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TWO RESULTS ON THE COMMUTATIVE PRODUCT OF DISTRIBUTIONS AND FUNCTIONS

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Abstract. Let f and g be distributions and let $f_n = (f * \delta_n x)$ and $g_n = (g * \delta_n)(x)$, where $\delta_n(x)$ is a certain sequence converging to the Dirac delta-function. The product $f.g$ of f and g is defined to be the limit of the sequence $\{f_n g_n\}$, provided its limit h exists in the sense that

$$\lim_{n\to\infty} \langle f_n(x)g_n(x), \varphi(x)\rangle = \langle h(x), \varphi(x)\rangle$$

for all functions φ in $\mathcal{D}$. It is proved that

$$(\operatorname{sgn} x|x|^{-r}\ln^p|x|).(|x|^\mu \ln^q|x|) = \operatorname{sgn} x|x|^{-r+\mu}\ln^{p+q}|x|,$$
$$(|x|^{-r}\ln^p|x|).(\operatorname{sgn} x|x|^\mu \ln^q|x|) = \operatorname{sgn} x|x|^{-r+\mu}\ln^{p+q}|x|$$

for $-2 < -r+\mu < -1$, $r = 1, 2, \ldots$ and $p, q = 0, 1, 2.\ldots$.

In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$.

We define the distributions x_+^{-r} and x_-^{-r} by

$$x_+^{-r} = \frac{(-1)^{r-1}(\ln x_+)^{(r)}}{(r-1)!}, \quad x_-^{-r} = -\frac{(\ln x_-)^{(r)}}{(r-1)!}$$

for $r = 1, 2, \ldots$ and not as in Gel'fand and Shilov [6].

Further, we define the distributions $x_+^{-1}\ln^p x_+$ and $x_-^{-1}\ln^p x_-$ by

$$x_+^{-1}\ln^p x_+ = \frac{(\ln^{p+1} x_+)'}{p+1}, \quad x_-^{-1}\ln^p x_- = -\frac{(\ln^{p+1} x_-)'}{p+1}$$

for $p = 1, 2, \ldots$ and we define the distributions $x_+^{-r}\ln^p x_+$ and $x_-^{-r}\ln^p x_-$ inductively by the equations

$$(x_+^{-r+1}\ln^{p+1} x_+)' = (-r+1)x_+^{-r}\ln^{p+1} x_+ + (p+1)x_+^{-r}\ln^p x_+,$$
$$(x_-^{-r+1}\ln^{p+1} x_-)' = (r-1)x_-^{-r}\ln^{p+1} x_- - (p+1)x_-^{-r}\ln^p x_-$$

for $r, p = 1, 2, \ldots$.

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The distributions $|x|^{-r}\ln^p|x|$ and $\operatorname{sgn} x|x|^{-r}\ln^p|x|$ are now defined by

$$|x|^{-r}\ln^p|x| = x_+^{-r}\ln^p x_+ + x_-^{-r}\ln^p x_-,$$
$$\operatorname{sgn} x|x|^{-r}\ln^p|x| = x_+^{-r}\ln^p x_+ - x_-^{-r}\ln^p x_-$$

for $r = 1, 2, \ldots$ and $p = 0, 1, 2, \ldots$.

In particular, we have

$$\operatorname{sgn} x|x|^{2r-1}\ln^p|x| = x^{2r-1}\ln^p|x|, \quad |x|^{2r}\ln^p|x| = x^{2r}\ln^p|x|$$

for $r = 1, 2, \ldots$ and $p = 0, 1, 2, \ldots$. The definitions here of $x^{2r-1}\ln^p|x|$ and $x^{2r}\ln^p|x|$ are in agreement with Gel'fand and Shilov's definition. The distributions $|x|^\mu\ln^p|x|$ and $\operatorname{sgn} x|x|^\mu\ln^p|x|$ are defined as by Gel'fand and Shilov.

It follows that

$$(|x|^\mu\ln^p|x|)' = \mu \operatorname{sgn} x|x|^{\mu-1}\ln^p|x| + p \operatorname{sgn} x|x|^{\mu-1}\ln^{p-1}|x|,$$
$$(\operatorname{sgn} x|x|^\mu\ln^p|x|)' = \mu|x|^{\mu-1}\ln^p|x| + p|x|^{\mu-1}\ln^{p-1}|x|$$

for all μ and $p = 0, 1, 2, \ldots$.

The definition of the product of a distribution and an infinitely differentiable function is the following, see for example [6].

Definition 1. *Let f be a distribution in $\mathcal{D}'$ and let g be an infinitely differentiable function. The product fg is defined by*

$$\langle fg, \varphi\rangle = \langle f, g\varphi\rangle$$

for all functions φ in $\mathcal{D}$.

A first extension of the product of a distribution and an infinitely differentiable function is the following, see for example [1].

Definition 2. *Let f and g be distributions in $\mathcal{D}'$ for which on the interval (a, b), f is the k-th derivative of a locally summable function F in $L^p(a, b)$ and $g^{(k)}$ is a locally summable function in $L^q(a, b)$ with $1/p + 1/q = 1$. Then the product $fg = gf$ of f and g is defined on the interval (a, b) by*

$$fg = \sum_{i=0}^{k} \binom{k}{i} (-1)^i [Fg^{(i)}]^{(k-i)}.$$

It follows easily from Definition 2 that

$$\ln^p|x|(|x|^\mu\ln^q|x|) = |x|^\mu\ln^{p+q}|x|, \tag{1}$$
$$(\operatorname{sgn} x\ln^p|x|)(\operatorname{sgn} x|x|^\mu\ln^q|x|) = |x|^\mu\ln^{p+q}|x| \tag{2}$$

for $\mu > -1$ and $p, q = 0, 1, 2. \ldots$ and

$$(\operatorname{sgn} x|x|^{-r} \ln^p |x|)(|x|^\mu \ln^q |x|) = \operatorname{sgn} x|x|^{-r+\mu} \ln^{p+q} |x|,$$

$$(|x|^{-r} \ln^p |x|)(\operatorname{sgn} x|x|^\mu \ln^q |x|) = \operatorname{sgn} x|x|^{-r+\mu} \ln^{p+q} |x|,$$

$$(|x|^{-r} \ln^p |x|)(|x|^\mu \ln^q |x| = |x|^{-r+\mu} \ln^{p+q} |x|, \tag{3}$$

$$(\operatorname{sgn} x|x|^{-r} \ln^p |x|)(\operatorname{sgn} x|x|^\mu \ln^q |x|) = |x|^{-r+\mu+1} \ln^{p+q} |x| \tag{4}$$

for $-r + \mu > -1$, $r = 1, 2, \ldots$ and $p, q = 0, 1, 2. \ldots$.

Now let $\rho(x)$ be a function in $\mathcal{D}$ having the following properties:

(i) $\rho(x)$ =0 for $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1} \rho(x)\, dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

If now f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t) \rangle$$

for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

The following definition for the commutative product of two distributions was given in [1] and generalizes Definition 2.

Definition 3. *Let f and g be distributions in $\mathcal{D}'$ and let $f_n(x) = (f * \delta_n)(x)$ and $g_n(x) = (g * \delta_n)(x)$. We say that the commutative product $f.g$ of f and g exists and is equal to the distribution h on the interval (a, b) if*

$$\lim_{n\to\infty} \langle f_n(x) g_n(x), \varphi(x) \rangle = \langle h(x), \varphi(x) \rangle$$

for all functions φ in $\mathcal{D}$ with support contained in the interval (a, b).

It was proved that if the product fg exists by Definition 2, then it exists by Definition 3 and $fg = f.g$.

Some results on the commutative product were proved in [1] and further results were proved in [2] and [3].

The following theorem is easily proved.

Theorem 1. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the commutative products $f.g$ and $f.g'$ (or $f'.g$) exists. Then the product $f'.g$ (or $f.g'$) exists and*

$$(f.g)' = f'.g + f.g'. \tag{5}$$

The next theorem was proved in [5].

Theorem 2. *The commutative product* $(|x|^\lambda \ln^p|x|).(\operatorname{sgn} x|x|^\mu \ln^q|x|)$ *exists and*

$$(|x|^\lambda \ln^p|x|).(\operatorname{sgn} x|x|^\mu \ln^q|x|) = \operatorname{sgn} x|x|^{\lambda+\mu} \ln^{p+q}|x| \tag{6}$$

for $-2 < \lambda + \mu < -1$; $\lambda, \mu \neq -1, -2, \dots$ *and* $p, q = 0, 1, 2. \dots$.

We now prove the following extension of Theorem 2.

Theorem 3. *The commutative products* $(\operatorname{sgn} x|x|^{-r} \ln^p|x|).(|x|^\mu \ln^q|x|)$ *and* $(|x|^{-r} \ln^p|x|).(\operatorname{sgn} x|x|^\mu \ln^q|x|)$ *exist and*

$$(\operatorname{sgn} x|x|^{-r} \ln^p|x|).(|x|^\mu \ln^q|x|) = \operatorname{sgn} x|x|^{-r+\mu} \ln^{p+q}|x|, \tag{7}$$

$$(|x|^{-r} \ln^p|x|).(\operatorname{sgn} x|x|^\mu \ln^q|x|) = \operatorname{sgn} x|x|^{-r+\mu} \ln^{p+q}|x| \tag{8}$$

for $-2 < -r + \mu < -1$, $r = 1, 2, \dots$ *and* $p, q = 0, 1, 2. \dots$.

Proof. With $-1 < \mu < 0$, we have from equation (1)

$$(\ln^{p+1}|x|)|x|^\mu = |x|^\mu \ln^{p+1}|x|. \tag{9}$$

Differentiating equation (9) and using Theorem 1, we get

$$\begin{aligned}(p+1)(x^{-1}\ln^p|x|).|x|^\mu + \mu \ln^{p+1}|x|.(\operatorname{sgn} x|x|^{-1+\mu}) \\ = \mu \operatorname{sgn} x|x|^{-1+\mu}\ln^{p+1}|x| + (p+1)\operatorname{sgn} x|x|^{-1+\mu}\ln^p|x| \\ = (p+1)(x^{-1}\ln^p|x|).|x|^\mu + \mu \operatorname{sgn} x|x|^{-1+\mu}\ln^{p+1}|x|\end{aligned}$$

on using equation (6) with $\lambda = q = 0$. It follows that

$$(x^{-1}\ln^p|x|).|x|^\mu = \operatorname{sgn} x|x|^{-1+\mu}\ln^p|x| \tag{10}$$

when $-1 < \mu < 0$ and $p = 0, 1, 2, \dots$.

Similarly, with $-1 < \mu < 0$, we have from equation (2)

$$(\operatorname{sgn} x \ln^{p+1}|x|)(\operatorname{sgn} x|x|^\mu) = |x|^\mu \ln^{p+1}|x|. \tag{11}$$

Differentiating equation (11) and using Theorem 1, we get

$$\begin{aligned}(p+1)(|x|^{-1}\ln^p|x|).(\operatorname{sgn} x|x|^\mu) + \mu(\operatorname{sgn} x \ln^{p+1}|x|).|x|^{-1+\mu}) \\ = \mu \operatorname{sgn} x|x|^{-1+\mu}\ln^{p+1}|x| + (p+1)\operatorname{sgn} x|x|^{-1+\mu}\ln^p|x| \\ = (p+1)(|x|^{-1}\ln^p|x|).(\operatorname{sgn} x|x|^\mu) + \mu \operatorname{sgn} x|x|^{-1+\mu})\ln^{p+1}|x|\end{aligned}$$

on using equation (6) with $\mu = p = 0$. It follows that

$$(|x|^{-1}\ln^p|x|).(\operatorname{sgn} x|x|^\mu) = \operatorname{sgn} x|x|^{-1+\mu}\ln^p|x| \tag{12}$$

when $-1 < \mu < 0$ and $p = 0, 1, 2, \dots$. □

Now suppose that

$$(\operatorname{sgn} x|x|^{-r}).|x|^{\mu} = \operatorname{sgn} x|x|^{-r+\mu}, \tag{13}$$

$$|x|^{-r}.(\operatorname{sgn} x|x|^{\mu}) = \operatorname{sgn} x|x|^{-r+\mu} \tag{14}$$

for some positive integer r and $-2 < -r+\mu < -1$. This is true when $r = 1$. From equation (3) we have

$$|x|^{-r}|x|^{\mu+1} = |x|^{-r+\mu+1}. \tag{15}$$

Differentiating equation (15) and using Theorem 1, we get

$$\begin{aligned} -r(\operatorname{sgn} x|x|^{-r-1}).|x|^{\mu+1} + (\mu+1)|x|^{-r}.(\operatorname{sgn} x|x|^{\mu}) &= (-r+\mu+1)\operatorname{sgn} x|x|^{-r+\mu} \\ &= -r(\operatorname{sgn} x|x|^{-r-1}).|x|^{\mu+1} + (\mu+1)\operatorname{sgn} x|x|^{-r+\mu} \end{aligned}$$

on using our assumption (14). It follows that

$$(\operatorname{sgn} x|x|^{-r-1}).|x|^{\mu+1} = \operatorname{sgn} x|x|^{-r+\mu}$$

and so equation (13) holds for $r+1$ and$-2 < -r+\mu-1 < -1$. Equation (13) follows by induction for $-2 < -r+\mu < -1$ and $r = 1, 2, \ldots$.

Similarly, from equation (4), we have

$$(\operatorname{sgn} x|x|^{-r})(\operatorname{sgn} x|x|^{\mu+1}) = |x|^{-r+\mu+1}. \tag{16}$$

Differentiating equation (16) and using Theorem 1, we get

$$\begin{aligned} -r|x|^{-r-1}.(\operatorname{sgn} x|x|^{\mu+1}) + (\mu+1)(\operatorname{sgn} x|x|^{-r}).|x|^{\mu} &= (-r+\mu+1)\operatorname{sgn} x|x|^{-r+\mu} \\ &= -r|x|^{-r-1}.(\operatorname{sgn} x|x|^{\mu+1}) + (\mu+1)\operatorname{sgn} x|x|^{-r+\mu} \end{aligned}$$

on using our assumption (13). It follows that

$$|x|^{-r-1}.(\operatorname{sgn} x|x|^{\mu+1}) = \operatorname{sgn} x|x|^{-r+\mu}$$

and so equation (14) holds for $r+1$ and$-2 < -r+\mu-1 < -1$. Equation (14) follows by induction for $-2 < -r+\mu < -1$ and $r = 1, 2, \ldots$.

Now suppose that

$$(\operatorname{sgn} x|x|^{-r}\ln^{p}|x|).|x|^{\mu} = \operatorname{sgn} x|x|^{-r+\mu}\ln^{p}|x|, \tag{17}$$

$$(|x|^{-r}\ln^{p}|x|).(\operatorname{sgn} x|x|^{\mu}) = \operatorname{sgn} x|x|^{-r+\mu}\ln^{p}|x| \tag{18}$$

for some positive integer p and $-2 < -r+\mu < -1$ and $r = 1, 2, \ldots$. This is true when $p = 0$.

Also suppose that with this p

$$(\operatorname{sgn} x|x|^{-r}\ln^{p+1}|x|).|x|^{\mu} = \operatorname{sgn} x|x|^{-r+\mu}\ln^{p+1}|x|, \tag{19}$$

$$(|x|^{-r}\ln^{p+1}|x|).(\operatorname{sgn} x|x|^{\mu}) = \operatorname{sgn} x|x|^{-r+\mu}\ln^{p+1}|x| \tag{20}$$

for some positive integer r and $-2 < -r+\mu < -1$. This is true when $r = 0$.

With this r and p, we have from equation (3)

$$(|x|^{-r}\ln^{p+1}|x|)|x|^{\mu+1} = |x|^{-r+\mu+1}\ln^{p+1}|x|. \tag{21}$$

Differentiating equation (21) and using Theorem 1, we get

$$\begin{aligned} &-r(\operatorname{sgn} x|x|^{-r-1}\ln^{p+1}|x|).|x|^{\mu+1} + (p+1)(\operatorname{sgn} x|x|^{-r-1}\ln^{p}|x|).|x|^{\mu+1} \\ &\qquad + (\mu+1)(|x|^{-r}\ln^{p+1}|x|).(\operatorname{sgn} x|x|^{\mu}) \\ &= (-r+\mu+1)\operatorname{sgn} x|x|^{-r+\mu}\ln^{p+1}|x| + (p+1)\operatorname{sgn} x|x|^{-r+\mu}\ln^{p}|x| \\ &= -r(\operatorname{sgn} x|x|^{-r-1}\ln^{p+1}|x|).|x|^{\mu+1} + (p+1)\operatorname{sgn} x|x|^{-r+\mu}\ln^{p}|x| \\ &\qquad + (\mu+1)\operatorname{sgn} x|x|^{-r+\mu}\ln^{p+1}|x| \end{aligned}$$

on using our assumptions (17) and (20). It follows that

$$(\operatorname{sgn} x|x|^{-r-1}\ln^{p+1}|x|).|x|^{\mu+1} = \operatorname{sgn} x|x|^{-r+\mu}\ln^{p+1}|x|$$

and so equation (19) holds for $-2 < -r+\mu-1 < -1$. Equation (19) follows by induction for $-2 < -r+\mu < -1$: $r = 1.2.\ldots$ and $p = 0, 1, 2, \ldots$.

Similarly, from equation (4) we have

$$(\operatorname{sgn} x|x|^{-r}\ln^{p+1}|x|)(\operatorname{sgn} x|x|^{\mu+1}) = |x|^{-r+\mu+1}\ln^{p+1}|x|. \tag{22}$$

Differentiating equation (22) and using Theorem 1, we get

$$\begin{aligned} &-r(|x|^{-r-1}\ln^{p+1}|x|).(\operatorname{sgn} x|x|^{\mu+1}) + (p+1)(|x|^{-r-1}\ln^{p}|x|).(\operatorname{sgn} x|x|^{\mu+1}) \\ &\qquad + (\mu+1)(\operatorname{sgn} x|x|^{-r}\ln^{p+1}|x|).|x|^{\mu} \\ &= (-r+\mu+1)\operatorname{sgn} x|x|^{-r+\mu}\ln^{p+1}|x| + (p+1)\operatorname{sgn} x|x|^{-r+\mu}\ln^{p}|x| \\ &= -r(|x|^{-r-1}\ln^{p+1}|x|).(\operatorname{sgn} x|x|^{\mu+1}) + (p+1)\operatorname{sgn} x|x|^{-r+\mu}\ln^{p}|x| \\ &\qquad + (\mu+1)\operatorname{sgn} x|x|^{-r+\mu}\ln^{p+1}|x| \end{aligned}$$

on using our assumptions (18) and (19). It follows that

$$(|x|^{-r-1}\ln^{p+1}|x|).(\operatorname{sgn} x|x|^{\mu+1}) = \operatorname{sgn} x|x|^{-r+\mu}\ln^{p+1}|x|$$

and so equation (20) holds for $-2 < -r+\mu-1 < -1$. Equation (19) follows by induction for $-2 < -r+\mu < -1$: $r = 1.2.\ldots$ and $p = 0, 1, 2, \ldots$.

The result of the theorem now follows on differentiating equations (16) and (17) q times partially with respect to μ.

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