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FRACTIONAL CALCULUS OPERATOR AND CERTAIN APPLICATIONS IN GEOMETRIC FUNCTION THEORY

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Abstract. Using a operator involving fractional calculus introduced by Owa and Srivastava [8], two novel families:

$$\mathcal{V}_{\delta}^{\alpha,\beta}(\nu;\gamma) \text{ and } \mathcal{W}_{\delta}^{\alpha,\beta}(\mu;\gamma)$$

$$(\delta \neq 0,\ \alpha < 1,\ \beta < 1,\ \gamma < 1,\ \mu \geq 0,\ \nu \in (-1,0) \cup (0,1))$$

of functions $f(z)$ which are analytic and univalent in the open unit disk $\mathcal{U}$ are defined. Moreover some consequences of main results are shown.

1. Introduction and definitions

Let $\mathcal{T}(n)$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \quad (n \in \mathcal{N} = \{1,\ 2,\ 3,\ \dots\}), \tag{1.1}$$

that are *analytic* in the open unit disk

$$\mathcal{U} = \big\{z : z \in \mathbf{C} \text{ and } |z| < 1\big\}.$$

Also let $\mathcal{S}(n)$ denote the class of all functions which are *univalent* in $\mathcal{U}$.

A function $f(z) \in \mathcal{T}(n)$ is said to be *starlike of order* Δ in $\mathcal{U}$, if it satisfies the inequality:

$$\Re e\left(\frac{zf'(z)}{f(z)}\right) > \Delta \quad (z \in \mathcal{U};\ 0 \leq \Delta < 1). \tag{1.2}$$

We denote by $\mathcal{S}(\Delta)$ the subclass of $\mathcal{T}(n)$ consisting of functions which are starlike of order Δ $(0 \leq \Delta < 1)$ in $\mathcal{U}$.

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Further, a function $f(z) \in \mathcal{T}(n)$ is said to be *convex of order* Δ in $\mathcal{U}$, if it satisfies the inequality:

$$\Re e\left(1+\frac{zf''(z)}{f'(z)}\right) > \Delta \quad (z \in \mathcal{U};\ 0 \leq \Delta < 1). \tag{1.3}$$

The subclass of $\mathcal{T}(n)$ of such functions is denoted by $\mathcal{K}(\Delta)$.

We note that

$$f(z) \in \mathcal{K}_n(\Delta) \iff zf'(z) \in \mathcal{S}_n(\Delta), \tag{1.4}$$

and

$$\mathcal{S}^*(\Delta) \subset \mathcal{S}^*(0) \equiv \mathcal{S}^* \text{ and } \mathcal{K}(\Delta) \subset \mathcal{K}(0) \equiv \mathcal{K},$$

where $\mathcal{S}^*$ and $\mathcal{K}$ are the subclasses of $\mathcal{T}(n)$ consisting of functions being starlike and convex in $\mathcal{U}$, respectively. See [3], [4], and [11] for the details of definitions in (1.2)-(1.4).

Various definitions of fractional calculus operators are given by many authors, see [10]. We use here the following definitions due to Owa and Srivastava [8], and see also [7, 1].

Definition 1. *Let a function $f(z)$ be analytic in a simply-connected region of the z-plane containing the origin. The fractional integral of order μ $(\mu > 0)$ is defined by*

$$D_z^{-\mu}\{f(z)\} = \frac{1}{\Gamma(\mu)}\int_0^z f(\xi)(z-\xi)^{\mu-1}d\xi, \tag{1.5}$$

and the fractional derivative of order μ $(0 \leq \mu < 1)$ is defined by

$$D_z^{\mu}\{f(z)\} = \frac{1}{\Gamma(1-\mu)}\frac{d}{dz}\int_0^z f(\xi)(z-\xi)^{-\mu}d\xi, \tag{1.6}$$

where the multiplicity of $(z-\xi)^{\mu-1}$ involved in (1.5) *and that of $(z-\xi)^{-\mu}$ in* (1.6) *are removed by requiring $\log(z-\xi)$ to be real when $z-\xi > 0$.*

Definition 2. *Using Definition* 1, *the fractional derivative of order $m + \mu$ $(m \in \mathcal{N}_0 = \mathcal{N} \cup \{0\}; 0 \leq \mu < 1)$ is defined by*

$$D_z^{m+\mu}\{f(z)\} = \frac{d^m}{dz^m}D_z^{\mu}\{f(z)\}. \tag{1.7}$$

With the help of the definitions in (1.6) and (1.7), Owa and Srivastava [8] defined a modification of the fractional calculus operator $\mathcal{J}_z^{\lambda}$ $(\lambda \neq 2,\ 3,\ 4,\ \ldots)$ by

$$\mathcal{J}_z^{\lambda}\{f(z)\} = \Gamma(2-\lambda)z^{\lambda}\mathcal{D}_z^{\lambda}\{f(z)\} \tag{1.8}$$

for functions (1.1) belonging to the class $\mathcal{T}(n)$.

By making use of the fractional calculus operator $\mathcal{J}_z^{\lambda}$, we now define two important and novel families $\mathcal{V}_{\delta}^{\alpha,\beta}(\nu;\gamma)$ and $\mathcal{W}_{\delta}^{\alpha,\beta}(\nu;\gamma)$ of functions $f(z)$ in the class $\mathcal{T}(n)$ below.

Definition 3. *A function $f(z) \in \mathcal{T}(n)$ is said to belong to $\mathcal{V}_\delta^{\alpha,\beta}(\nu;\gamma)$, if it satisfies the inequality:*

$$\Re e\left\{\frac{\alpha-\beta+(1-\alpha)\frac{\mathcal{J}_z^{1+\alpha}\{f(z)\}}{\mathcal{J}_z^{\alpha}\{f(z)\}}-(1-\beta)\frac{\mathcal{J}_z^{1+\beta}\{f(z)\}}{\mathcal{J}_z^{\beta}\{f(z)\}}}{1-\gamma\frac{J_z^{\beta}\{f(z)\}}{J_z^{\alpha}\{f(z)\}}}\right\}$$
$$\left\{\begin{matrix} < \frac{\nu(\nu-1)}{\delta(1+\nu)^2} & \textit{when } \ (\delta>0, \ -1<\nu<0) \\ > \frac{\nu(1+\nu)}{\delta(1-\nu)^2} & \textit{when } \ (\delta<0, \quad 0<\nu<1) \end{matrix}\right\}, \tag{1.9}$$

where $\delta \in \mathbf{R}^* := \mathbf{R}, \ \alpha<1, \ \beta<1, \ \gamma<1,$ and $z \in \mathcal{U}$.

Definition 4. *A function $f(z) \in \mathcal{T}(n)$ is said to belong to $\mathcal{W}_\delta^{\alpha,\beta}(\mu;\gamma)$, if it satisfies the inequality:*

$$\Re e\left\{\left[\frac{1}{1-\gamma}\left(\frac{\mathcal{J}_z^{\alpha}\{f(z)\}}{\mathcal{J}_z^{\beta}\{f(z)\}}-\gamma\right)\right]^\delta\right\} > \mu, \tag{1.10}$$

where $\delta \in \mathbf{R}^*, \ \alpha<2, \ \beta<2, \ \gamma<1, \ \mu \geq 0, \ z \in \mathcal{U}$ and the value of

$$\left[\frac{1}{1-\gamma}\left(\frac{\mathcal{J}_z^{\alpha}\{f(z)\}}{\mathcal{J}_z^{\beta}\{f(z)\}}-\gamma\right)\right]^\delta$$

is taken to be its principal value.

Noting that

$$\mathcal{W}_1^{\alpha,\beta}(0;\gamma) =: \mathcal{A}(\alpha,\beta,\gamma),$$
$$\mathcal{W}_1^{0,0}(0;\gamma) =: \mathcal{A}(1,0,\gamma) =: \mathcal{S}^*(\gamma)$$

and

$$\mathcal{W}_1^{1+\alpha,0}(0;\gamma) =: \mathcal{A}(1+\alpha,0,\gamma) =: \mathcal{S}^*(\gamma,\alpha),$$

where the class $\mathcal{A}(\alpha,\beta,\gamma)$ was studied Choi et al. [2] and the classes $\mathcal{S}^*(\gamma)$ and $\mathcal{S}^*(\gamma,\alpha)$ were studied by Owa and Shen [9] when $n=1$. We also denote:

$$\mathcal{V}_1^{\alpha,\beta}(0;\gamma) =: \mathcal{B}(\alpha,\beta,\gamma),$$
$$\mathcal{V}_1^{0,0}(0;\gamma) =: \mathcal{B}(1,0,\gamma) =: \mathcal{E}(\gamma)$$

and

$$\mathcal{V}_1^{1+\alpha,0}(0;\gamma) =: \mathcal{B}(1+\alpha,0,\gamma) =: \mathcal{F}(\gamma,\alpha).$$

In this paper, we also point out some certain relationships between the classes $\mathcal{V}_\delta^{\alpha,\beta}(\nu;\gamma)$, $\mathcal{W}_\delta^{\alpha,\beta}(\mu;\gamma)$ and their subclasses.

2. Main results

Now, we mention the following result which is used in the sequel.

Lemma 1. *Let the function* $f(z)$ *defined by* (1.1) *and let* $\lambda < 1$. *Then*

$$z(\mathcal{J}_z^{\lambda}\{f(z)\})' = (1-\lambda)\mathcal{J}_z^{1+\lambda}\{f(z)\} + \lambda\mathcal{J}_z^{\lambda}\{f(z)\} \quad (z \in \mathcal{U}). \tag{2.1}$$

Lemma 2. ([5, 6]) *Let* $w(z)$ *be an analytic function in the unit disk* $\mathcal{U}$ *with* $w(0) = 0$ *and let* $0 < r < 1$. *If* $|w(z)|$ *attains at* z_0 *its maximum value on the circle* $|z| = r$, *then*

$$z_0 w'(z_0) = cw(z_0) \qquad (c \geq 1). \tag{2.2}$$

Making use of Lemmas 1 and 2, we first give the following theorem:

Theorem 1. *Let* $f(z) \in \mathcal{T}(n)$, $\delta \in \mathbf{R}^*$, $\alpha < 1$, $\beta < 1$, $\gamma < 1$ *and* $z \in \mathcal{U}$. *If* $f(z) \in \mathcal{V}_\delta^{\alpha,\beta}(\nu;\gamma)$ *then* $f(z) \in \mathcal{W}_\delta^{\alpha,\beta}(\mu;\gamma)$, *where* $\mu = 1 - |\nu|$.

Proof. Using the definition of fractional calculus, we have

$$\mathcal{J}_z^{\lambda}\{f(z)\} = z + \sum_{k=n+1}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)}\, a_k z^k \qquad (n \in \mathcal{N}).$$

Since

$$\frac{1}{1-\gamma}\left(\frac{\mathcal{J}_z^{\alpha}\{f(z)\}}{\mathcal{J}_z^{\beta}\{f(z)\}} - \gamma\right) = 1 + d_1 z + d_2 z^2 + \cdots$$

define a function $w(z)$ by

$$1 + \nu w(z) = \left[\frac{1}{1-\gamma}\left(\frac{\mathcal{J}_z^{\alpha}\{f(z)\}}{\mathcal{J}_z^{\beta}\{f(z)\}} - \gamma\right)\right]^{\delta}. \tag{2.3}$$

Clearly, $w(z)$ is an analytic function in $\mathcal{U}$, and $w(0) = 0$. Differentiation of (2.3) gives

$$\frac{\frac{z(\mathcal{J}_z^{\alpha}\{f(z)\})'\mathcal{J}_z^{\beta}\{f(z)\} - z(\mathcal{J}_z^{\beta}\{f(z)\})'\mathcal{J}_z^{\alpha}\{f(z)\}}{(\mathcal{J}_z^{\beta}\{f(z)\})^2}}{\frac{\mathcal{J}_z^{\alpha}\{f(z)\}}{\mathcal{J}_z^{\beta}\{f(z)\}} - \gamma} = \frac{\nu z w'(z)}{\delta[1 + vw(z)]} \tag{2.4}$$

and using (2.1) in (2.4) once again :

$$\mathcal{G}(z) := \frac{\alpha - \beta + (1-\alpha)\frac{\mathcal{J}_z^{1+\alpha}\{f(z)\}}{\mathcal{J}_z^{\alpha}\{f(z)\}} - (1-\beta)\frac{\mathcal{J}_z^{1+\beta}\{f(z)\}}{\mathcal{J}_z^{\beta}\{f(z)\}}}{1 - \gamma\frac{J_z^{\beta}\{f(z)\}}{J_z^{\alpha}\{f(z)\}}},$$

or, equivalently,

$$\mathcal{G}(z) = \frac{\nu z w'(z)}{\delta[1 + \nu w(z)]} \tag{2.5}$$

Now, suppose that there exists a point $z_0 \in \mathcal{U}$ such that

$$\max \{ \ |w(z)| \ \} = |w(z_0)| = 1 \ \ when \ \ |z| \leq |z_0| \ \ (z \in \mathcal{U}).$$

Then, applying Lemma 2, we can write $z_0 w'(z_0) = cw(z_0) \quad (c \geq 1)$ and $w(z_0) = e^{i\theta}$. Thus, from (2.5) we obtain

$$\Re e\{\mathcal{G}(z_0)\} = \Re e\left(\frac{\nu z_0 w'(z_0)}{\delta[1+\nu w(z_0)]}\right)$$
$$= \frac{c\nu}{\delta}\,\Re e\left(\frac{e^{i\theta}}{1+\nu e^{i\theta}}\right) = \frac{c\nu(\nu+cos\theta)}{\delta(1+2\nu cos\theta+\nu^2)}. \quad (2.6)$$

and further, from (2.6) yields that

$$\Re e\,\{\mathcal{G}(z_0)\}\begin{cases} \geq \frac{\nu(\nu-1)}{\delta(1+\nu)^2} & \text{if } (\delta>0,\ -1<\nu<0) \\ \leq \frac{\nu(1+\nu)}{\delta(1-\nu)^2} & \text{if } (\delta<0,\ \ \ 0<\nu<1) \end{cases}, \quad (2.7)$$

where $c \geq 1$. But, the inequalities in (2.7) contradict the inequalities (1.9) relating to our assumptions that $f(z) \in \mathcal{V}_\delta^{\alpha,\beta}(\nu;\gamma)$, and hence, we conclude that $|w(z)| < 1$ for all $z \in \mathcal{U}$. Consequently, it follows from (2.3) that

$$\left|\left[\frac{1}{1-\gamma}\left(\frac{\mathcal{J}_z^\alpha\{f(z)\}}{\mathcal{J}_z^\beta\{f(z)\}}-\gamma\right)\right]^\delta - 1\right| < |v|, \quad (2.8)$$

which implies that

$$\Re e\left\{\left[\frac{1}{1-\gamma}\left(\frac{\mathcal{J}_z^\alpha\{f(z)\}}{\mathcal{J}_z^\beta\{f(z)\}}-\gamma\right)\right]^\delta\right\} > \mu = 1-|v|, \quad (2.9)$$

i.e., $f(z) \in \mathcal{W}_\delta^{\alpha,\beta}(\mu;\gamma)$. Therefore, the proof of Theorem 1 is completed. □

If we put $zf'(z)$ in stead of $f(z)$ in the Theorem 1, we then obtain the following theorem.

Theorem 2. *Let* $\delta \in \mathbf{R}^*,\ \alpha<1,\ \beta<1\ \gamma<1$ *and* $z \in \mathcal{U}$. *If the function* $f(z) \in \mathcal{T}(n)$ *satisfies the following conditions:*

$$\Re e\left\{\frac{\alpha-\beta+(1-\alpha)\frac{\mathcal{J}_z^{1+\alpha}\{zf'(z)\}}{\mathcal{J}_z^{\alpha}\{zf'(z)\}}-(1-\beta)\frac{\mathcal{J}_z^{1+\beta}\{zf'(z)\}}{\mathcal{J}_z^{\beta}\{zf'(z)\}}}{1-\gamma\frac{J_z^\beta\{zf'(z)\}}{J_z^\alpha\{zf'(z)\}}}\right\}$$
$$\begin{cases} < \frac{\nu(\nu-1)}{\delta(1+\nu)^2} & when\ \ (\delta>0,\ -1<\nu<0) \\ > \frac{\nu(1+\nu)}{\delta(1-\nu)^2} & when\ \ (\delta<0,\ \ \ 0<\nu<1) \end{cases}, \quad (2.10)$$

then

$$\Re e\left\{\left[\frac{1}{1-\gamma}\left(\frac{\mathcal{J}_z^\alpha\{zf'(z)\}}{\mathcal{J}_z^\beta\{zf'(z)\}}-\gamma\right)\right]^\delta\right\} > 1-|\nu|. \quad (2.11)$$

3. Certain consequences of the main results

Now we introduce two subclasses $\mathbf{V}_{\delta}^{\alpha,\beta}(\mu;\gamma)$ and $\mathbf{W}_{\delta}^{\alpha,\beta}(\mu;\gamma)$ of functions $f(z) \in \mathcal{T}(n)$, respectively, satisfying the inequalities (2.10) and (2.11). In particular, we define some certain subclasses of the these classes as following:

$$\mathbf{W}_{1}^{\alpha,\beta}(0;\gamma) =: \mathbf{B}(\alpha,\beta,\gamma),$$

$$\mathbf{W}_{1}^{0,0}(0;\gamma) =: \mathbf{B}(1,0,\gamma) =: \mathbf{E}(\gamma),$$

$$\mathbf{W}_{1}^{1+\alpha,0}(0;\gamma) =: \mathcal{B}(1+\alpha,0,\gamma) =: \mathbf{F}(\gamma,\alpha),$$

$$\mathbf{V}_{1}^{\alpha,\beta}(0;\gamma) =: \mathbf{A}(\alpha,\beta,\gamma),$$

$$\mathbf{V}_{1}^{0,0}(0;\gamma) =: \mathbf{A}(1,0,\gamma) =: \mathcal{K}(\gamma),$$

$$\mathbf{V}_{1}^{1+\alpha,0}(0;\gamma) =: \mathcal{A}(1+\alpha,0,\gamma) =: \mathcal{K}(\gamma,\alpha).$$

Next, we can give some interesting and/or important results for the above subclasses below:

Corollary 1. *If $f(z) \in \mathcal{B}(\alpha,\beta,\gamma)$ then $f(z) \in \mathcal{A}(\alpha,\beta,\gamma)$.*

Corollary 2. *If $f(z) \in \mathcal{F}(\gamma,\alpha)$ then $f(z) \in \mathcal{S}^*(\gamma,\alpha)$.*

Corollary 3. *If $f(z) \in \mathcal{E}(\gamma)$ then $f(z) \in \mathcal{S}^*(\gamma)$, i.e., $f(z)$ is starlike function of order γ $(0 \leq \gamma < 1)$ in $\mathcal{U}$.*

Corollary 4. *If $f(z) \in \mathbf{B}(\alpha,\beta,\gamma)$ then $f(z) \in \mathbf{A}(\alpha,\beta,\gamma)$.*

Corollary 5. *If $f(z) \in \mathbf{F}(\gamma,\alpha)$ then $f(z) \in \mathcal{K}(\gamma,\alpha)$.*

Corollary 6. *If $f(z) \in \mathbf{E}(\gamma)$ then $f(z) \in \mathcal{K}(\gamma)$, i.e., $f(z)$ is convex function of order γ $(0 \leq \gamma < 1)$ in $\mathcal{U}$.*

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References

[1] M. P. Chen, H. Irmak and H. M. Srivastava, *Some families multivalently analytic functions with negative coefficients*, J. Math. Anal. Appl., 214 (1997), 674-690.

[2] J. H. Choi, Y. C. Kim and S. Owa, *Fractional calculus operator and its applications in the univalent functions theory*, Frac. Calc. Appl. Anal., 4 (3) (2001), 367-378.

[3] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften 259, Springer-Verlag, New York, Berlin, Heidelberg, and Tokyo, 1983.

[4] A. W. Goodman, *Univalent Functions*, Vols. I and II, Polygonal Publishing House, Washington, New Jersey, 1983.

[5] I. S. Jack, *Functions starlike and convex of order α*, J. London Math. Soc., 3 (1971), 469-474.

[6] S. S. Miller and P. T. Mocanu, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl., 65 (1978), 289-305.

[7] S. Owa, *On the distortion theorems I*, Kyungpook Math. J., 18 (1978), 53-59.

[8] S. Owa, and H. M. Srivastava, *Univalent and starlike generalized hypergeometric functions*, Can. J. Math., 39 (1987), 1057-1077.

[9] S. Owa and C. Y. Shen, *Generalized classes of starlike and convex functions of order α*, Inter. J. Math. Math. Sci., 8 (1985), 455-467.

[10] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integral and Derivatives, Theory and Applications*, Gordon and Breach, Yverdon (Switzerland), 1993.

[11] H. M. Srivastava, and S. Owa, (Editors), *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, New Jersey, London, and Hong Kong, 1992.

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