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ON A NONLINEAR VOLTERRA INTEGRAL EQUATION IN TWO VARIABLES

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Abstract. The aim of this paper is to study the existence, uniqueness and other properties of solutions of a certain nonlinear Volterra integral equation in two variables. The fundamental tools employed in the analysis are based on applications of the Banach fixed point theorem and a certain variant of the integral inequality with explicit estimate on the unknown function.

1. Introduction

Let R denote the set of real numbers, R^n the real n-dimensional Euclidean space with appropriate norm denoted by $|.|$ and $C(S_1, S_2)$ the class of continuous functions from the set S_1 to the set S_2. We denote by $R_+ = [0,\infty), E = R_+ \times R_+,\ E_1 = \{(x,y,s) : 0 \le s \le x < \infty, y \in R_+\}$, and $E_2 = \{(x,y,s,t) : 0 \le s \le x < \infty,\ 0 \le t \le y < \infty\}$. The partial derivatives of a function $z = z(x,y) : E \to R^n$ with respect to x, y and xy are denoted by $D_1 z$ (or z_x), $D_2 z$ (or z_y) and $D_2 D_1 z = D_1 D_2 z$ (or z_{xy}). In [5, p. 20] C. Corduneanu pointed out that, by means of the substitution $u = v \exp\left(-\int_0^x b_0(y,t)\,dy\right)$, the following hyperbolic equation

$$u_{xt} + a_0(x,t)\,u_x + b_0(x,t)\,u_t = c_0(x,t,u), \tag{1.1}$$

considered on the semi-strip $0 \le x \le l, 0 \le t < \infty$, with the given characteristic data

$$u(x,0) = u_1(x), u(0,t) = u_0(t), \tag{1.2}$$

takes the form

$$v_{xt} + a(x,t)\,v_x = c(x,t,v), \tag{1.3}$$

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where $a(x,t)$ and $c(x,t,v)$ are like $a_0(x,t)$ and $c_0(x,t,u)$ and the data on the characteristics preserve their form

$$v(x,0) = u_1(x)\exp\left(\int_0^x b_0(y,0)dy\right) = v_1(x), v(0,t) = u_0(t). \tag{1.4}$$

Furthermore, by taking $z(x,t) = v_{xt}(x,t)$ it is easy to observe that the equation (1.3) with characteristic data (1.4) takes the form

$$z(x,t) + a(x,t)\left(D_1 v_1(x) + \int_0^t z(x,\tau)\,d\tau\right) = c\left(x,t,u_0(t) + v_1(x) - u_1(0) + \int_0^x \int_0^t z(y,\tau)\,d\tau dy\right). \tag{1.5}$$

In view of the fact that many physical problems arising in a wide variety of applications governed by such equations and the above observations, in this paper we consider the integral equation of the form

$$u(x,y) = f(x,y) + \int_0^x g(x,y,\xi,u(\xi,y))\,d\xi + \int_0^x \int_0^y h(x,y,\sigma,\tau,u(\sigma,\tau))\,d\tau d\sigma, \tag{1.6}$$

which belongs to the type (1.5), where f,g,h are given functions and u is the unknown function to be found. Throughout, we assume that $f \in C(E,R^n)$, $g \in C(E_1 \times R^n, R^n)$, $h \in C(E_2 \times R^n, R^n)$ .The main objective of the present paper is to study the existence, uniqueness and other properties of solutions of equation (1.6) under various assumptions on the functions in (1.6). The analysis used in the proofs is based on the applications of Banach fixed point theorem coupled with Bielecki type norm (see [1,4,5,7]) and a new variant of the integral inequality with explicit estimate given in [12, p. 74]. Our approach here is elementary and provide some basic results for future advanced studies in the field.

2. Existence and uniqueness

Let S be the space of functions $z \in C(E,R^n)$ which fulfill the condition

$$|z(x,y)| = O(\exp(\lambda(x+y))), \tag{2.1}$$

where λ is a positive constant. In the space S we define the norm

$$|z|_S = \sup_{(x,y)\in E} [|z(x,y)|\exp(-\lambda(x+y))]. \tag{2.2}$$

It is easy to see that S with norm defined in (2.2) is a Banach space. We note that the condition (2.1) implies that there exists a constant $M_0 \geq 0$ such that $|z(x,y)| \leq M_0 (\exp(\lambda(x+y)))$. Using this fact in (2.2) we observe that

$$|z|_S \leq M_0. \tag{2.3}$$

The following theorem ensures the existence of a unique solution to equation (1.6).

Theorem 1. *Suppose that*

(i) *the functions g, h in equation* (1.6) *satisfy the conditions*

$$|g(x,y,\xi,u) - g(x,y,\xi,\bar{u})| \leq a(x,y,\xi)|u-\bar{u}|, \tag{2.4}$$

$$|h(x,y,\sigma,\tau,u) - h(x,y,\sigma,\tau,\bar{u})| \leq b(x,y,\sigma,\tau)|u-\bar{u}|, \tag{2.5}$$

where $a \in C(E_1, R_+)$, $b \in C(E_2, R_+)$,

(ii) *for λ as in* (2.1),

(j_1) *there exists a nonnegative constant α such that $\alpha < 1$ and*

$$\int_0^x a(x,y,\xi)\exp(\lambda(\xi+y))\,d\xi + \int_0^x\int_0^y b(x,y,\sigma,\tau)\exp(\lambda(\sigma+\tau))\,d\tau d\sigma \leq \alpha\exp(\lambda(x+y)), \tag{2.6}$$

(j_2) *there exists a nonnegative constant β such that*

$$\left|f(x,y) + \int_0^x g(x,y,\xi,0)\,d\xi + \int_0^x\int_0^y h(x,y,\sigma,\tau,0)\,d\tau d\sigma\right| \leq \beta\exp(\lambda(x+y)), \tag{2.7}$$

where f, g, h are the functions in equation (1.6).

Under the assumptions (i) *and* (ii) *the equation* (1.6) *has a unique solution on E in S.*

Proof. Let $u \in S$ and define the operator T by

$$(Tu)(x,y) = f(x,y) + \int_0^x g(x,y,\xi,u(\xi,y))\,d\xi + \int_0^x\int_0^y h(x,y,\sigma,\tau,u(\sigma,\tau))\,d\tau d\sigma. \tag{2.8}$$

Now we shall show that T maps S into itself. Evidently, Tu is continuous on E and $Tu \in R^n$. We verify that (2.1) is fulfilled. From (2.8) and using the hypotheses and (2.3), we have

$$
\begin{aligned}
|(Tu)(x,y)| &\le \left| f(x,y) + \int_0^x g(x,y,\xi,0)\,d\xi + \int_0^x \int_0^y h(x,y,\sigma,\tau,0)\,d\tau d\sigma \right| \\
&\quad + \int_0^x |g(x,y,\xi,u(\xi,y)) - g(x,y,\xi,0)| d\xi \\
&\quad + \int_0^x \int_0^y |h(x,y,\sigma,\tau,u(\sigma,\tau)) - h(x,y,\sigma,\tau,0)|\,d\tau d\sigma \\
&\le \beta \exp(\lambda(x+y)) + \int_0^x a(x,y,\xi)\,|u(\xi,y)|\,d\xi \\
&\quad + \int_0^x \int_0^y b(x,y,\sigma,\tau)\,|u(\sigma,\tau)|\,d\tau d\sigma \\
&\le \beta \exp(\lambda(x+y)) + |u|_S \left[\int_0^x a(x,y,\xi) \exp(\lambda(\xi+y))\,d\xi \right. \\
&\quad \left. + \int_0^x \int_0^y b(x,y,\sigma,\tau) \exp(\lambda(\sigma+\tau))\,d\tau d\sigma \right] \\
&\le [\beta + M_0 \alpha] \exp(\lambda(x+y)). \qquad (2.9)
\end{aligned}
$$

From (2.9) it follows that $Tu \in S$. This proves that T maps S into itself.

Now, we verify that the operator T is a contraction map. Let $u, v \in S$. From (2.8) and using the hypotheses, we have

$$
\begin{aligned}
|(Tu)(x,y) - (Tv)(x,y)| &\le \int_0^x |g(x,y,\xi,u(\xi,y)) - g(x,y,\xi,v(\xi,y))| d\xi \\
&\quad + \int_0^x \int_0^y |h(x,y,\sigma,\tau,u(\sigma,\tau)) - h(x,y,\sigma,\tau,v(\sigma,\tau))| d\tau d\sigma \\
&\le \int_0^x a(x,y,\xi)\,|u(\xi,y) - v(\xi,y)|\,d\xi \\
&\quad + \int_0^x \int_0^y b(x,y,\sigma,\tau)\,|u(\sigma,\tau) - v(\sigma,\tau)| d\tau d\sigma \\
&\le |u-v|_S \left[\int_0^x a(x,y,\xi) \exp(\lambda(\xi+y)) d\xi \right. \\
&\quad \left. + \int_0^x \int_0^y b(x,y,\sigma,\tau) \exp(\lambda(\sigma+\tau)) d\tau d\sigma \right] \\
&\le \alpha\,|u-v|_S \exp(\lambda(x+y)). \qquad (2.10)
\end{aligned}
$$

From (2.10), we obtain

$$
|Tu - Tv|_S \le \alpha\,|u-v|_S .
$$

Since $\alpha < 1$, it follows from Banach fixed point theorem (see [5, p. 37]) that T has a unique fixed point in S. The fixed point of T is however a solution of equation (1.6). The proof is complete. $\square$

Remark 1. We note that, Theorem 1 given above provides a simple way to establish the existence and uniqueness for solutions of equation (1.6) in the space of continuous functions. The existence and uniqueness result in L^p spaces for more general version of equation (1.6) is analyzed by M.Kwapisz [8, Theorem 1] using weighted norm introduced first by Bielecki [1]. Our approach here applies also for the equation considered in [8, p. 246]. For a survey on the results proved by Bielecki's method for integral and integrodifferential equations, see [4].

3. Properties of solutions

In this section we study some fundamental properties of solutions of equation (1.6) under various assumptions on the functions involved therein. The analysis is based on the application of the following variant of the inequality due to the present author given in [12, Theorem 2.3.1, p. 74]. For similar results, see [10].

Lemma 1. *Let $w \in C(E, R_+)$, $q, D_1q \in C(E_1, R_+)$, $r, D_1r, D_2r, D_2D_1r \in C(E_2, R_+)$ and $c \geq 0$ is a constant. If*

$$w(x,y) \leq c + \int_0^x q(x,y,\xi)\, w(\xi,y) d\xi + \int_0^x \int_0^y r(x,y,\sigma,\tau)\, w(\sigma,\tau) d\tau d\sigma, \tag{3.1}$$

for $x, y \in R_+$, then

$$w(x,y) \leq cA(x,y) \exp\left(\int_0^x \int_0^y B(s,t)\, dt ds\right), \tag{3.2}$$

for $x, y \in R_+$, where

$$A(x,y) = \exp(Q(x,y)), \tag{3.3}$$

in which

$$Q(x,y) = \int_0^x \left[q(\eta,y,\eta) + \int_0^\eta D_1q(\eta,y,\xi)\, d\xi\right] d\eta, \tag{3.4}$$

and

$$B(x,y) = r(x,y,x,y)\, A(x,y) + \int_0^x D_1r(x,y,\sigma,y)\, A(\sigma,y)\, d\sigma$$
$$+ \int_0^y D_2r(x,y,x,\tau)\, A(x,\tau)\, d\tau + \int_0^x \int_0^y D_2D_1r(x,y,\sigma,\tau)\, A(\sigma,\tau)\, d\tau d\sigma. \tag{3.5}$$

Proof. Define the function $z(x,y)$ by

$$z(x,y) = c + \int_0^x \int_0^y r(x,y,\sigma,\tau)\, w(\sigma,\tau)\, d\tau d\sigma. \tag{3.6}$$

Then (3.1) can be restated as

$$w(x,y) \le z(x,y) + \int_0^x q(x,y,\xi) w(\xi,y)\, d\xi. \tag{3.7}$$

From the hypotheses, it is easy to observe that $z(x,y)$ is nonnegative and nondecreasing for $x, y \in R_+$. Treating (3.7) as a one-dimensional integral inequality for any fixed $y \in R_+$ and a suitable application of the inequality given in [12, Theorem 1.2.1, Remark 1.2.1, p. 11] yields

$$w(x,y) \le A(x,y)\, z(x,y)\,. \tag{3.8}$$

From (3.6) and (3.8), we have

$$z(x,y) \le c + \int_0^x \int_0^y r(x,y,\sigma,\tau)\, A(\sigma,\tau)\, z(\sigma,\tau)\, d\tau d\sigma. \tag{3.9}$$

Now a suitable application of the inequality given in [12, Theorem 2.2.1, Remark 2.2.1, p. 66] to (3.9) yields

$$z(x,y) \le c \exp\left(\int_0^x \int_0^y B(s,t)\, dt\, ds\right). \tag{3.10}$$

Using (3.10) in (3.8), we get the required inequality in (3.2). □

First, we shall give the following theorem concerning the estimate on the solution of equation (1.6).

Theorem 2. *Suppose that the functions f, g, h in equation (1.6) satisfy the conditions*

$$|g(x,y,\xi,u) - g(x,y,\xi,\bar{u})| \le q(x,y,\xi)\, |u - \bar{u}|\,, \tag{3.11}$$

$$|h(x,y,\sigma,\tau,u) - h(x,y,\sigma,\tau,\bar{u})| \le r(x,y,\sigma,\tau)\, |u - \bar{u}|\,, \tag{3.12}$$

where $q, D_1 q \in C(E_1, R_+)$ and $r, D_1 r, D_2 r, D_2 D_1 r \in C(E_2, R_+)$ and

$$c = \sup_{(x,y)\in E} \left| f(x,y) + \int_0^x g(x,y,\xi,0)\, d\xi + \int_0^x \int_0^y h(x,y,\sigma,\tau,0)\, d\tau d\sigma \right| < \infty. \tag{3.13}$$

If $u(x,y)$ is any solution of equation (1.6) on E, then

$$|u(x,y)| \le c A(x,y) \exp\left(\int_0^x \int_0^y B(s,t)\, dtds\right), \tag{3.14}$$

for $(x,y) \in E$, where $A(x,y)$ and $B(x,y)$ are given by (3.3) and (3.5).

Proof. Using the fact that $u(x,y)$ is a solution of equation (1.6) and hypotheses, we have

$$
\begin{aligned}
|u(x,y)| \leq & \left| f(x,y) + \int_0^x g(x,y,\xi,0)\, d\xi + \int_0^x \int_0^y h(x,y,\sigma,\tau,0) d\tau d\sigma \right| \\
& + \int_0^x |g(x,y,\xi,u(\xi,y)) - g(x,y,\xi,0)| d\xi \\
& + \int_0^x \int_0^y |h(x,y,\sigma,\tau,u(\sigma,\tau)) - h(x,y,\sigma,\tau,0)| d\tau d\sigma \\
\leq & \; c + \int_0^x q(x,y,\xi)\, |u(\xi,y)| d\xi + \int_0^x \int_0^y r(x,y,\sigma,\tau)\, |u(\sigma,\tau)| d\tau d\sigma. \quad (3.15)
\end{aligned}
$$

Now an application of Lemma 1 to (3.15) yields (3.14). □

Remark 2. In Theorem 2, if we assume that (i) $Q(x,y) < \infty$ and (ii) $\int_0^\infty \int_0^\infty B(s,t)\, dtds < \infty$, then the solution $u(x,y)$ of equation (1.6) is bounded on E.

A slight variant of Theorem 2 is embodied in the following theorem.

Theorem 3. *Suppose that the functions g, h in equation* (1.6) *satisfy the conditions* (3.11), (3.12) *and*

$$
\int_0^x |g(x,y,\xi,f(\xi,y))|\, d\xi + \int_0^x \int_0^y |h(x,y,\sigma,\tau,f(\sigma,\tau))| d\tau d\sigma \leq d, \quad (3.16)
$$

for $x, y \in R_+$, where f is the function involved in (1.6) *and $d \geq 0$ is a real constant. If $u(x,y)$ is any solution of equation* (1.6) *on E, then*

$$
|u(x,y) - f(x,y)| \leq dA(x,y) \exp\left(\int_0^x \int_0^y B(s,t)\, dtds \right), \quad (3.17)
$$

for $(x,y) \in E$, where $A(x,y)$ and $B(x,y)$ are given by (3.3) *and* (3.5).

Proof. Let $e(x,y) = |u(x,y) - f(x,y)|$ for $(x,y) \in E$. Using the fact that $u(x,y)$ is a solution of equation (1.6) and the hypotheses, we have

$$
\begin{aligned}
e(x,y) \leq & \int_0^x |g(x,y,\xi,f(\xi,y))|\, d\xi + \int_0^x \int_0^y |h(x,y,\sigma,\tau,f(\sigma,\tau))| d\tau d\sigma \\
& + \int_0^x |g(x,y,\xi,u(\xi,y)) - g(x,y,\xi,f(\xi,y))| d\xi \\
& + \int_0^x \int_0^y |h(x,y,\sigma,\tau,u(\sigma,\tau)) - h(x,y,\sigma,\tau,f(\sigma,\tau))| d\tau d\sigma \\
\leq & \; d + \int_0^x q(x,y,\xi)\, e(\xi,y) d\xi + \int_0^x \int_0^y r(x,y,\sigma,\tau)\, e(\sigma,\tau) d\tau d\sigma. \quad (3.18)
\end{aligned}
$$

Now an application of Lemma 1 to (3.18) yields (3.17).

We call the function $u \in C(E, R^n)$ an ε-approximate solution to equation (1.6), if there exists a constant $\varepsilon \geq 0$ such that

$$\left| u(x,y) - \left\{ f(x,y) + \int_0^x g(x,y,\xi,u(\xi,y))\,d\xi \right.\right.$$
$$\left.\left. + \int_0^x \int_0^y h(x,y,\sigma,\tau,u(\sigma,\tau))\,d\tau d\sigma \right\} \right| \leq \varepsilon,$$

for all $x, y \in R_+$. □

The next theorem deals with the estimate on the difference between two approximate solutions of equation (1.6).

Theorem 4. *Let $u_1(x,y)$ and $u_2(x,y)$ be respectively, ε_1 and ε_2 approximate solutions of equation* (1.6) *on E. Suppose that the functions g, h in equation* (1.6) *satisfy the conditions* (3.11), (3.12)*. Then*

$$|u_1(x,y) - u_2(x,y)| \leq (\varepsilon_1 + \varepsilon_2) A(x,y) \exp\left(\int_0^x \int_0^y B(s,t)\,dtds \right), \tag{3.19}$$

for $(x,y) \in E$, where $A(x,y)$ and $B(x,y)$ are given by (3.3) *and* (3.5)*.*

Proof. Since $u_1(x,y)$ and $u_2(x,y)$ are respectively, ε_1 and ε_2 approximate solutions to equation (1.6), we have

$$\left| u_i(x,y) - \left\{ f(x,y) + \int_0^x g(x,y,\xi,u_i(\xi,y))\,d\xi \right.\right.$$
$$\left.\left. + \int_0^x \int_0^y h(x,y,\sigma,\tau,u_i(\sigma,\tau))\,d\tau d\sigma \right\} \right| \leq \varepsilon_i, \tag{3.20}$$

for $i = 1, 2$. From (3.20) and using the elementary inequalities $|v - z| \leq |v| + |z|$ and $|v| - |z| \leq |v - z|$, we observe that

$$\varepsilon_1 + \varepsilon_2 \geq \left| u_1(x,y) - \left\{ f(x,y) + \int_0^x g(x,y,\xi,u_1(\xi,y))\,d\xi \right.\right.$$
$$\left.\left. + \int_0^x \int_0^y h(x,y,\sigma,\tau,u_1(\sigma,\tau))\,d\tau d\sigma \right\} \right|$$
$$+ \left| u_2(x,y) - \left\{ f(x,y) + \int_0^x g(x,y,\xi,u_2(\xi,y))\,d\xi \right.\right.$$
$$\left.\left. + \int_0^x \int_0^y h(x,y,\sigma,\tau,u_2(\sigma,\tau))\,d\tau d\sigma \right\} \right|$$
$$\geq \left| [u_1(x,y) - u_2(x,y)] - \left[\left\{ f(x,y) + \int_0^x g(x,y,\xi,u_1(\xi,y))\,d\xi \right.\right.\right.$$

$$
\begin{aligned}
&\qquad + \int_0^x \int_0^y h\left(x, y, \sigma, \tau, u_1(\sigma, \tau)\right) d\tau d\sigma \Big\} \\
&\qquad - \Big\{ f(x, y) + \int_0^x g\left(x, y, \xi, u_2(\xi, y)\right) d\xi \\
&\qquad + \int_0^x \int_0^y h\left(x, y, \sigma, \tau, u_2(\sigma, \tau)\right) d\tau d\sigma \Big\} \Big] \Big| \\
&\geq \left|u_1(x, y) - u_2(x, y)\right| - \Big| \int_0^x \left\{ g\left(x, y, \xi, u_1(\xi, y)\right) - g\left(x, y, \xi, u_2(\xi, y)\right) \right\} d\xi \\
&\quad + \int_0^x \int_0^y \left\{ h\left(x, y, \sigma, \tau, u_1(\sigma, \tau)\right) - h\left(x, y, \sigma, \tau, u_2(\sigma, \tau)\right) \right\} d\tau d\sigma \Big| . \quad (3.21)
\end{aligned}
$$

Let $w(x, y) = \left|u_1(x, y) - u_2(x, y)\right|$, $(x, y) \in E$. From (3.21) and using conditions (3.11), (3.12), we have

$$
\begin{aligned}
w(x, y) &\leq (\varepsilon_1 + \varepsilon_2) + \int_0^x \left| g\left(x, y, \xi, u_1(\xi, y)\right) - g\left(x, y, \xi, u_2(\xi, y)\right) \right| d\xi \\
&\quad + \int_0^x \int_0^y \left| h\left(x, y, \sigma, \tau, u_1(\sigma, \tau)\right) - h\left(x, y, \sigma, \tau, u_2(\sigma, \tau)\right) \right| d\tau d\sigma \\
&\leq (\varepsilon_1 + \varepsilon_2) + \int_0^x q(x, y, \xi)\, w(\xi, y)\, d\xi \\
&\quad + \int_0^x \int_0^y r(x, y, \sigma, \tau)\, w(\sigma, \tau)\, d\tau d\sigma. \quad (3.22)
\end{aligned}
$$

Now an application of Lemma 1 to (3.22) yields (3.19). □

Remark 3. In case, if $u_1(x, y)$ is a solution of equation (1.6), then we have $\varepsilon_1 = 0$ and from (3.19) we see that $u_1(x, y) \to u_2(x, y)$ as $\varepsilon_2 \to 0$. Moreover, from (3.19) the uniqueness of solutions of equation (1.6) follows if $\varepsilon_i \equiv 0$ $(i = 1, 2)$.

We next consider the following variants of equation (1.6):

$$
\begin{aligned}
u(x, y) = f(x, y) &+ \int_0^x g\left(x, y, \xi, u(\xi, y), \mu\right) d\xi \\
&+ \int_0^x \int_0^y h\left(x, y, \sigma, \tau, u(\sigma, \tau), \mu\right) d\tau d\sigma, \quad (3.23)
\end{aligned}
$$

and

$$
\begin{aligned}
u(x, y) = f(x, y) &+ \int_0^x g\left(x, y, \xi, u(\xi, y), \mu_0\right) d\xi \\
&+ \int_0^x \int_0^y h\left(x, y, \sigma, \tau, u(\sigma, \tau), \mu_0\right) d\tau d\sigma, \quad (3.24)
\end{aligned}
$$

for $(x,y) \in E$, where $f \in C(E, R^n)$, $g \in C(E_1 \times R^n \times R, R^n)$, $h \in C(E_2 \times R^n \times R, R^n)$ and μ, μ_0 are parameters.

The following theorem shows the dependency of solutions of equations (3.23) and (3.24) on parameters.

Theorem 5. *Suppose that the functions* g, h *in equations* (3.23), (3.24) *satisfy the conditions*

$$|g(x,y,\xi,u,\mu) - g(x,y,\xi,\bar{u},\mu)| \le q(x,y,\xi)|u - \bar{u}|, \tag{3.25}$$

$$|g(x,y,\xi,u,\mu) - g(x,y,\xi,u,\mu_0)| \le p_1(x,y,\xi)|\mu - \mu_0|, \tag{3.26}$$

$$|h(x,y,\sigma,\tau,u,\mu) - h(x,y,\sigma,\tau,\bar{u},\mu)| \le r(x,y,\sigma,\tau)|u - \bar{u}|, \tag{3.27}$$

$$|h(x,y,\sigma,\tau,u,\mu) - h(x,y,\sigma,\tau,u,\mu_0)| \le p_2(x,y,\sigma,\tau)|\mu - \mu_0|, \tag{3.28}$$

where $p_1, q, D_1 q \in C(E_1, R_+)$, $p_2, r, D_1 r, D_2 r, D_2 D_1 r \in C(E_2, R_+)$ *and*

$$\int_0^x p_1(x,y,\xi)\,d\xi \le M_1, \tag{3.29}$$

$$\int_0^x \int_0^y p_2(x,y,\sigma,\tau) d\tau d\sigma \le M_2, \tag{3.30}$$

in which M_1, M_2 *are nonnegative constants. Let* $u_1(x,y)$ *and* $u_2(x,y)$ *be the solutions of equations* (3.23) *and* (3.24) *respectively. Then*

$$|u_1(x,y) - u_2(x,y)| \le (M_1 + M_2)|\mu - \mu_0| A(x,y) \exp\left(\int_0^x \int_0^y B(s,t)\,dtds\right), \tag{3.31}$$

for $(x,y) \in E$, *where* $A(x,y)$ *and* $B(x,y)$ *are given by* (3.3) *and* (3.5).

Proof. Let $w(x,y) = |u_1(x,y) - u_2(x,y)|$, $(x,y) \in E$. Using the facts that $u_1(x,y)$ and $u_2(x,y)$ are the solutions of equations (3.23) and (3.24) and hypotheses, we have

$$\begin{aligned}
w(x,y) \le & \int_0^x |g(x,y,\xi,u_1(\xi,y),\mu) - g(x,y,\xi,u_2(\xi,y),\mu)|\,d\xi \\
& + \int_0^x |g(x,y,\xi,u_2(\xi,y),\mu) - g(x,y,\xi,u_2(\xi,y),\mu_0)|\,d\xi \\
& + \int_0^x \int_0^y |h(x,y,\sigma,\tau,u_1(\sigma,\tau),\mu) - h(x,y,\sigma,\tau,u_2(\sigma,\tau),\mu)| d\tau d\sigma \\
& + \int_0^x \int_0^y |h(x,y,\sigma,\tau,u_2(\sigma,\tau),\mu) - h(x,y,\sigma,\tau,u_2(\sigma,\tau),\mu_0)| d\tau d\sigma \\
\le & \int_0^x q(x,y,\xi) w(\xi,y)\,d\xi + \int_0^x p_1(x,y,\xi)|\mu - \mu_0|\,d\xi \\
& + \int_0^x \int_0^y r(x,y,\sigma,\tau) w(\sigma,\tau)\,d\tau d\sigma + \int_0^x \int_0^y p_2(x,y,\sigma,\tau)|\mu - \mu_0|\,d\tau d\sigma
\end{aligned}$$

$$\leq (M_1 + M_2)\,|\mu - \mu_0| + \int_0^x q\,(x, y, \xi) w\,(\xi, y)\, d\xi$$
$$+ \int_0^x \int_0^y r\,(x, y, \sigma, \tau) w\,(\sigma, \tau)\, d\tau d\sigma. \quad (3.32)$$

Now an application of Lemma 1 to (3.32) yields (3.31), which shows the dependency of solutions of equations (3.23) and (3.24) on parameters. □

Remark 4. We note that the idea employed above can be extended to study the integrodifferential equation of the form

$$D_2 D_1 u\,(x, y) = F\,(x, y, u\,(x, y)\,, (Ku)\,(x, y)\,, (Lu)\,(x, y))\,, \quad (3.33)$$

with the given data

$$u\,(x, 0) = \sigma\,(x)\,, u\,(0, y) = \tau\,(y)\,, \quad (3.34)$$

for $x, y \in R_+$, where

$$(Ku)\,(x, y) = \int_0^x g\,(x, y, \xi, u\,(\xi, y)) d\xi, \quad (3.35)$$

$$(Lu)\,(x, y) = \int_0^x \int_0^y h\,(x, y, \sigma, \tau, u\,(\sigma, \tau)) d\tau d\sigma, \quad (3.36)$$

by making use of a suitable variant of the inequality given in [12, Theorem 2.5.1, p. 96] (see also [10]). Here, we omit the details.

4. Discrete analogues

Let N denote the set of natural numbers, $N_0 = \{0, 1, 2, \ldots\}$ and $D\,(S_1, S_2)$ the class of discrete functions from the set S_1 to the set S_2. We denote by $G = N_0 \times N_0$, $G_1 = \{(m, n, \xi) : 0 \leq \xi \leq m < \infty, n \in N_0\}$ and $G_2 = \{(m, n, \sigma, \tau) : 0 \leq \sigma \leq m < \infty,\ 0 \leq \tau \leq n < \infty\}$. For any function $w : G \to R^n$ we define the operators Δ_1, Δ_2 by $\Delta_1 w\,(m, n) = w\,(m + 1, n) - w\,(m, n)\,,$ $\Delta_2 w\,(m, n) = w\,(m, n + 1) - w\,(m, n)$ and $\Delta_2 \Delta_1 w\,(m, n) = \Delta_2\,(\Delta_1 w\,(m, n))\,.$ We use the usual conventions that empty sums and products are taken to be 0 and 1 respectively. The sum-difference equation which constitutes the discrete analogue of equation (1.6) can be written as

$$u\,(m, n) = \bar{f}\,(m, n) + \sum_{\xi=0}^{m-1} \bar{g}\,(m, n, \xi, u\,(\xi, n))$$
$$+ \sum_{\sigma=0}^{m-1} \sum_{\tau=0}^{n-1} \bar{h}\,(m, n, \sigma, \tau, u\,(\sigma, \tau)), \quad (4.1)$$

for $m, n \in N_0$, where $\bar{f} \in D\,(G, R^n)\,,$ $\bar{g} \in D\,(G_1 \times R^n, R^n)\,,$ $\bar{h} \in D(G_2 \times R^n, R^n)$.

In this section, we formulate in brief the discrete versions of Lemma 1, Theorems 2 and 3 only, concerning the solutions of equation (4.1). One can formulate results similar to those in Theorems 1, 4 and 5 for the solutions of equation (4.1). For a detailed account on the study of such equations, see [11,12].

Lemma 2. *Let* $w \in D(G, R_+)$, $\bar{q}, \Delta_1 \bar{q} \in D(G_1, R_+)$, $\bar{r}, \Delta_1 \bar{r}, \Delta_2 \bar{r}, \Delta_2 \Delta_1 \bar{r} \in D(G_2, R_+)$, *and* $\bar{c} \geq 0$ *is a constant. If*

$$w(m,n) \leq \bar{c} + \sum_{\xi=0}^{m-1} \bar{q}(m,n,\xi)\, w(\xi,n) + \sum_{\sigma=0}^{m-1} \sum_{\tau=0}^{n-1} \bar{r}(m,n,\sigma,\tau)\, w(\sigma,\tau), \quad (4.2)$$

for $m, n \in N_0$, *then*

$$w(m,n) \leq \bar{c}\bar{A}(m,n) \prod_{s=0}^{m-1} \left[1 + \sum_{t=0}^{n-1} \bar{B}(s,t)\right], \quad (4.3)$$

for $m, n \in N_0$, *where*

$$\bar{A}(m,n) = \prod_{\xi=0}^{m-1} \left[1 + \bar{Q}(\xi,n)\right], \quad (4.4)$$

in which

$$\bar{Q}(m,n) = \bar{q}(m+1,n,m) + \sum_{\eta=0}^{m-1} \Delta_1 \bar{q}(m,n,\eta), \quad (4.5)$$

and

$$\bar{B}(m,n) = \bar{r}(m+1,n+1,m,n)\, \bar{A}(m,n) + \sum_{\sigma=0}^{m-1} \Delta_1 \bar{r}(m,n+1,\sigma,n)\, \bar{A}(\sigma,n)$$
$$+ \sum_{\tau=0}^{n-1} \Delta_2 \bar{r}(m+1,n,m,\tau)\, \bar{A}(m,\tau) + \sum_{\sigma=0}^{m-1} \sum_{\tau=0}^{n-1} \Delta_2 \Delta_1 \bar{r}(m,n,\sigma,\tau)\, \bar{A}(\sigma,\tau). \quad (4.6)$$

Proof. Define a function $z(m,n)$ by

$$z(m,n) = \bar{c} + \sum_{\sigma=0}^{m-1} \sum_{\tau=0}^{n-1} \bar{r}(m,n,\sigma,\tau) w(\sigma,\tau). \quad (4.7)$$

Then (4.2) can be restated as

$$w(m,n) \leq z(m,n) + \sum_{\xi=0}^{m-1} \bar{q}(m,n,\xi)\, w(\xi,n). \quad (4.8)$$

From the hypotheses, it is easy to observe that $z(m,n)$ is nonnegative and nondecreasing in $m,n \in N_0$. Treating (4.8) as one dimensional inequality for any fixed $n \in N_0$ and following the proof of Theorem 4.3.1 part (a_1) given in [12, p. 206], we get

$$w(m,n) \leq z(m,n)\bar{A}(m,n). \tag{4.9}$$

From (4.7) and (4.9), we have

$$z(m,n) \leq \bar{c} + \sum_{\sigma=0}^{m-1}\sum_{\tau}^{n-1} \bar{r}(m,n,\sigma,\tau)\bar{A}(\sigma,\tau)z(\sigma,\tau). \tag{4.10}$$

Now by following the proof of Theorem 5.2.2 part (b_1) given in [12, p. 246] with suitable modifications, we get

$$z(m,n) \leq \bar{c}\prod_{s=0}^{m-1}\left[1 + \sum_{t=0}^{n-1}\bar{B}(s,t)\right]. \tag{4.11}$$

Using (4.11) in (4.9) we get the required inequality in (4.3). □

Theorem 6. *Suppose that the functions $\bar{f},\bar{g},\bar{h}$ in equation (4.1) satisfy the conditions*

$$|\bar{g}(m,n,\xi,u) - \bar{g}(m,n,\xi,\bar{u})| \leq \bar{q}(m,n,\xi)|u-\bar{u}|, \tag{4.12}$$

$$\left|\bar{h}(m,n,\sigma,\tau,u) - \bar{h}(m,n,\sigma,\tau,\bar{u})\right| \leq \bar{r}(m,n,\sigma,\tau)|u-\bar{u}|, \tag{4.13}$$

where $\bar{q},\Delta_1\bar{q} \in D(G_1,R_+)$, $\bar{r},\Delta_1 r,\Delta_2\bar{r},\Delta_2\Delta_1\bar{r} \in D(G_2,R_+)$ and

$$\bar{c} = \left|\bar{f}(m,n) + \sum_{\xi=0}^{m-1}\bar{g}(m,n,\xi,0) + \sum_{\sigma=0}^{m-1}\sum_{\tau=0}^{n-1}\bar{h}(m,n,\sigma,\tau,0)\right| < \infty. \tag{4.14}$$

If $u(m,n)$ is any solution of equation (4.1) on G, then

$$|u(m,n)| \leq \bar{c}\bar{A}(m,n)\prod_{s=0}^{m-1}\left[1 + \sum_{t=0}^{n-1}\bar{B}(s,t)\right], \tag{4.15}$$

for $(m,n) \in G$, where $\bar{A}(m,n)$ and $\bar{B}(m,n)$ are given by (4.4) and (4.6).

The proof follows by the arguments as in the proof of Theorem 2 given above and using Lemma 2. We omit the details.

By the similar way as Theorem 3 we have

Theorem 7. *Suppose that the functions $\bar{f},\bar{g},\bar{h}$ in equation (4.1) satisfy the conditions (4.12), (4.13) and*

$$\sum_{\xi=0}^{m-1}\left|\bar{g}\left(m,n,\xi,\bar{f}(\xi,n)\right)\right| + \sum_{\sigma=0}^{m-1}\sum_{\tau=0}^{n-1}\left|\bar{h}\left(m,n,\sigma,\tau,\bar{f}(\sigma,\tau)\right)\right| \leq \bar{d}, \tag{4.16}$$

for $m, n \in N_0$*, where* $\bar{d} \geq 0$ *is a real constant. If* $u(m,n)$ *is any solution of equation* (4.1) *on* G*, then*

$$\left|u\left(m,n\right)-\bar{f}\left(m,n\right)\right| \leq \bar{d}\bar{A}\left(m,n\right)\prod_{s=0}^{m-1}\left[1+\sum_{t=0}^{n-1}\bar{B}\left(s,t\right)\right], \tag{4.17}$$

for $(m,n) \in G$*, where* $\bar{A}\left(m,n\right)$ *and* $\bar{B}\left(m,n\right)$ *are given by* (4.4) *and* (4.6)*.*

In concluding we note that, if we take $h = 0$ and treat the variable y as a constant, then the equation (1.6) reduces to the Volterra integral equation in one independent variable and by taking $g = 0$ it reduces to the Volterra integral equation in two independent variables. Indeed, a particular feature of our approach is that it presents conditions under which we can offer simple, unified and concise proofs of some of the important qualitative properties of solutions of equations (1.6) and (4.1). For detailed account on such equations, see [2-9, 14].

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