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JOST SOLUTION OF THE MATRIX DIFFERENCE EQUATIONS

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ABSTRACT. In this paper, we investigate the Jost solution and the analytical properties of the Jost solution of the non-selfadjoint matrix difference equation of second order.

1. INTRODUCTION

Let us consider the Sturm-Liouville equation

$$-y'' + q(x)y = \lambda^2 y, \quad x \in \mathbb{R}_+ := [0, \infty), \tag{1.1}$$

where $q : \mathbb{R}_+ \longrightarrow \mathbb{C}$ and $\lambda \in \mathbb{C}$ is a spectral parameter. The bounded solution of the equation (1.1) satisfying the condition

$$\lim_{x \longrightarrow \infty} y(x, \lambda) e^{-i\lambda x} = 1, \ \lambda \in \bar{\mathbb{C}}_+ := \{\lambda : \lambda \in \mathbb{C}, \ \mathrm{Im}\ \lambda \geq 0\},$$

will be denoted by $e(x, \lambda)$. The solution $e(x, \lambda)$ is called the Jost solution of the equation (1.1). The Jost solution satisfies the integral equation

$$e(x, \lambda) = e^{i\lambda x} + \int_x^\infty \frac{\sin(t - x)}{\lambda} q(t) e(t, \lambda) dt. \tag{1.2}$$

It has been shown that [12,13] under the condition $\int_0^\infty x\,|q(x)|\,dx < \infty$, the Jost solution has the representation

$$\ell(x, \lambda) = e^{i\lambda x} + \int_x^\infty \mathcal{K}(x, t) e^{i\lambda t}\, dz, \tag{1.3}$$

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where the function $\mathcal{K}(x,t)$ satisfies the following integral equation

$$\mathcal{K}(x,t) = \frac{1}{2}\int_{\frac{x+t}{2}}^{\infty} q(s)ds + \frac{1}{2}\int_{x}^{\frac{x+t}{2}} \int_{t+x-s}^{t+s-x} q(s)\mathcal{K}(s,u)\,du\,ds + \frac{1}{2}\int_{\frac{x+t}{2}}^{\infty} \int_{s}^{t+s-x} q(s)\mathcal{K}(s,u)\,du\,ds. \quad (1.4)$$

The representations (1.3) and (1.4) of the Jost solution of the equation (1.1) play an important role in the solutions of direct and inverse problems of quantum scattering theory [12,13]. Therefore, the similar representations for the Dirac systems, quadratic pencil of Schrödinger, Klein-Gordon, discrete Schrödinger and discrete Dirac equations have been obtained [6-8,11]. Using the analytical properties of the Jost solutions, the spectral analysis of non-selfadjoint Schrödinger, discrete Schrödinger and discrete Dirac equations have been investigated in [1-4,9,10].

Let us consider the matrix difference equation of second order

$$A_{n-1}Y_{n-1} + B_nY_n + A_nY_{n+1} = \lambda Y_n, \quad n \in \mathbb{N} = \{1, 2, \dots\}, \quad (1.5)$$

where $\{A_n\}, n \in \mathbb{N}\cup\{0\}, \{B_n\}, n \in \mathbb{N}$ are the non-selfadjoint square $m \times m$ matrix sequences in m-dimensional complex Euclidean space $\mathbb{C}^m$ and λ is a complex parameter. In the following we will assume that, $\det A_n \neq 0$, $n \in \mathbb{N}\cup\{0\}$, hold. It is clear that the equation (1.5) is non-selfadjoint.

In this paper we find the Jost solution of (1.1) and study the analytical properties of the Jost solution. We also investigate the asymptotics of the Jost solution.

2. Jost solution

We will assume that the matrix sequences $\{A_n\}$ and $\{B_n\}, n \in \mathbb{N}$ satisfy

$$\sum_{n=1}^{\infty} (\|I - A_n\| + \|B_n\|) < \infty, \quad (2.1)$$

where $\|.\|$ denote the matrix norm in $\mathbb{C}^m$ and I the identity matrix.

Let $E(z) := \{E_n(z)\}, n \in \mathbb{N}\cup\{0\}$ denote the matrix solution of the equation

$$A_{n-1}Y_{n-1} + B_nY_n + A_nY_{n+1} = 2\cos zY_n \quad (2.2)$$

satisfying the condition

$$\lim_{n\to\infty} E_n(z).e^{-inz} = I, \; z \in \bar{\mathbb{C}}_+ := \{z : z \in\in \mathbb{C}, \; \operatorname{Im} z \geq 0\}. \quad (2.3)$$

The solution $E(z)=\{E_n(z)\}, n \in \mathbb{N}\cup\{0\}$ is called the Jost solution of (2.2).

Theorem 1. *Under the condition* (2.1) *the solution* $E(z)$ *exists and satisfies the equation*

$$E_n(z) = e^{inz}I + \sum_{k=n+1}^{\infty} \frac{\sin(k-n)z}{\sin z} \left[(I - A_{k-1})Y_{k-1} - B_kY_k + (I - A_k)Y_k\right]. \tag{2.4}$$

Proof. From (2.2) we get that

$$Y_{n-1} + Y_{n+1} - (e^{iz} + e^{-iz})Y_n = F_n, \tag{2.5}$$

where

$$F_n = (I - A_{n-1})Y_{n-1} - B_nY_n + (I - A_n)Y_n.$$

By the method of variation of parameters we obtain that, the general solution of (2.5) is

$$Y_n(z) = Ce^{inz} + De^{-inz} + \sum_{k=n+1}^{\infty} \frac{\sin(k-n)z}{\sin z} F_n. \tag{2.6}$$

using the condition (2.3) and (2.6) we have (2.4). □

Theorem 2. *If*

$$\sum_{n=1}^{\infty} n\left(\|I - A_n\| + \|B_n\|\right) < \infty \tag{2.7}$$

hold, then the Jost solution $E(z) = \{E_n(z)\}$, $n \in \mathbb{N} \cup \{0\}$ *has a representation*

$$E_n(z) = T_n e^{inz}\left[I + \sum_{m=1}^{\infty} K_{n,m}e^{imz}\right], \ n \in \mathbb{N} \cup \{0\}, \ z \in \bar{\mathbb{C}}_+, \tag{2.8}$$

where T_n *and* $K_{n,m}$ *are expressed in terms* $\{A_n\}$ *and* $\{B_n\}$.

Proof. Substituting $E_n(z)$ defined by (2.8) in (2.2) we get the following,

$$T_n = \prod_{p=n}^{\infty} A_n^{-1}, \quad n \in \mathbb{N} \cup \{0\}, \tag{2.9}$$

$$K_{n,1} = -\sum_{p=n+1}^{\infty} T_p^{-1}B_pT_p, \quad n \in \mathbb{N} \cup \{0\}, \tag{2.10}$$

$$K_{n,2} = -\sum_{p=n+1}^{\infty} T_p^{-1}(I - A_p^2) + \sum_{p=n+1}^{\infty} T_p^{-1}B_pT_pK_{p,1}, \quad n \in \mathbb{N} \cup \{0\}, \tag{2.11}$$

$$K_{n,m+2} = +\sum_{p=n+1}^{\infty} T_p^{-1}(I - A_p^2) - \sum_{p=n+1}^{\infty} T_p^{-1}B_pT_pK_{p,m+1} + K_{n+1,m} \tag{2.12}$$

where $n \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$.

Due to the condition (2.7), the infinite product and the series in the definition of T_n and $K_{n,m}$ are absolutely convergent. □

By mathematical induction, using (2.10)-(2.12) we obtain

$$\|K_{n,m}\| \leqslant c \sum_{p=n+[|\frac{m}{2}|]}^{\infty} \left(\|I - A_p\| + \|B_p\|\right), \tag{2.13}$$

where $\left[\left|\frac{m}{2}\right|\right]$ is the integer part of $\frac{m}{2}$ and $c > 0$ is a constant. It follows from (2.13) that $E_n(z)$, $n \in \mathbb{N} \cup \{0\}$ is analytic with respect to z in $\mathbb{C}_+ := \{z : z \in\in \mathbb{C},\ \text{Im } z \geq 0\}$ and continuous in $\bar{\mathbb{C}}_+$.

Theorem 3. *The Jost solution satisfies the following asimptotics:*

$$E_n(z) = e^{inz}\left[I + o(1)\right],\ z \in \bar{\mathbb{C}}_+,\ n \to \infty, \tag{2.14}$$

$$E_n(z) = T_n e^{inz}\left[I + o(1)\right],\ z \in \bar{\mathbb{C}}_+,\ z = \xi + i\tau,\ \tau \to \infty. \tag{2.15}$$

Proof. It follows from (2.7), (2.9), (2.13) that

$$\lim_{n\to\infty} \|T_n - I\| = 0, \tag{2.16}$$

and

$$\sum_{m=1}^{\infty} K_{n,m} e^{imz} = o(1),\ z \in \bar{\mathbb{C}}_+,\ n \to \infty, \tag{2.17}$$

hold. From (2.8), (2.16) and (2.17) we get (2.14). Using (2.7) and (2.13) we have

$$\sum_{m=1}^{\infty} K_{n,m} e^{imz} = o(1),\ z \in \bar{\mathbb{C}}_+,\ z = \xi + i\tau,\ \tau \to \infty. \tag{2.18}$$

From (2.8) and (2.18) we find (2.15). □

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