L}$, with bottom (or smallest) element 0 and top (or largest) element 1, such that L is a sublattice of $\overline{L}$, $\overline{L} = L \cup \{0, 1\}$, $\inf L = 0$ and $\sup L = 1$.

It can be observed that every complete lattice is locally complete. For any set X, $(\mathfrak{P}(X), \subseteq)$, $(L^X, \leq)$ (where L is locally complete σ-continuous lattice) and $(\mathbb{I}, \leq)$ (where $\mathbb{I}$ is the closed unit interval [0, 1] of the real line $\mathbb{R}$) are locally complete σ-continuous lattices. For more examples of locally complete lattices, we refer to [1].

2.2 ([10]; see also [17]). An *orthomodular poset* (*OMP*, in short) is a bounded poset $(P, \leq, ', 0, 1)$ with a unary operation $' : P \to P$ (an orthocomplementation) such that the following conditions are satisfied for all $a, b, c \in P$: (i) If $a \leq b$ then $b' \leq a'$, (ii) $(a')' = a$, (iii) $a \vee a' = 1$, (iv) If $a \leq b'$ then $a \vee b$ exists in P, (v) (orthomodular law) If $a \leq b$, then there is a $c \in P$ such that $c \leq a'$ and $a \vee c = b$. Two elements $a, b \in P$ are called orthogonal (written as $a \perp b$) if $a \leq b'$ or equivalently $b \leq a'$. An *orthomodular lattice* (*OML*) L can be defined as a lattice ordered OMP. A subset M is called a *suborthomodular lattice* (or *sub OML*) of an OML L if it contains 0 and 1, and is closed under the operations $'$, $\wedge$ and $\vee$, i.e. it is an orthomodular lattice with respect to induced operations. An OMP is called a *quantum logic* if it is a σ-complete lattice.

2.3 ([7, 9, 10, 13, 24]; see also [4, 5, 14]). An *effect algebra* $(L; \oplus, 0, 1)$ is a structure consisting of a set L, two special elements 0 and 1, and a partially defined binary operation $\oplus$ on $L \times L$ satisfying the following conditions for every $a, b, c \in L$:

(1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.
(2) If $b \oplus c$ and $a \oplus (b \oplus c)$ are defined, then $a \oplus b$ and $(a \oplus b) \oplus c$ are defined and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

(3) For every $a \in L$, there exists a unique $a^{\perp} \in L$ such that $a \oplus a^{\perp}$ is defined and $a \oplus a^{\perp} = 1$.
(4) If $a \oplus 1$ is defined, then $a = 0$.

A subset F is called a *subeffect algebra* of an effect algebra L if (i) $0, 1 \in F$, (ii) for $a \in F$, $a^{\perp} \in F$, (iii) for $a, b \in F$ with $a \perp b$, $a \oplus b \in F$.

In every effect algebra L, a dual operation $\ominus$ to $\oplus$ can be defined as follows: $a \ominus c$ exists and equals b if and only if $b \oplus c$ exists and equals a. We say that two elements $a, b \in L$ are *orthogonal* and we write $a \perp b$, if $a \oplus b$ exists. Also for $a, b \in L$, define $a \leq b$ if there exists $c \in L$ such that $a \perp c$ and $a \oplus c = b$. If $(L, \leqslant)$ is a lattice, we say that the effect algebra L is a *lattice effect algebra*, or a *D-lattice*. Any OMP may be regarded as a D-poset (which is equivalent to effect algebra) by defining $b \ominus a = b \wedge a^{'}$ precisely when $a \leq b$.

For $a_1, \dots, a_n \in L$, we inductively define $a_1 \oplus \cdots \oplus a_n = (a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n$, provided that the right hand side exists. The definition is independent of permutation of the elements. A finite subset $\{a_1, \dots, a_n\}$ of L is said to be *orthogonal* if $a_1 \oplus \cdots \oplus a_n$ exists. A sequence $\{a_n\}$ in L is called orthogonal if, for every n, $\bigoplus_{i \leqslant n} a_i$ exists. If, moreover $\sup_n \bigoplus_{i \leqslant n} a_i$ exists, the *sum* $\bigoplus_{n \in \mathbb{N}} a_n$ of an orthogonal sequence $\{a_n\}$ in L is defined as $\sup_n \bigoplus_{i \leqslant n} a_i$. An effect algebra L is called a *σ-complete effect algebra* if every orthogonal sequence in L has its sum.

2.4. Let $a, b, c \in L$ (where L is an OML) such that $a \perp b$ and $b \leq c$. Then $a \vee b \geq c$ if and only if $a \geq c \wedge b^{'}$.

2.5. A function $\mu : L \to [0, \infty]$ (where L is a lattice) is said to be *modular* if, for every $a, b \in L$, $\mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b)$. We say that μ is *submodular* if for every $a, b \in L$, we have $\mu(a) + \mu(b) \geq \mu(a \vee b) + \mu(a \wedge b)$; μ is called *supermodular* if for every $a, b \in L$, we have $\mu(a) + \mu(b) \leq \mu(a \vee b) + \mu(a \wedge b)$. A function μ is called *subadditive* if for every $a, b \in L$, we have $\mu(a \vee b) \leq \mu(a) + \mu(b)$; μ is called *superadditive* if for every $a, b \in L$, we have $\mu(a \vee b) \geq \mu(a) + \mu(b)$.

2.6 ([9, 10, 13]). Assume that a, b are elements of an effect algebra L.

(i) If $a \leq b$, then $b = a \oplus (b \ominus a)$.
(ii) If $a \leq b \leq c$, then $b \ominus a \leq c \ominus a$.
(iii) If $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$.
(iv) If $c \leq a$, $d \leq b$ and $a \perp b$, then $c \perp d$ and $c \oplus d \leq a \oplus b$.

3. Extension of semi-continuous measures to quasi-measures

Throughout in this section, let L be a locally complete σ-continuous lattice, let C be a non-empty subset of L, and M be a proper sublattice of L containing 0. We denote $M_\sigma = \{b \in L : \text{there exists a sequence } \{a_n\} \text{ in } M \text{ such that } a_n \uparrow b\}$, and $M_\delta = \{b \in L : \text{there exists a sequence } \{a_n\} \text{ in}$

M such that $a_n \downarrow b\}$. We may also describe M_σ as the family of all countable joins of elements from M and M_δ as the family of all countable meets of elements from M.

Definition 3.1. *A function $\mu : C \to [0, \infty]$ is called a semi-continuous measure (or non-additive measure) on C, if it satisfies the following conditions:*

(i) $\mu(0) = 0$, *whenever* $0 \in C$,
(ii) (*monotone*) *if* $a \le b$, $a, b \in C$, *then* $\mu(a) \le \mu(b)$,
(iii) (*semi-continuous from below*) *if* $a_n \uparrow a$, $a \in C$, $a_n \in C$ $(n \in \mathbb{N})$, *then* $\lim_{n\to\infty} \mu(a_n) = \mu(a)$,
(iv) (*semi-continuous from above*) *if* $a_n \downarrow a$, $a \in C$, $a_n \in C$ $(n \in \mathbb{N})$, $\mu(a_1) < \infty$, *then* $\lim_{n\to\infty} \mu(a_n) = \mu(a)$.

The function μ is said to be a lower semi-continuous measure (or lsc-measure) if it satisfies (i), (ii) and (iii), while μ is said to be an upper semi-continuous measure (or usc-measure) if it satisfies (i), (ii) and (iv).

Definition 3.2. *A function $\mu : C \to [0, \infty]$ is called finite-stable if, for all $c_1, c_2 \in C$, $\max(\mu(c_1), \mu(c_2)) < \infty$ implies $\mu(c_1 \vee c_2) < \infty$.*

Define $\nu(c) = \sup\{\mu(d) : d \le c, d \in C, \mu(d) < \infty\}$, $c \in C$. If $\mu = \nu$, then μ is called semifinite.

If, in addition, L is an orthomodular lattice (OML), *M is a proper sub-orthomodular lattice of L and $\rho : L \to [0, \infty]$ with $\rho(0) = 0$, we write $\mathcal{M}(\rho; M) = \{a \in L : \rho(b) = \rho(b \wedge a) + \rho(b \wedge (b \wedge a)')$ for all $b \in M\}$.*

Definition 3.3. *Let $\mu : M \to [0, \infty]$ with $\mu(0) = 0$. Define $\mu_* : L \to [0, \infty]$ and $\mu^* : L \to [0, \infty]$ by*

$$\mu_*(a) = \sup\{\mu(b) : b \le a, b \in M\}, \ a \in L$$

and

$$\mu^*(a) = \inf\{\mu(b) : a \le b, b \in M\}, \ a \in L,$$

μ_* is called the lower envelope and μ^* the upper envelope of μ.

We get: (*i*) $\mu^*(0) = 0$, $\mu_*(0) = 0$, (*ii*) both μ^* and μ_* are monotone, (*iii*) $\mu^*|M \le \mu \le \mu_*|M$, (*iv*) μ is semifinite if and only if μ is monotone if and only if $\mu^*|M = \mu = \mu_*|M$.

Proposition 3.1.

(i) *If μ is finite-stable, then μ^* is finite-stable.*
(ii) *If μ is supermodular, then μ_* is supermodular (see* [3]*).*
(iii) *If μ is submodular, then μ^* is submodular (see* [3]*).*
(iv) *If μ is superadditive, then μ_* is superadditive.*
(v) *If μ is subadditive, then μ^* is subadditive.*
(vi) *Let M be a proper suborthomodular lattice of an orthomodular lattice L. Then we have:*

(a) *μ is superadditive* $\Rightarrow \mathcal{M}(\mu_*; M) = \{a \in L : \mu(b) \leq \mu_*(b \wedge a) + \mu_*(b \wedge (b \wedge a)')$ *for all* $b \in M\}$.

(b) *If μ is subadditive* $\Rightarrow \mathcal{M}(\mu^*; M) = \{a \in L : \mu(b) \geq \mu^*(b \wedge a) + \mu^*(b \wedge (b \wedge a)')$ *for all* $b \in M\}$.

Proof. We shall prove only (i), (ii) and (v).

(i) Let $a_1, a_2 \in L$ with $\mu^*(a_1) < \infty$, $\mu^*(a_2) < \infty$. Let $\varepsilon > 0$. Then there exist $b_1, b_2 \in M$, $a_1 \leq b_1$, $a_2 \leq b_2$, $\mu(b_1) < \mu^*(a_1) + \varepsilon/2$ and $\mu(b_2) < \mu^*(a_2) + \varepsilon/2$. Since $a_1 \vee a_2 \leq b_1 \vee b_2$ and μ is finite-stable, we obtain $\mu^*(a_1 \vee a_2) < \infty$. Hence μ^* is finite-stable.

(ii) Let $a, b \in L$. Let $\varepsilon > 0$. Then there exist $c, d \in M$, $c \leq a$, $d \leq b$, $\mu_*(a) - \varepsilon/2 < \mu(c)$ and $\mu_*(b) - \varepsilon/2 < \mu(d)$. It follows that $\mu_*(a) + \mu_*(b) - \varepsilon < \mu(c) + \mu(d)$. Since μ is supermodular, we obtain $\mu_*(a) + \mu_*(b) - \varepsilon < \mu(c \vee d) + \mu(c \wedge d)$. Now, we have $\mu_*(a) + \mu_*(b) - \varepsilon < \mu_*(a \vee b) + \mu_*(a \wedge b)$. Thus μ_* is supermodular.

(v) Let $a, b \in L$. Let $\varepsilon > 0$. Then there exist $c, d \in M$, $a \leq c$, $b \leq d$, $\mu(c) < \mu^*(a) + \varepsilon/2$ and $\mu(d) < \mu^*(b) + \varepsilon/2$. Since $a \vee b \leq c \vee d$ and μ^* is monotone, we get $\mu^*(a \vee b) \leq \mu(c \vee d) \leq \mu(c) + \mu(d) < \mu^*(a) + \varepsilon/2 + \mu^*(b) + \varepsilon/2 = \mu^*(a) + \mu^*(b) + \varepsilon$. Hence μ^* is subadditive. □

Lemma 3.1. (*see* [3])

(a) *Let μ be a $[0, \infty]$-valued finite-stable usc-measure on M. Then for any $b \in M_\delta$ with $\mu^*(b) < \infty$, there exists a sequence $\{b_n\}_{n=1}^\infty$ in M such that $b_n \downarrow b$ and for each such sequence $\mu^*(b) = \inf_n \mu(b_n)$.*

(b) *Let μ be a $[0, \infty]$-valued lsc-measure on M. Then for any $b \in M_\sigma$, there exists a sequence $\{b_n\}_{n=1}^\infty$ in M such that $b_n \uparrow b$ and for each such sequence $\mu_*(b) = \sup_n \mu(b_n)$.*

Proof. (a) Let $b \in M_\delta$ with $\mu^*(b) < \infty$. Then there exists a sequence $\{b_n\}_{n=1}^\infty$ in M such that $b_n \downarrow b$. Choose $c \in M$ such that $b \leq c$ and $\mu(c) < \infty$. Replace b_n by $b_n \wedge d$ (where $d \in M$, $b \leq d$ and $\mu(d) < \infty$). Then we have $b_n \vee c \downarrow c$ (as L is σ-continuous), and $\inf_n \mu(b_n \vee c) < \infty$. Again since μ is semi-continuous from above, we get $\lim_{n\to\infty} \mu(b_n \vee c) = \inf_n \mu(b_n \vee c) = \mu(c)$. Also $b_n \leq b_n \vee c$ for each n, yields $\inf_n \mu(b_n) \leq \mu^*(b)$. On the other hand, $b \leq b_n$ $(n \in \mathbb{N})$ and μ^* is monotone, imply $\mu^*(b) \leq \inf_n \mu(b_n)$. Hence $\mu^*(b) = \inf_n \mu(b_n)$.

(b) It follows using similar arguments as in (a). □

Theorem 3.1.

(a) *If μ is a $[0, \infty]$-valued finite-stable usc-measure on M, then $\mu^*|M_\delta$ is a usc-measure.*

(b) *If μ is a $[0,\infty]$-valued lsc-measure on M, then $\mu_*|M_\sigma$ is a lsc-measure.*

Proof. (a) Let $\{a_n\}_{n=1}^\infty$ be a decreasing sequence in M_δ with $\mu^*(a_1) < \infty$ and $a_n \downarrow a$ $(n \in \mathbb{N})$, $a \in M_\delta$. Then there exists a sequence $\{a_{ni}\}_{i=1}^\infty$ in M such that $a_{ni} \downarrow a_n$ and $\mu^*(a_n) = \lim_{i\to\infty} \mu(a_{ni})$. For $i \in \mathbb{N}$, set $b_i = a_{1i} \wedge a_{2i} \ldots \wedge a_{ii}$. Then $b_i \in M$, $\{b_i\}_{i=1}^{\infty}$ is a decreasing sequence and $b_i \geq a_i \geq a$ for all i, which yield that $b = \lim_{i\to\infty} b_i = \bigwedge b_i \geq \bigwedge a_i = a$. It may be noted that $b \in M_\delta$. Also $a_{ki} \geq b_i$ for $1 \leq k \leq i$. Therefore $a_k = \lim_{i\to\infty} a_{ki} \geq \lim_{i\to\infty} b_i = b$. It follows that $a = \bigwedge a_k \geq b$. Thus $a = b$. Now, by Lemma 3.1(a), we have $\mu^*(a) = \lim_{i\to\infty} \mu(b_i) = \lim_{i\to\infty} \mu^*(b_i) \geq \lim_{i\to\infty} \mu^*(a_i)$. Also since $a \leq a_n$, the result follows.

(b) It follows using similar arguments as in (a). $\square$

Definition 3.4. *Let $\mu^- : L \to [0, \infty]$ be defined by*

$$\mu^-(a) = \sup\{\mu^*(b) : b \leq a, b \in M_\delta, \mu^*(b) < \infty\},\ a \in L.$$

We obtain: (i) $\mu^-(0) = 0$, (ii) μ^- is monotone, (iii) $\mu^- \leq \mu^*$, (iv) $\mu^-|M_\delta = \mu^*|M_\delta$; particularly $\mu^-|M = \mu^*|M$, (v) if μ is monotone, then $\mu^-|M = \mu$, i.e. μ^- is an extension of μ, (vi) if μ is subadditive, then μ^- is subadditive.

Proposition 3.2. *Let $M = M_\delta$. Then*

(i) $\mu^- \leq \mu_*$,
(ii) $\mu^- = \mu_*$ *on M, provided μ is semifinite.*

Proof. (i) Let $a \in L$ and $b \in M$ with $b \leq a$. Since $\mu^*|M \leq \mu_*|M$ and μ_* is monotone, we obtain $\mu^*(b) \leq \mu_*(b) \leq \mu_*(a)$. Hence $\mu^-(a) \leq \mu_*(a)$.

(ii) Since $M = M_\delta$ and μ is semifinite, so $\mu^- = \mu^*$. Using Definition 3.3 (iv), we obtain $\mu^- = \mu_*$ on M. $\square$

Theorem 3.2. *Let μ be a finite-stable usc-measure on M. Then $(i) \Rightarrow (ii) \Rightarrow (iii)$, where*

(i) *μ is supermodular.*
(ii) *$\mu^*|M_\delta$ is supermodular.*
(iii) *(a) μ^- is supermodular.* *(b) μ^- is semi-continuous from above.*

In addition, if L is an orthomodular lattice, M is a proper suborthomodular lattice of L and μ is subadditive, then $(iii) \Rightarrow (iv) \Rightarrow (v)$, where

(iv) $\mathcal{M}(\mu^-; M) = \{a \in L : \mu(b) \leq \mu^-(b \wedge a) + \mu^-(b \wedge (b \wedge a)')$ *for all* $b \in M$ *with* $\mu(b) < \infty\}$.
(v) $\mathcal{M}(\mu_*; M) \subseteq \mathcal{M}(\mu^-; M)$.

Proof. (i)$\Rightarrow$(ii) Let μ be supermodular. Let $a, b \in M_\delta$ with $\mu^*(a) < \infty$, $\mu^*(b) < \infty$. Then, by Lemma 3.1(a) there exist sequences $\{a_n\}$ and $\{b_n\}$ in M such that $a_n \downarrow a, b_n \downarrow b$, $\mu^*(a) = \lim_{n\to\infty} \mu(a_n)$ and $\mu^*(b) = \lim_{n\to\infty} \mu(b_n)$. Since L is σ-continuous, so $a_n \vee b_n \downarrow a \vee b$ and also we have $a_n \wedge b_n \downarrow a \wedge b$.

Therefore

$$\begin{aligned}\mu^*(a)+\mu^*(b) &= \lim_{n\to\infty}\mu(a_n)+\lim_{n\to\infty}\mu(b_n)\\ &= \lim_{n\to\infty}(\mu(a_n)+\mu(b_n))\\ &\le \lim_{n\to\infty}(\mu(a_n\vee b_n)+\mu(a_n\wedge b_n))\\ &= \lim_{n\to\infty}\mu(a_n\vee b_n)+\lim_{n\to\infty}\mu(a_n\wedge b_n)\\ &= \mu^*(a\vee b)+\mu^*(a\wedge b).\end{aligned}$$

(ii)⇒(iii) (a) Let $a,b\in L$. For $\varepsilon>0$, we have $c,d\in M_\delta$ such that $c\le a, d\le b$, with $\mu^*(c)<\infty$, $\mu^*(d)<\infty$. Then we have $\mu^-(a)-\varepsilon/2<\mu^*(c)$ and $\mu^-(b)-\varepsilon/2<\mu^*(d)$. Thus

$$\begin{aligned}\mu^-(a)+\mu^-(b)-\varepsilon &< \mu^*(c)+\mu^*(d)\\ &\le \mu^*(c\vee d)+\mu^*(c\wedge d)\\ &\le \mu^-(a\vee b)+\mu^-(a\wedge b).\end{aligned}$$

Since ε is arbitrary, we have $\mu^-(a)+\mu^-(b)\le\mu^-(a\vee b)+\mu^-(a\wedge b)$.

(ii)⇒(iii) (b) Let $\{a_n\}_{n=1}^\infty$ be a sequence in L with $\mu^-(a_1)<\infty$ and $a_n\downarrow a$ ($n\in\mathbb{N}$), $a\in L$. Let $\varepsilon>0$. For each n, we choose $b_n\in M_\delta$ such that $b_n\le a_n$, $\mu^*(b_n)<\infty$ and

$$\mu^*(b_n)>\mu^-(a_n)-\varepsilon/2^n. \tag{3.2.1}$$

Put $c_n=b_1\wedge b_2\wedge\ldots\wedge b_n$. Then $c_n\in M_\delta$ and $c_n\downarrow c$, $c\in M_\delta$. Now, by (3.2.1), we get $\mu^*(c_1)=\mu^*(b_1)>\mu^-(a_1)-\varepsilon/2$. Since $\mu^*|M_\delta$ is supermodular, it follows that

$$\begin{aligned}\mu^*(c_2) &= \mu^*(b_1\wedge b_2)\\ &\ge \mu^*(b_1)+\mu^*(b_2)-\mu^*(b_1\vee b_2)\\ &> \mu^-(a_1)-\varepsilon/2+\mu^-(a_2)-\varepsilon/2^2-\mu^-(a_1\vee a_2)\\ &= \mu^-(a_2)-(\varepsilon/2+\varepsilon/2^2).\end{aligned}$$

Suppose that $\mu^*(c_m)\ge\mu^-(a_m)-\sum_{i=1}^m\varepsilon/2^i$. Then

$$\begin{aligned}\mu^*(c_{m+1}) &= \mu^*(c_m\wedge b_{m+1})\\ &\ge \mu^*(c_m)+\mu^*(b_{m+1})-\mu^*(c_m\vee b_{m+1})\\ &> \mu^-(a_m)-\sum_{i=1}^m\varepsilon/2^i+\mu^-(a_{m+1})-\varepsilon/2^{m+1}-\mu^-(a_m\vee a_{m+1})\\ &= \mu^-(a_{m+1})-\sum_{i=1}^{m+1}\varepsilon/2^i.\end{aligned}$$

Thus by induction, we obtain that $\mu^*(c_n) \geq \mu^-(a_n) - \sum_{i=1}^n \varepsilon/2^i$, for all n. As $b_n \leq a_n$ ($n \in \mathbb{N}$), and L is locally complete, therefore $c = \bigwedge_{n=1}^\infty c_n = \bigwedge_{n=1}^\infty b_n \leq \bigwedge_{n=1}^\infty a_n = a$, which yield, using Theorem 3.1(a) that $\mu^-(a) \geq \mu^*(c) = \lim_{n\to\infty} \mu^*(c_n) \geq \lim_{n\to\infty}(\mu^-(a_n) - \sum_{i=1}^n \varepsilon/2^i) = \lim_{n\to\infty} \mu^-(a_n) - \varepsilon$. So $\mu^-(a) \geq \lim_{n\to\infty} \mu^-(a_n)$. Now since $a \leq a_n$ for all n and μ^- is monotone the result follows.

(iii)⇒(iv) Put $\mathcal{D} = \{a \in L : \mu(b) \leq \mu^-(b \wedge a) + \mu^-(b \wedge (b \wedge a)^{'})$ for all $b \in M$ with $\mu(b) < \infty\}$. Since μ^- is subadditive, so $\mathcal{M}(\mu^-; M) \subseteq \mathcal{D}$. Let $a \in \mathcal{D}$. Let $c \in L$, $b \in M_\delta$ with $b \leq c$ and $\mu^*(b) < \infty$. Then, by Lemma 3.1(a) there exists a sequence $\{b_n\}$ in M such that $b_n \downarrow b$ and $\mu^*(b) = \lim_{n\to\infty} \mu(b_n)$. Now, we have $\mu(b_n) \leq \mu^-(b_n \wedge a) + \mu^-(b_n \wedge (b_n \wedge a)^{'})$. Since L is σ-continuous, and μ^- is monotone, using 2.6 we obtain,

$$\begin{aligned}
\mu^*(b) &= \lim_{n\to\infty} \mu(b_n) \\
&\leq \lim_{n\to\infty} \mu^-(b_n \wedge a) + \lim_{n\to\infty} \mu^-(b_n \wedge (b_n \wedge a)^{'}) \\
&= \mu^-(b \wedge a) + \mu^-(b \wedge (b \wedge a)^{'}) \\
&\leq \mu^-(c \wedge a) + \mu^-(c \wedge (c \wedge a)^{'}).
\end{aligned}$$

Now by definition of μ^-, we have $\mu^-(c) \leq \mu^-(c \wedge a) + \mu^-(c \wedge (c \wedge a)^{'})$. The reverse inequality follows from the supermodularity of μ^- and the orthomodular law. Thus $a \in \mathcal{M}(\mu^-; M)$. Hence $\mathcal{D} \subseteq \mathcal{M}(\mu^-; M)$.

(iv)⇒(v) Let $a \in \mathcal{M}(\mu_*; M)$. Let $b \in M$ with $\mu(b) < \infty$. $\mu(b) = \mu_*(b) \leq \mu_*(b \wedge a) + \mu_*(b \wedge (b \wedge a)^{'}) \leq \mu^-(b \wedge a) + \mu^-(b \wedge (b \wedge a)^{'})$. Thus $a \in \mathcal{M}(\mu^-; M)$. □

Definition 3.5. *Define $\mu_- : L \to [0, \infty]$ by*

$$\mu_-(a) = \inf\{\mu_*(b) : a \leq b, b \in M_\sigma\},\ a \in L.$$

We observe that: (i) $\mu_-(0) = 0$, (ii) μ_- is monotone, (iii) $\mu_- \geq \mu_*$, (iv) $\mu_-|M_\sigma = \mu_*|M_\sigma$; particularly $\mu_-|M = \mu_*|M$, (v) if μ is monotone, then $\mu_-|M = \mu$, i.e. μ_- is an extension of μ, (vi) if μ is superadditive, then μ_- is superadditive.

Proposition 3.3. *Let $M = M_\sigma$. Then $\mu^* \leq \mu_-$.*

Proof. Let $a \in L$ and $b \in M$ with $a \leq b$. Since $\mu^*|M \leq \mu_*|M$ and μ^* is monotone, we obtain $\mu^*(a) \leq \mu^*(b) \leq \mu_*(b)$. Hence $\mu^*(a) \leq \mu_-(a)$. □

Using similar arguments as in Theorem 3.2, we obtain the following:

Theorem 3.3. *Let μ be an lsc-measure on M. Then $(i) \Rightarrow (ii) \Rightarrow (iii)$, where (i) μ is submodular. (ii) $\mu_*|M_\sigma$ is submodular. (iii) (a) μ_- is submodular. (b) μ_- is semi-continuous from below.*

In addition, if L is an orthomodular lattice, M is a proper suborthomodular lattice of L and μ is superadditive, then (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v), where (iv) $\mathcal{M}(\mu_-; M) = \{a \in L : \mu(b) \geq \mu_-(b \wedge a) + \mu_-(b \wedge (b \wedge a)') \text{ for all } b \in M\}$. (v) $\mathcal{M}(\mu^*; M) \subseteq \mathcal{M}(\mu_-; M)$.

Definition 3.6. *Let $C \subseteq L$. We call a function $\rho : L \to [0, \infty]$ C-inner regular if for any $a \in E$, $\rho(a) = \sup\{\rho(b) : b \leq a, b \in C,\ \mu(b) < \infty\}$, and is said to be C-outer regular if for any $a \in E$, $\rho(a) = \inf\{\rho(b) : a \leq b, b \in C\}$. The function ρ is said to be quasi*-measure (respectively, quasi$_*$-measure) on L, if it is supermodular usc-measure (respectively, submodular lsc-measure); the pair (L, ρ) is called quasi*-measure space (respectively, quasi$_*$-measure space). The function ρ is said to be quasi-measure on L, if it is either M_δ-inner regular quasi*-measure or M_σ-outer regular quasi$_*$-measure.*

Since $\mu_-|M_\sigma = \mu_*|M_\sigma$, the function μ_- is M_σ-outer regular. Also, since $\mu^-|M_\delta = \mu^*|M_\delta$, the function μ^- is M_δ-inner regular.

Theorem 3.4.

(a) *Every finite-stable, supermodular, usc-measure defined on a proper sublattice M of L, can be extended to an M_δ-inner regular quasi*-measure on L.*
(b) *Every submodular, lsc-measure defined on a proper sublattice M of L, can be extended to an M_σ-outer regular quasi$_*$-measure on L.*

Proof. Follows from Theorem 3.2 and Theorem 3.3. □

Theorem 3.5. *Let μ be a semi-continuous measure on M. Then we have:*

(i) *μ^- is semifinite and M_δ-inner regular.*
(ii) *μ^- is a semiextension of μ (i.e. $\mu(a) = \mu^-(a)$ for $a \in M$, $\mu(a) < \infty$), and μ^- is the smallest semiextension of μ within the class of all monotone, $[0, \infty]$-valued functions on L which are semi-continuous from above on M_δ.*
(iii) *μ^- is an extension of μ if and only if μ is semifinite.*
(iv) *μ_- is the largest extension of μ within the class of all monotone, $[0, \infty]$-valued functions on L which are semi-continuous from below on M_σ.*

Proof. We shall prove only (ii) and (iii).

Obviously μ^- is a semiextension of μ. Now, let $\eta : L \to [0, \infty]$ be a monotone semiextension of μ, and is semi-continuous from above on M_δ. Let $a \in L$, $b \in M_\delta$ with $b \leq a$ and $\mu^*(b) < \infty$. Then there exists a sequence $\{b_n\}_{n=1}^{\infty}$ in M such that $b_n \downarrow b$ and $\mu^*(b) = \inf_n \mu(b_n)$. Thus $\mu^*(b) = \inf_n \eta(b_n) = \eta(b) \leq \eta(a)$, and so $\mu^-(a) \leq \eta(a)$, which proves (ii).

Define a function η on M by $\eta(a) = \sup\{\mu(b) : b \leq a, b \in M, \mu(b) < \infty\}$. Now, μ is semifinite if and only if $\mu = \eta$. Since μ^- is an extension of η, (iii) follows. □

4. Measuring envelopes in D-lattices

In this Section we shall consider L to be a σ-complete σ-continuous D-lattice and study a few properties of the function μ defined on L, and its measuring envelopes μ_* and μ^* (introduced in Section 3), in the context of some weaker forms of additivity of the function μ.

Definition 4.1. *Let E be a difference poset (i.e. a D-poset) or effect algebra and $\mu : E \to [0, \infty]$ be a function. Then μ is called:*

(i) *null-additive, if $\mu(b \oplus c) = \mu(b)$ provided $b, c \in E$, $b \perp c$ and $\mu(c) = 0$.*

(ii) *converse null-additive, if $\mu(c \ominus b) = 0$ provided $b, c \in E$, $b \leq c$, $\mu(b) = \mu(c) < \infty$.*

(iii) *superadditive, if $\mu(b \oplus c) \geq \mu(b) + \mu(c)$ provided $b, c \in E$, $b \perp c$.*

Observe that, μ is null-additive if and only if $\mu(b \ominus c) = \mu(b)$ provided $b, c \in M$, $c \leq b$ and $\mu(c) = 0$. Also, observe that if μ is superadditive, then μ is converse null-additive. The converse of this statement need not be true as μ_2 (given below) is converse null-additive but not superadditive.

Example 4.1. Let $E_1 = \{0, a, b, c, d, e, 1\}$. Let us define: $a \oplus b = b \oplus a = c$, $b \oplus c = c \oplus b = a \oplus d = d \oplus a = e \oplus e = 1$ and let $x \oplus 0 = 0 \oplus x = x$ for all $x \in E_1$. Then E_1 is an effect algebra. Define functions μ_1 and μ_2 on E_1 as follows:

(I) $\mu_1(x) = 0$ if $x \in \{0, b, c, d, 1\}$ and $\mu_1(x) = 1$ if $x \in \{a, e\}$;

(II) $\mu_2(x) = 0$ if $x \in \{0, a, b, c, d, 1\}$ and $\mu_2(e) = 1$.

Then we have,

(i) μ_2 is null-additive, but μ_1 is not;

(ii) μ_1 and μ_2 are not superadditive;

(iii) μ_2 is converse null-additive, but μ_1 is not.

Example 4.2. Let $E_2 = \{0, a, b, c, 1\}$. Let us define: $a \oplus b = b \oplus a = c$, $b \oplus c = c \oplus b = a \oplus a = 1$ and let $x \oplus 0 = 0 \oplus x = x$ for all $x \in E_2$. Then E_2 is an effect algebra. Define functions μ_3, μ_4 and μ_5 on E_2 as follows:

(I) $\mu_3(x) = 0$ if $x \in \{0, b\}$, and $\mu_3(x) = 1$ if $x \in \{a, c, 1\}$.

(II) $\mu_4(x) = 0$ if $x \in \{0, a, b\}$, and $\mu_4(x) = 1$ if $x \in \{c, 1\}$.

(III) $\mu_5(x) = 0$ for all $x \in E_2$.

Then we have,

(i) μ_3 and μ_5 are null-additive, but μ_4 is not;

(ii) μ_4 and μ_5 are superadditive, but μ_3 is not;

(iii) μ_4 and μ_5 are converse null-additive, but μ_3 is not.

Proposition 4.1. *Let μ_1 and μ_2 be two $[0,\infty)$-valued functions defined on an effect algebra E. Define $(\mu_1+\mu_2)(a)=\mu_1(a)+\mu_2(a)$, and $(\mu_1-\mu_2)(a)=\mu_1(a)-\mu_2(a), a\in E$. Then the following hold:*

(i) *If μ_1 and μ_2 are supermodular (submodular, respectively) then $\mu_1+\mu_2$ is supermodular (submodular, respectively).*
(ii) *If μ_1 and μ_2 are modular, then so are $\mu_1+\mu_2$ and $\mu_1-\mu_2$.*
(iii) *If μ_1 and μ_2 are null-additive, then so are $\mu_1+\mu_2$ and $\mu_1-\mu_2$.*

Proof. (i) Let μ_1 and μ_2 be supermodular. Let $a,b\in E$. Then

$$\begin{aligned}(\mu_1+\mu_2)(a)+(\mu_1+\mu_2)(b) &= \mu_1(a)+\mu_2(a)+\mu_1(b)+\mu_2(b)\\ &\le \mu_1(a\vee b)+\mu_1(a\wedge b)+\mu_2(a\vee b)+\mu_2(a\wedge b)\\ &= (\mu_1+\mu_2)(a\vee b)+(\mu_1+\mu_2)(a\wedge b).\end{aligned}$$

If μ_1 and μ_2 are submodular, then by similar arguments, $\mu_1+\mu_2$ is submodular.

(ii) The proof is obvious.

(iii) Let $a,b\in E$ and $a\perp b$. Since μ_1 and μ_2 are null-additive, therefore, we have $\mu_1(a\oplus b)=\mu_1(a)$ and $\mu_2(a\oplus b)=\mu_2(a)$, whenever $\mu_1(b)=0$ and $\mu_2(b)=0$. So $(\mu_1+\mu_2)(b)=0$ and $(\mu_1+\mu_2)(a\oplus b)=\mu_1(a\oplus b)+\mu_2(a\oplus b)=(\mu_1+\mu_2)(a)$.

By similar arguments, we can show that $\mu_1-\mu_2$ is also null-additive. □

From now onwards, let L be a σ-complete σ-continuous D-lattice, M be a proper D-sublattice of L and μ be a $[0,\infty]$-valued function defined on M.

Proposition 4.2. *Let μ be a superadditive function defined on M. Then the function μ_* is superadditive.*

Proof. Let $a,b\in L$ and $a\perp b$. By definition of μ_*, we have $c,d\in M$, $c\le a$, $d\le b$, $\mu_*(a)-\varepsilon/2<\mu(c)$ and $\mu_*(b)-\varepsilon/2<\mu(d)$. Now, by 2.8 (iv) we have $c\oplus d\le a\oplus b$. Since μ_* is monotone, we have $\mu_*(a\oplus b)\ge\mu_*(c\oplus d)\ge\mu(c\oplus d)\ge\mu(c)+\mu(d)>\mu_*(a)-\varepsilon/2+\mu_*(b)-\varepsilon/2=\mu_*(a)+\mu_*(b)-\varepsilon$. Since ε is arbitrary, the result follows. □

Definition 4.2. *Let μ be a monotone null-additive function on M. Let $N_\mu=\{a\in L:\exists\, a_1,a_2\in M \text{ such that } a_1\le a\le a_2 \text{ and } \mu(a_2\ominus a_1)=0\}$. The function $\overline{\mu}:N_\mu\to[0,+\infty]$ is defined as $\overline{\mu}(a)=\mu(a_1)$, where $a_1,a_2\in M$, $a_1\le a\le a_2$ and $\mu(a_2\ominus a_1)=0$. Elements of N_μ are sometimes also referred to as μ-measurable.*

Since $a_1\le a_2$, so by 2.8(i), we have $a_2=a_1\oplus(a_2\ominus a_1)$. Hence by null-additivity of μ, we get $\mu(a_2)=\mu(a_1)$. So $\overline{\mu}(a)$ may equally be defined as $\mu(a_2)$. Further, if $d\le a$ and $d\in M$, then $\mu(d)\le\mu(a_2)=\mu(a_1)$. Hence

$\mu(a_1) = \sup\{\mu(d) : d \leq a, d \in M\}$. Thus $\overline{\mu}$ is well-defined. Also $\overline{\mu}$ is an extension of μ on N_μ; in particular $\overline{\mu}(0) = \mu(0)$.

Also if $a_1, a_2 \in N_\mu$ and $a_1 \leq a_2$, then $\overline{\mu}(a_1) \leq \overline{\mu}(a_2)$: Since $a_1, a_2 \in N_\mu$, then there exist $b_1, c_1, b_2, c_2 \in M$ such that $b_1 \leq a_1 \leq c_1, \mu(c_1 \ominus b_1) = 0$ and $b_2 \leq a_2 \leq c_2, \mu(c_2 \ominus b_2) = 0$. Again, since $b_1 \leq a_1 \leq a_2 \leq c_2$, so $\overline{\mu}(a_1) = \mu(b_1) \leq \mu(c_2) = \overline{\mu}(a_2)$.

Let μ and ν be monotone null-additive functions defined on M. If ker μ = ker ν, then $N_\mu = N_\nu$, where ker $\mu = \{a \in M : \mu(a) = 0\}$.

Proposition 4.3. *Let μ be a monotone null-additive function defined on M. If μ is converse null-additive, then the function $\overline{\mu}$ is also converse null-additive on N_μ.*

Proof. Let $b, c \in N_\mu$ with $b \leq c$ and $\overline{\mu}(b) = \overline{\mu}(c) < \infty$. Then there exist $b_1, c_1, b_2, c_2 \in M$ such that $b_1 \leq b \leq b_2, \mu(b_2 \ominus b_1) = 0$ and $c_1 \leq c \leq c_2, \mu(c_2 \ominus c_1) = 0$. Since μ is converse null-additive on M and $\mu(b_1) = \overline{\mu}(b) = \overline{\mu}(c) = \mu(c_2)$, so $\mu(c_2 \ominus b_1) = 0$. Since $b \leq c \leq c_2$, then by 2.8 (ii) and (iii) we have $c \ominus b \leq c_2 \ominus b \leq c_2 \ominus b_1$. Again since $\overline{\mu}$ is monotone, $\overline{\mu}(c \ominus b) \leq \mu(c_2 \ominus b_1) = 0$, so $\overline{\mu}(c \ominus b) = 0$. Hence $\overline{\mu}$ is converse null-additive on N_μ. □

Definition 4.3. *Let μ be a function defined on M. Let $d \in L$ such that $d \notin M$ and $\mu^*(d) < \infty$. A function μ is called weak converse null-additive (on M) with respect to d, if for all $a, b \in M$ such that $a \leq d \leq b$ and $\mu(a) = \mu(b) < \infty$, there exist elements $a_0, b_0 \in M$ with $a_0 \leq d \leq b_0$ such that $\mu(b_0 \ominus a_0) = 0$; μ is called weak converse null-additive (on M), if it is weak converse null-additive (on M) with respect to d, whenever $d \notin M$ and $\mu^*(d) < \infty$.*

Observe that if μ is converse null-additive, then it is weak converse null-additive, which yields that every superadditive function is weak converse null-additive. The converse of this statement need not be true as μ_2 in Example 4.1 is weak converse null-additive but not superadditive.

Theorem 4.1. *Let μ be a monotone null-additive function defined on M. Then the following hold:*

(i) *If $a \in N_\mu$, then $\mu_*(a) = \mu^*(a)$.*
(ii) *If for every $d \in L$ such that $\mu_*(d) = \mu^*(d) < \infty$, $d \in N_\mu$, then μ is weak converse null-additive.*

Proof. (i) Let $a \in N_\mu$. Then there exist $b, c \in M$ such that $b \leq a \leq c$ and $\mu(c \ominus b) = 0$. So $\mu(b) \leq \mu_*(a) \leq \mu^*(a) \leq \mu(c)$. Since $\mu(b) = \mu(c)$, we have $\mu_*(a) = \mu^*(a)$.

(ii) Suppose that μ is not weak converse null-additive (on M), then there exists an element $d \in L$ such that $\mu^*(d) < \infty$ and if for all $b, c \in M$,

$b \leq d \leq c$ and $\mu(b) = \mu(c) < \infty$, $\mu(c \ominus b) \neq 0$. Since $\mu_*(d) = \mu^*(d) < \infty$, we have that $d \in N_\mu$. Then there exist $b_0, c_0 \in M$ such that $b_0 \leq d \leq c_0$ and $\mu(c_0 \ominus b_0) = 0$, a contradiction. □

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