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SOME NEW GRÜSS' TYPE INEQUALITIES FOR FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

S. S. DRAGOMIR

Abstract. Some new inequalities of Grüss' type for functions of selfadjoint operators in Hilbert spaces, under suitable assumptions for the involved operators, are given. Several examples for particular functions of interest are provided as well.

1. Introduction

In 1935, G. Grüss [13] proved the following integral inequality which gives an approximation of the integral of the product in terms of the product of the integrals as follows

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx\right| \leq \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma), \quad (1.1)$$

where $f,\ g:[a,b]\to\mathbb{R}$ are integrable on $[a,b]$ and satisfy the condition

$$\phi\leq f(x)\leq\Phi,\ \gamma\leq g(x)\leq\Gamma \quad (1.2)$$

for each $x\in[a,b]$, where ϕ,Φ,γ,Γ are given real constants.

Moreover, the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

For a simple proof of (1.1) as well as for some other integral inequalities of Grüss type, see Chapter X of the book [15] and the papers [1]-[7] and [11].

In 1950, M. Biernacki, H. Pidek and C. Ryll-Nardjewski [15, Chapter X] established the following discrete version of Grüss' inequality:

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Theorem 1. *Let $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ be two n-tuples of real numbers such that $r \leq a_i \leq R$ and $s \leq b_i \leq S$ for $i = 1, \ldots, n$. Then one has*

$$\left| \frac{1}{n} \sum_{i=1}^{n} a_i b_i - \frac{1}{n} \sum_{i=1}^{n} a_i \cdot \frac{1}{n} \sum_{i=1}^{n} b_i \right| \leq \frac{1}{n} \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right) (R - r)(S - s), \tag{1.3}$$

where $[x]$ denotes the integer part of x, $x \in \mathbb{R}$.

A weighted version of the discrete Grüss inequality was proved by J. E. Pečarić in 1979 [15, Chapter X]:

Theorem 2. *Let a and b be two monotonic n-tuples and p a positive one. Then*

$$\left| \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i b_i - \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i \cdot \frac{1}{P_n} \sum_{i=1}^{n} p_i b_i \right| \leq |a_n - a_1| \, |b_n - b_1| \max_{1 \leq k \leq n-1} \left[\frac{P_k \bar{P}_{k+1}}{P_n^2} \right], \tag{1.4}$$

where $P_n := \sum_{i=1}^{n} p_i$, and $\bar{P}_{k+1} = P_n - P_{k+1}$.

In 1981, A. Lupaş, [15, Chapter X] proved some similar results for the first difference of a as follows.

Theorem 3. *Let a, b be two monotonic $n-$tuples in the same sense and p a positive n-tuple. Then*

$$\begin{aligned} &\min_{1 \leq i \leq n-1} |\Delta a_i| \min_{1 \leq i \leq n-1} |\Delta b_i| \left[\frac{1}{P_n} \sum_{i=1}^{n} i^2 p_i - \left(\frac{1}{P_n} \sum_{i=1}^{n} i p_i \right)^2 \right] \\ &\leq \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i b_i - \frac{1}{P_n} \sum_{i=1}^{n} p_i a_i \cdot \frac{1}{P_n} \sum_{i=1}^{n} p_i b_i \\ &\leq \max_{1 \leq i \leq n-1} |\Delta a_i| \max_{1 \leq i \leq n-1} |\Delta b_i| \left[\frac{1}{P_n} \sum_{i=1}^{n} i^2 p_i - \left(\frac{1}{P_n} \sum_{i=1}^{n} i p_i \right)^2 \right], \end{aligned} \tag{1.5}$$

where $\Delta a_i := a_{i+1} - a_i$ is the forward first difference. If there exist the numbers $\bar{a}, \bar{a}_1, r, r_1$ $(rr_1 > 0)$ such that $a_k = \bar{a} + kr$ and $b_k = \bar{a}_1 + kr_1$, then equality holds in (1.5).

Similar integral inequalities can be stated, however they will not be presented here.

2. Operator versions of the Grüss inequality

In order to state the operator version of the Grüss inequality we recall briefly in the following the *Gelfand functional calculus.*

Let A be a selfadjoint linear operator on a complex Hilbert space $(H;\langle .,.\rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, an the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [12, p. 3]):

For any $f,g\in C(Sp(A))$ and any $\alpha,\beta\in\mathbb{C}$ we have

(i) $\Phi(\alpha f+\beta g)=\alpha\Phi(f)+\beta\Phi(g)$;
(ii) $\Phi(fg)=\Phi(f)\Phi(g)$ and $\Phi\left(\bar{f}\right)=\Phi(f)^*$;
(iii) $\|\Phi(f)\|=\|f\|:=\sup_{t\in Sp(A)}|f(t)|$;
(iv) $\Phi(f_0)=1_H$ and $\Phi(f_1)=A$, where $f_0(t)=1$ and $f_1(t)=t$, for $t\in Sp(A)$.

With this notation we define

$$f(A):=\Phi(f) \text{ for all } f\in C(Sp(A))$$

and call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t)\geq 0$ for any $t\in Sp(A)$ implies that $f(A)\geq 0$, *i.e.* $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$f(t)\geq g(t) \text{ for any } t\in Sp(A) \text{ implies that } f(A)\geq g(A) \tag{P}$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [12] and the references therein. For other results, see [17], [14] and [18].

The following operator version of the Grüss inequality was obtained by Mond & Pečarić in [16]:

Theorem 4. [Mond-Pečarić, 1993, [16]] *Let C_j, $j\in\{1,\ldots,n\}$ be selfadjoint operators on the Hilbert space $(H,\langle .,.\rangle)$ and such that $m_j\cdot 1_H\leq C_j\leq M_j\cdot 1_H$ for $j\in\{1,\ldots,n\}$, where 1_H is the identity operator on H. Further, let $g_j,h_j:[m_j,Mj]\to\mathbb{R}$, $j\in\{1,\ldots,n\}$ be functions such that*

$$\varphi\cdot 1_H\leq g_j(C_j)\leq\Phi\cdot 1_H \text{ and } \gamma\cdot 1_H\leq h_j(C_j)\leq\Gamma\cdot 1_H \tag{2.1}$$

for each $j\in\{1,\ldots,n\}$.

If $x_j \in H,\ j \in \{1, \dots, n\}$ *are such that* $\sum_{j=1}^{n} \|x_j\|^2 = 1,$ *then*

$$\left| \sum_{j=1}^{n} \langle g_j(C_j) h_j(C_j) x_j, x_j \rangle - \sum_{j=1}^{n} \langle g_j(C_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle h_j(C_j) x_j, x_j \rangle \right| \le \frac{1}{4} \cdot (\Phi - \varphi)(\Gamma - \gamma). \quad (2.2)$$

If $C_j, j \in \{1, \dots, n\}$ are selfadjoint operators such that $Sp(C_j) \subseteq [m, M]$ for $j \in \{1, \dots, n\}$ and for some scalars $m < M$ and if $g, h : [m, M] \longrightarrow \mathbb{R}$ are continuous then by the Mond-Pečarić inequality we deduce the following version of the Grüss inequality for operators

$$\left| \sum_{j=1}^{n} \langle g(C_j) h(C_j) x_j, x_j \rangle - \sum_{j=1}^{n} \langle g(C_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle h(C_j) x_j, x_j \rangle \right| \le \frac{1}{4} \cdot (\Phi - \varphi)(\Gamma - \gamma), \quad (2.3)$$

where $x_j \in H,\ j \in \{1, \dots, n\}$ are such that $\sum_{j=1}^{n} \|x_j\|^2 = 1$ and $\varphi = \min_{t\in[m,M]} g(t),\ \Phi = \max_{t\in[m,M]} g(t),\ \gamma = \min_{t\in[m,M]} h(t),$ and $\Gamma = \max_{t\in[m,M]} h(t).$

In particular, if the selfadjoint operator C satisfy the condition $Sp(C) \subseteq [m, M]$ for some scalars $m < M$, then

$$|\langle g(C) h(C) x, x \rangle - \langle g(C) x, x \rangle \cdot \langle h(C) x, x \rangle| \le \frac{1}{4} \cdot (\Phi - \varphi)(\Gamma - \gamma), \quad (2.4)$$

for any $x \in H$ with $\|x\| = 1$.

In the recent paper [9] the following refinement of (2.4) has been obtained:

Theorem 5 (Dragomir, 2008, [9])**.** *Let* A *be a selfadjoint operator on the Hilbert space* $(H; \langle .,. \rangle)$ *and assume that* $Sp(A) \subseteq [m, M]$ *for some scalars* $m < M$. *If* f *and* g *are continuous on* $[m, M]$ *and* $\gamma := \min_{t\in[m,M]} f(t)$, $\Gamma := \max_{t\in[m,M]} f(t)$, $\delta := \min_{t\in[m,M]} g(t)$ *and* $\Delta := \max_{t\in[m,M]} g(t)$ *then*

$$|\langle f(A) g(A) x, x \rangle - \langle f(A) x, x \rangle \langle g(A) x, x \rangle| \le \frac{1}{2}(\Gamma - \gamma)\left[\|g(A) x\|^2 - \langle g(A) x, x \rangle^2\right]^{1/2} \left[\le \frac{1}{4} \cdot (\Gamma - \gamma)(\Delta - \delta)\right] \quad (2.5)$$

for each $x \in H$ *with* $\|x\| = 1$.

A version of n operators that generalise this result and improves (2.3) is incorporated in:

Theorem 6 (Dragomir, 2008, [9])**.** *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \dots, n\}$ and for some scalars $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are as in Theorem 5 then*

$$\left| \sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle - \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle \right|$$
$$\le \frac{1}{2} (\Gamma - \gamma) \left[\sum_{j=1}^{n} \|g(A_j) x_j\|^2 - \left(\sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle \right)^2 \right]^{1/2}$$
$$\left[\le \frac{1}{4} \cdot (\Gamma - \gamma)(\Delta - \delta) \right] \quad (2.6)$$

for each $x_j \in H, j \in \{1, \dots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

Motivated by the above results we investigate in this paper other Grüss' type inequalities for selfadjoint operators in Hilbert spaces. Some of the obtained results improve the inequalities (2.3) and (2.4) derived from the Mond-Pečarić inequality. Others provide different operator versions for the celebrated Grüss' inequality mentioned above. Examples for power functions and the logarithmic function are given as well.

3. Some vectorial Grüss' type inequalities

The following lemmas, that are of interest in their own right, collect some Grüss type inequalities for vectors in inner product spaces obtained earlier by the author:

Lemma 1. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$, $u, v, e \in H$, $\|e\| = 1$, and $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ such that*

$$\operatorname{Re} \langle \beta e - u, u - \alpha e \rangle \ge 0, \ \operatorname{Re} \langle \delta e - v, v - \gamma e \rangle \ge 0 \quad (3.1)$$

or equivalently,

$$\left\| u - \frac{\alpha + \beta}{2} e \right\| \le \frac{1}{2} |\beta - \alpha|, \ \left\| v - \frac{\gamma + \delta}{2} e \right\| \le \frac{1}{2} |\delta - \gamma|. \quad (3.2)$$

Then

$$|\langle u, v \rangle - \langle u, e \rangle \langle e, v \rangle|$$
$$\le \frac{1}{4} \cdot |\beta - \alpha| |\delta - \gamma| - \begin{cases} [\operatorname{Re} \langle \beta e - u, u - \alpha e \rangle \operatorname{Re} \langle \delta e - v, v - \gamma e \rangle]^{\frac{1}{2}}, \\ \left| \langle u, e \rangle - \frac{\alpha+\beta}{2} \right| \left| \langle v, e \rangle - \frac{\gamma+\delta}{2} \right|. \end{cases} \quad (3.3)$$

The first inequality has been obtained in [2] (see also [8, p. 44]) while the second result was established in [3] (see also [8, p. 90]). They provide refinements of the earlier result from [1] where only the first part of the bound, i.e., $\frac{1}{4}|\beta-\alpha||\delta-\gamma|$ has been given. Notice that, as pointed out in [3], the upper bounds for the Grüss functional incorporated in (3.3) cannot be compared in general, meaning that one is better than the other depending on appropriate choices of the vectors and scalars involved.

Another result of this type is the following one:

Lemma 2. *With the assumptions in Lemma* 1 *and if* $\operatorname{Re}(\beta\overline{\alpha})>0, \operatorname{Re}(\delta\overline{\gamma})>0$ *then*

$$|\langle u,v\rangle-\langle u,e\rangle\langle e,v\rangle| \leq \begin{cases} \frac{1}{4}\cdot\frac{|\beta-\alpha||\delta-\gamma|}{[\operatorname{Re}(\beta\overline{\alpha})\operatorname{Re}(\delta\overline{\gamma})]^{\frac{1}{2}}}|\langle u,e\rangle\langle e,v\rangle|, \\ \left[\left(|\alpha+\beta|-2[\operatorname{Re}(\beta\overline{\alpha})]^{\frac{1}{2}}\right)\left(|\delta+\gamma|-2[\operatorname{Re}(\delta\overline{\gamma})]^{\frac{1}{2}}\right)\right]^{\frac{1}{2}} \\ \cdot[|\langle u,e\rangle\langle e,v\rangle|]^{\frac{1}{2}}. \end{cases} \tag{3.4}$$

The first inequality has been established in [4] (see [8, p. 62]) while the second one can be obtained in a canonical manner from the reverse of the Schwarz inequality given in [5]. The details are omitted.

Finally, another inequality of Grüss type that has been obtained in [6] (see also [8, p. 65]) can be stated as:

Lemma 3. *With the assumptions in Lemma* 1 *and if* $\beta\neq-\alpha$, $\delta\neq-\gamma$ *then*

$$|\langle u,v\rangle-\langle u,e\rangle\langle e,v\rangle| \leq \frac{1}{4}\cdot\frac{|\beta-\alpha||\delta-\gamma|}{[|\beta+\alpha||\delta+\gamma|]^{\frac{1}{2}}}[(\|u\|+|\langle u,e\rangle|)(\|v\|+|\langle v,e\rangle|)]^{\frac{1}{2}}. \tag{3.5}$$

4. Some inequalities of Grüss' type for one operator

The following results incorporates some new inequalities of Grüss' type for two functions of a selfadjoint operator.

Theorem 7. *Let A be a selfadjoint operator on the Hilbert space $(H;\langle .,.\rangle)$ and assume that $Sp(A)\subseteq[m,M]$ for some scalars $m<M$. If f and g are continuous on $[m,M]$ and $\gamma:=\min_{t\in[m,M]}f(t)$, $\Gamma:=\max_{t\in[m,M]}f(t)$,*

$\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$ then

$$|\langle f(A) g(A) x, x\rangle - \langle f(A) x, x\rangle \langle g(A) x, x\rangle| \le \frac{1}{4} \cdot (\Gamma-\gamma)(\Delta-\delta)$$
$$- \begin{cases} [\langle \Gamma x - f(A) x, f(A) x - \gamma x\rangle \langle \Delta x - g(A) x, g(A) x - \delta x\rangle]^{\frac{1}{2}}, \\ \left|\langle f(A) x, x\rangle - \frac{\Gamma+\gamma}{2}\right| \left|\langle g(A) x, x\rangle - \frac{\Delta+\delta}{2}\right|. \end{cases} \tag{4.1}$$

for each $x \in H$ with $\|x\| = 1$.

Moreover if γ and δ are positive, then we also have

$$|\langle f(A) g(A) x, x\rangle - \langle f(A) x, x\rangle \langle g(A) x, x\rangle|$$
$$\le \begin{cases} \frac{1}{4} \cdot \frac{(\Gamma-\gamma)(\Delta-\delta)}{\sqrt{\Gamma\gamma\Delta\delta}} \langle f(A) x, x\rangle \langle g(A) x, x\rangle, \\ \left(\sqrt{\Gamma}-\sqrt{\gamma}\right)\left(\sqrt{\Delta}-\sqrt{\delta}\right) [\langle f(A) x, x\rangle \langle g(A) x, x\rangle]^{\frac{1}{2}}. \end{cases} \tag{4.2}$$

while for $\Gamma+\gamma, \Delta+\delta \ne 0$ we have

$$|\langle f(A) g(A) x, x\rangle - \langle f(A) x, x\rangle \langle g(A) x, x\rangle|$$
$$\le \frac{1}{4} \cdot \frac{(\Gamma-\gamma)(\Delta-\delta)}{[|\Gamma+\gamma| |\Delta+\delta|]^{\frac{1}{2}}} [(\|f(A) x\| + |\langle f(A) x, x\rangle|)$$
$$(\|g(A) x\| + |\langle g(A) x, x\rangle|)]^{\frac{1}{2}} \tag{4.3}$$

for each $x \in H$ with $\|x\| = 1$.

Proof. Since $\gamma := \min_{t\in[m,M]} f(t)$, $\Gamma := \max_{t\in[m,M]} f(t)$, $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$, the by the property (P) we have that

$$\gamma \cdot 1_H \le f(A) \le \Gamma \cdot 1_H \text{ and } \delta \cdot 1_H \le g(A) \le \Delta \cdot 1_H$$

in the operator order, which imply that

$$[f(A) - \gamma \cdot 1][\Gamma \cdot 1_H - f(A)] \ge 0 \text{ and } [\Delta \cdot 1_H - g(A)][g(A) - \delta \cdot 1_H] \ge 0 \tag{4.4}$$

in the operator order.

We then have from (4.4)

$$\langle [f(A) - \gamma \cdot 1][\Gamma \cdot 1_H - f(A)] x, x\rangle \ge 0$$

and

$$\langle [\Delta \cdot 1_H - g(A)][g(A) - \delta \cdot 1_H] x, x\rangle \ge 0,$$

for each $x \in H$ with $\|x\| = 1$, which, by the fact that the involved operators are selfadjoint, are equivalent with the inequalities

$$\langle \Gamma x - f(A) x, f(A) x - \gamma x\rangle \ge 0 \text{ and } \langle \Delta x - g(A) x, g(A) x - \delta x\rangle \ge 0, \tag{4.5}$$

for each $x \in H$ with $\|x\| = 1$.

Now, if we apply Lemma 1 for $u = f(A)x$, $v = g(A)x$, $e = x$, and the real scalars Γ, γ, Δ and δ defined in the statement of the theorem, then we can state the inequality

$$|\langle f(A)x, g(A)x\rangle - \langle f(A)x, x\rangle \langle x, g(A)x\rangle| \le \frac{1}{4} \cdot (\Gamma - \gamma)(\Delta - \delta)$$
$$- \begin{cases} [\operatorname{Re}\langle \Gamma x - f(A)x, f(A)x - \gamma x\rangle \operatorname{Re}\langle \Delta x - g(A)x, g(A)x - \delta x\rangle]^{\frac{1}{2}}, \\ \left|\langle f(A)x, x\rangle - \frac{\Gamma+\gamma}{2}\right| \left|\langle g(A)x, x\rangle - \frac{\Delta+\delta}{2}\right|, \end{cases} \tag{4.6}$$

for each $x \in H$ with $\|x\| = 1$, which is clearly equivalent with the inequality (4.1).

The inequalities (4.2) and (4.3) follow by Lemma 2 and Lemma 3 respectively and the details are omitted. □

Remark 1. The first inequality in (4.2) can be written in a more convenient way as

$$\left|\frac{\langle f(A)g(A)x, x\rangle}{\langle f(A)x, x\rangle \langle g(A)x, x\rangle} - 1\right| \le \frac{1}{4} \cdot \frac{(\Gamma - \gamma)(\Delta - \delta)}{\sqrt{\Gamma\gamma\Delta\delta}} \tag{4.7}$$

for each $x \in H$ with $\|x\| = 1$, while the second inequality has the following equivalent form

$$\left|\frac{\langle f(A)g(A)x, x\rangle}{[\langle f(A)x, x\rangle \langle g(A)x, x\rangle]^{1/2}} - [\langle f(A)x, x\rangle \langle g(A)x, x\rangle]^{1/2}\right| \le \left(\sqrt{\Gamma} - \sqrt{\gamma}\right)\left(\sqrt{\Delta} - \sqrt{\delta}\right) \tag{4.8}$$

for each $x \in H$ with $\|x\| = 1$.

We know, from [10] that if f, g are synchronous (asynchronous) functions on the interval $[m, M]$, i.e., we recall that

$$[f(t) - f(s)][g(t) - g(s)] (\ge) \le 0 \text{ for each } t, s \in [m, M],$$

then we have the inequality

$$\langle f(A)g(A)x, x\rangle \ge (\le) \langle f(A)x, x\rangle \langle g(A)x, x\rangle \tag{4.9}$$

for each $x \in H$ with $\|x\| = 1$,provided f, g are continuous on $[m, M]$ and A is a selfadjoint operator with $Sp(A) \subseteq [m, M]$.

Therefore, if f, g are synchronous then we have from (4.7) and from (4.8) the following results:

$$0 \le \frac{\langle f(A)g(A)x, x\rangle}{\langle f(A)x, x\rangle \langle g(A)x, x\rangle} - 1 \le \frac{1}{4} \cdot \frac{(\Gamma - \gamma)(\Delta - \delta)}{\sqrt{\Gamma\gamma\Delta\delta}} \tag{4.10}$$

and

$$0 \leq \frac{\langle f(A)\, g(A)\, x, x\rangle}{[\langle f(A)\, x, x\rangle \langle g(A)\, x, x\rangle]^{1/2}} - [\langle f(A)\, x, x\rangle \langle g(A)\, x, x\rangle]^{1/2} \leq \left(\sqrt{\Gamma} - \sqrt{\gamma}\right)\left(\sqrt{\Delta} - \sqrt{\delta}\right) \quad (4.11)$$

for each $x \in H$ with $\|x\| = 1$, respectively.

If f, g are asynchronous then

$$0 \leq 1 - \frac{\langle f(A)\, g(A)\, x, x\rangle}{\langle f(A)\, x, x\rangle \langle g(A)\, x, x\rangle} \leq \frac{1}{4} \cdot \frac{(\Gamma - \gamma)(\Delta - \delta)}{\sqrt{\Gamma\gamma\Delta\delta}} \quad (4.12)$$

and

$$0 \leq [\langle f(A)\, x, x\rangle \langle g(A)\, x, x\rangle]^{1/2} - \frac{\langle f(A)\, g(A)\, x, x\rangle}{[\langle f(A)\, x, x\rangle \langle g(A)\, x, x\rangle]^{1/2}} \leq \left(\sqrt{\Gamma} - \sqrt{\gamma}\right)\left(\sqrt{\Delta} - \sqrt{\delta}\right) \quad (4.13)$$

for each $x \in H$ with $\|x\| = 1$, respectively.

It is obvious that all the inequalities from Theorem 7 can be used to obtain reverse inequalities of Grüss' type for various particular instances of operator functions, see for instance [9]. However we give here only a few provided by the inequalities (4.10) and (4.11) above.

Example 1. Let A be a selfadjoint operator with $Sp(A) \subseteq [m, M]$ for some scalars $m < M$.

If A is positive ($m \geq 0$) and $p, q > 0$, then

$$0 \leq \frac{\langle A^{p+q}x, x\rangle}{\langle A^p x, x\rangle \cdot \langle A^q x, x\rangle} - 1 \leq \frac{1}{4} \cdot \frac{(M^p - m^p)(M^q - m^q)}{M^{\frac{p+q}{2}} m^{\frac{p+q}{2}}} \quad (4.14)$$

and

$$0 \leq \frac{\langle A^{p+q}x, x\rangle}{[\langle A^p x, x\rangle \cdot \langle A^q x, x\rangle]^{1/2}} - [\langle A^p x, x\rangle \cdot \langle A^q x, x\rangle]^{1/2} \leq \left(M^{\frac{p}{2}} - m^{\frac{p}{2}}\right)\left(M^{\frac{q}{2}} - m^{\frac{q}{2}}\right) \quad (4.15)$$

for each $x \in H$ with $\|x\| = 1$.

If A is positive definite ($m > 0$) and $p, q < 0$, then

$$0 \leq \frac{\langle A^{p+q}x, x\rangle}{\langle A^p x, x\rangle \cdot \langle A^q x, x\rangle} - 1 \leq \frac{1}{4} \cdot \frac{(M^{-p} - m^{-p})(M^{-q} - m^{-q})}{M^{-\frac{p+q}{2}} m^{-\frac{p+q}{2}}} \quad (4.16)$$

and

$$0 \le \frac{\langle A^{p+q}x, x\rangle}{[\langle A^p x, x\rangle \cdot \langle A^q x, x\rangle]^{1/2}} - [\langle A^p x, x\rangle \cdot \langle A^q x, x\rangle]^{1/2} \le \frac{\left(M^{-\frac{p}{2}} - m^{-\frac{p}{2}}\right)\left(M^{-\frac{q}{2}} - m^{-\frac{q}{2}}\right)}{M^{-\frac{p+q}{2}} m^{-\frac{p+q}{2}}} \quad (4.17)$$

for each $x \in H$ with $\|x\| = 1$.

Similar inequalities may be stated for either $p > 0, q < 0$ or $p < 0, q > 0$. The details are omitted.

Example 2. Let A be a positive definite operator with $Sp(A) \subseteq [m, M]$ for some scalars $1 < m < M$. If $p > 0$ then

$$0 \le \frac{\langle A^p \ln Ax, x\rangle}{\langle A^p x, x\rangle \cdot \langle \ln Ax, x\rangle} - 1 \le \frac{1}{4} \cdot \frac{(M^p - m^p)\ln \frac{M}{m}}{M^{\frac{p}{2}} m^{\frac{p}{2}} \sqrt{\ln M \cdot \ln m}} \quad (4.18)$$

and

$$0 \le \frac{\langle A^p \ln Ax, x\rangle}{[\langle A^p x, x\rangle \cdot \langle \ln Ax, x\rangle]^{1/2}} - [\langle A^p x, x\rangle \cdot \langle \ln Ax, x\rangle]^{1/2} \le \left(M^{\frac{p}{2}} - m^{\frac{p}{2}}\right)\left[\sqrt{\ln M} - \sqrt{\ln m}\right], \quad (4.19)$$

for each $x \in H$ with $\|x\| = 1$.

5. Some inequalities of Grüss' type for n operators

The following extension for sequences of operators can be stated:

Theorem 8. *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \dots, n\}$ and for some scalars $m < M$. If f and g are continuous on $[m, M]$ and $\gamma := \min_{t\in[m,M]} f(t)$, $\Gamma := \max_{t\in[m,M]} f(t)$, $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$ then*

$$\left|\sum_{j=1}^n \langle f(A_j) g(A_j) x_j, x_j\rangle - \sum_{j=1}^n \langle f(A_j) x_j, x_j\rangle \cdot \sum_{j=1}^n \langle g(A_j) x_j, x_j\rangle\right| \le \frac{1}{4} \cdot (\Gamma - \gamma)(\Delta - \delta)$$
$$- \begin{cases} \left[\sum_{j=1}^n \langle \Gamma x_j - f(A_j) x_j, f(A_j) x_j - \gamma x_j\rangle \right. \\ \left. \cdot \sum_{j=1}^n \langle \Delta x_j - g(A_j) x_j, g(A_j) x - \delta x_j\rangle\right]^{\frac{1}{2}}, \\ \\ \left|\sum_{j=1}^n \langle f(A_j) x_j, x_j\rangle - \frac{\Gamma+\gamma}{2}\right| \left|\sum_{j=1}^n \langle g(A_j) x_j, x_j\rangle - \frac{\Delta+\delta}{2}\right|, \end{cases} \quad (5.1)$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

Moreover if γ *and* δ *are positive, then we also have*

$$\left| \sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle - \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle \right| \le \begin{cases} \frac{1}{4} \cdot \frac{(\Gamma-\gamma)(\Delta-\delta)}{\sqrt{\Gamma\gamma\Delta\delta}} \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle, \\ \left(\sqrt{\Gamma} - \sqrt{\gamma}\right)\left(\sqrt{\Delta} - \sqrt{\delta}\right) \\ \cdot \left[\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle\right]^{\frac{1}{2}}, \end{cases} \tag{5.2}$$

while for $\Gamma + \gamma, \Delta + \delta \neq 0$ *we have*

$$\left| \sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle - \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle \right| \le \frac{1}{4} \cdot \frac{(\Gamma - \gamma)(\Delta - \delta)}{[|\Gamma + \gamma| |\Delta + \delta|]^{\frac{1}{2}}} \cdot \left[\left(\left(\sum_{j=1}^{n} \|f(A_j) x_j\|^2 \right)^{1/2} + \left| \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \right| \right) \cdot \left(\left(\sum_{j=1}^{n} \|g(A_j) x_j\|^2 \right)^{1/2} + \left| \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle \right| \right) \right]^{1/2}, \tag{5.3}$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

Proof. As in [12, p. 6], if we put

$$\widetilde{A} := \begin{pmatrix} A_1 & . & . & . & 0 \\ & . & & & \\ & & . & & \\ & & & . & \\ 0 & . & . & . & A_n \end{pmatrix} \text{ and } \widetilde{x} = \begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix},$$

then we have $Sp\left(\widetilde{A}\right) \subseteq [m, M]$, $\|\widetilde{x}\| = 1$

$$\left\langle f\left(\widetilde{A}\right) g\left(\widetilde{A}\right) \widetilde{x}, \widetilde{x} \right\rangle = \sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle, \left\langle g\left(\widetilde{A}\right) \widetilde{x}, \widetilde{x} \right\rangle = \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle,$$

$$\left\langle f\left(\widetilde{A}\right)\widetilde{x},\widetilde{x}\right\rangle = \sum_{j=1}^{n}\left\langle f\left(A_j\right)x_j,x_j\right\rangle,\ \ \left\|f\left(\widetilde{A}\right)\widetilde{x}\right\|^2 = \sum_{j=1}^{n}\left\|f\left(A_j\right)x_j\right\|^2$$

and

$$\left\|g\left(\widetilde{A}\right)\widetilde{x}\right\|^2 = \sum_{j=1}^{n}\left\|g\left(A_j\right)x_j\right\|^2.$$

Applying Theorem 7 for $\widetilde{A}$ and $\widetilde{x}$ we deduce the desired results. The details are omitted. □

Remark 2. The first inequality in (5.2) can be written in a more convenient way as

$$\left|\frac{\sum_{j=1}^{n}\left\langle f\left(A_j\right)g\left(A_j\right)x_j,x_j\right\rangle}{\sum_{j=1}^{n}\left\langle f\left(A_j\right)x_j,x_j\right\rangle\cdot\sum_{j=1}^{n}\left\langle g\left(A_j\right)x_j,x_j\right\rangle}-1\right| \le \frac{1}{4}\cdot\frac{\left(\Gamma-\gamma\right)\left(\Delta-\delta\right)}{\sqrt{\Gamma\gamma\Delta\delta}} \tag{5.4}$$

for each $x_j \in H, j \in \{1,\ldots,n\}$ with $\sum_{j=1}^{n}\left\|x_j\right\|^2 = 1$, while the second inequality has the following equivalent form

$$\left|\frac{\sum_{j=1}^{n}\left\langle f\left(A_j\right)g\left(A_j\right)x_j,x_j\right\rangle}{\left[\sum_{j=1}^{n}\left\langle f\left(A_j\right)x_j,x_j\right\rangle\cdot\sum_{j=1}^{n}\left\langle g\left(A_j\right)x_j,x_j\right\rangle\right]^{1/2}} - \left[\sum_{j=1}^{n}\left\langle f\left(A_j\right)x_j,x_j\right\rangle\cdot\sum_{j=1}^{n}\left\langle g\left(A_j\right)x_j,x_j\right\rangle\right]^{1/2}\right| \le \left(\sqrt{\Gamma}-\sqrt{\gamma}\right)\left(\sqrt{\Delta}-\sqrt{\delta}\right) \tag{5.5}$$

for each $x_j \in H, j \in \{1,\ldots,n\}$ with $\sum_{j=1}^{n}\left\|x_j\right\|^2 = 1$.

We know, from [10] that if f,g are synchronous (asynchronous) functions on the interval $[m,M]$, then we have the inequality

$$\sum_{j=1}^{n}\left\langle f\left(A_j\right)g\left(A_j\right)x_j,x_j\right\rangle \ge (\le) \sum_{j=1}^{n}\left\langle f\left(A_j\right)x_j,x_j\right\rangle\cdot\sum_{j=1}^{n}\left\langle g\left(A_j\right)x_j,x_j\right\rangle \tag{5.6}$$

for each $x_j \in H, j \in \{1,\ldots,n\}$ with $\sum_{j=1}^{n}\left\|x_j\right\|^2 = 1$, provided f,g are continuous on $[m,M]$ and A_j are selfadjoint operators with $Sp\left(A_j\right) \subseteq [m,M]$, $j \in \{1,\ldots,n\}$.

Therefore, if f, g are synchronous then we have from (5.4) and from (5.5) the following results:

$$0 \leq \frac{\sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle}{\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle} - 1 \leq \frac{1}{4} \cdot \frac{(\Gamma - \gamma)(\Delta - \delta)}{\sqrt{\Gamma \gamma \Delta \delta}} \quad (5.7)$$

and

$$0 \leq \frac{\sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle}{\left[\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle\right]^{1/2}} - \left[\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle\right]^{1/2} \leq \left(\sqrt{\Gamma} - \sqrt{\gamma}\right)\left(\sqrt{\Delta} - \sqrt{\delta}\right) \quad (5.8)$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, respectively.

If f, g are asynchronous then

$$0 \leq 1 - \frac{\sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle}{\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle} \leq \frac{1}{4} \cdot \frac{(\Gamma - \gamma)(\Delta - \delta)}{\sqrt{\Gamma \gamma \Delta \delta}} \quad (5.9)$$

and

$$0 \leq \left[\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle\right]^{1/2} - \frac{\sum_{j=1}^{n} \langle f(A_j) g(A_j) x_j, x_j \rangle}{\left[\sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle \cdot \sum_{j=1}^{n} \langle g(A_j) x_j, x_j \rangle\right]^{1/2}} \leq \left(\sqrt{\Gamma} - \sqrt{\gamma}\right)\left(\sqrt{\Delta} - \sqrt{\delta}\right) \quad (5.10)$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$, respectively.

It is obvious that all the inequalities from Theorem 8 can be used to obtain reverse inequalities of Grüss' type for various particular instances of operator functions, see for instance [9]. However we give here only a few provided by the inequalities (5.7) and (5.8) above.

Example 3. Let A_j $j \in \{1, \ldots, n\}$ be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$, $j \in \{1, \ldots, n\}$ for some scalars $m < M$.

If A_j are positive ($m \geq 0$) and $p, q > 0$, then

$$0 \leq \frac{\sum_{j=1}^{n} \left\langle A_j^{p+q} x_j, x_j \right\rangle}{\sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle} - 1 \leq \frac{1}{4} \cdot \frac{(M^p - m^p)(M^q - m^q)}{M^{\frac{p+q}{2}} m^{\frac{p+q}{2}}} \quad (5.11)$$

and

$$0 \leq \frac{\sum_{j=1}^{n} \left\langle A_j^{p+q} x_j, x_j \right\rangle}{\left[\sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle\right]^{1/2}} - \left[\sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle\right]^{1/2} \leq \left(M^{\frac{p}{2}} - m^{\frac{p}{2}}\right)\left(M^{\frac{q}{2}} - m^{\frac{q}{2}}\right) \quad (5.12)$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

If A is positive definite ($m > 0$) and $p, q < 0$, then

$$0 \leq \frac{\sum_{j=1}^{n} \left\langle A_j^{p+q} x_j, x_j \right\rangle}{\sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle} - 1 \leq \frac{1}{4} \cdot \frac{(M^{-p} - m^{-p})(M^{-q} - m^{-q})}{M^{-\frac{p+q}{2}} m^{-\frac{p+q}{2}}} \quad (5.13)$$

and

$$0 \leq \left[\sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle\right]^{1/2} - \frac{\sum_{j=1}^{n} \left\langle A_j^{p+q} x, x \right\rangle}{\left[\sum_{j=1}^{n} \left\langle A_j^{p} x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle A_j^{q} x_j, x_j \right\rangle\right]^{1/2}} \leq \frac{\left(M^{-\frac{p}{2}} - m^{-\frac{p}{2}}\right)\left(M^{-\frac{q}{2}} - m^{-\frac{q}{2}}\right)}{M^{-\frac{p+q}{2}} m^{-\frac{p+q}{2}}} \quad (5.14)$$

for each $x_j \in H, j \in \{1, \ldots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

Similar inequalities may be stated for either $p > 0, q < 0$ or $p < 0, q > 0$. The details are omitted.

Example 4. Let A be a positive definite operator with $Sp(A) \subseteq [m, M]$ for some scalars $1 < m < M$. If $p > 0$ then

$$0 \le \frac{\sum_{j=1}^{n} \left\langle A_j^p \ln A_j x_j, x_j \right\rangle}{\sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle \ln A_j x_j, x_j \right\rangle} - 1 \le \frac{1}{4} \cdot \frac{(M^p - m^p) \ln \frac{M}{m}}{M^{\frac{p}{2}} m^{\frac{p}{2}} \sqrt{\ln M \cdot \ln m}} \quad (5.15)$$

and

$$0 \le \frac{\sum_{j=1}^{n} \left\langle A_j^p \ln A_j x_j, x_j \right\rangle}{\left[\sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle \ln A_j x_j, x_j \right\rangle\right]^{1/2}} - \left[\sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle \cdot \sum_{j=1}^{n} \left\langle \ln A_j x_j, x_j \right\rangle\right]^{1/2} \le \left(M^{\frac{p}{2}} - m^{\frac{p}{2}}\right) \left[\sqrt{\ln M} - \sqrt{\ln m}\right], \quad (5.16)$$

for each $x_j \in H, j \in \{1, \dots, n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

Similar inequalities may be stated for $p < 0$. The details are omitted.

The following result for n operators can be stated as well:

Corollary 1. *Let A_j be selfadjoint operators with $Sp(A_j) \subseteq [m, M]$ for $j \in \{1, \dots, n\}$ and for some scalars $m < M$. If f and g are continuous on $[m, M]$ and $\gamma := \min_{t \in [m,M]} f(t)$, $\Gamma := \max_{t \in [m,M]} f(t)$, $\delta := \min_{t \in [m,M]} g(t)$ and $\Delta := \max_{t \in [m,M]} g(t)$ then for any $p_j \ge 0, j \in \{1, \dots, n\}$ with $\sum_{j=1}^{n} p_j = 1$ we have*

$$\left| \left\langle \sum_{j=1}^{n} p_j f(A_j) g(A_j) x, x \right\rangle - \left\langle \sum_{j=1}^{n} p_j f(A_j) x, x \right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g(A_j) x, x \right\rangle \right| \le \frac{1}{4} \cdot (\Gamma - \gamma)(\Delta - \delta)$$

$$
-\begin{cases}
\left[\sum_{j=1}^{n} p_j \left\langle \Gamma x - f\left(A_j\right)x, f\left(A_j\right)x - \gamma x\right\rangle \right. \\
\left. \cdot \sum_{j=1}^{n} p_j \left\langle \Delta x - g\left(A_j\right)x, g\left(A_j\right)x - \delta x\right\rangle \right]^{\frac{1}{2}}, \\
\left|\left\langle \sum_{j=1}^{n} p_j f\left(A_j\right)x, x\right\rangle - \frac{\Gamma+\gamma}{2}\right| \left|\left\langle \sum_{j=1}^{n} p_j g\left(A_j\right)x, x\right\rangle - \frac{\Delta+\delta}{2}\right|,
\end{cases} \tag{5.17}
$$

for each $x \in H$, *with* $\|x\|^2 = 1$.

Moreover if γ *and* δ *are positive, then we also have*

$$
\left|\left\langle \sum_{j=1}^{n} p_j f\left(A_j\right) g\left(A_j\right)x, x\right\rangle - \left\langle \sum_{j=1}^{n} p_j f\left(A_j\right)x, x\right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g\left(A_j\right)x, x\right\rangle\right|
$$
$$
\leq \begin{cases}
\frac{1}{4} \cdot \frac{(\Gamma-\gamma)(\Delta-\delta)}{\sqrt{\Gamma\gamma\Delta\delta}} \left\langle \sum_{j=1}^{n} p_j f\left(A_j\right)x, x\right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g\left(A_j\right)x, x\right\rangle, \\
\left(\sqrt{\Gamma} - \sqrt{\gamma}\right)\left(\sqrt{\Delta} - \sqrt{\delta}\right) \left[\left\langle \sum_{j=1}^{n} p_j f\left(A_j\right)x, x\right\rangle \right. \\
\left. \cdot \left\langle \sum_{j=1}^{n} p_j g\left(A_j\right)x, x\right\rangle \right]^{\frac{1}{2}}.
\end{cases} \tag{5.18}
$$

while for $\Gamma + \gamma, \Delta + \delta \neq 0$ *we have*

$$
\left|\left\langle \sum_{j=1}^{n} p_j f\left(A_j\right) g\left(A_j\right)x, x\right\rangle - \left\langle \sum_{j=1}^{n} p_j f\left(A_j\right)x, x\right\rangle \cdot \left\langle \sum_{j=1}^{n} p_j g\left(A_j\right)x, x\right\rangle\right|
$$
$$
\leq \frac{1}{4} \cdot \frac{\left(\Gamma - \gamma\right)\left(\Delta - \delta\right)}{\left[\left|\Gamma + \gamma\right|\left|\Delta + \delta\right|\right]^{\frac{1}{2}}}
$$
$$
\cdot \left[\left(\left(\sum_{j=1}^{n} p_j \left\|f\left(A_j\right)x\right\|^2\right)^{1/2} + \left|\left\langle \sum_{j=1}^{n} p_j f\left(A_j\right)x, x\right\rangle\right|\right)\right.
$$
$$
\left. \cdot \left(\left(\sum_{j=1}^{n} p_j \left\|g\left(A_j\right)x\right\|^2\right)^{1/2} + \left|\left\langle \sum_{j=1}^{n} p_j g\left(A_j\right)x, x\right\rangle\right|\right)\right]^{1/2} \tag{5.19}
$$

for each $x \in H$, *with* $\|x\|^2 = 1$.

Proof. Follows from Theorem 8 on choosing $x_j = \sqrt{p_j} \cdot x$, $j \in \{1, \ldots, n\}$, where $p_j \geq 0, j \in \{1, \ldots, n\}$, $\sum_{j=1}^{n} p_j = 1$ and $x \in H$, with $\|x\| = 1$. The details are omitted. □

Remark 3. The first inequality in (5.18) can be written in a more convenient way as

$$\left|\frac{\left\langle\sum_{j=1}^{n}p_jf(A_j)g(A_j)x,x\right\rangle}{\left\langle\sum_{j=1}^{n}p_jf(A_j)x,x\right\rangle\cdot\left\langle\sum_{j=1}^{n}p_jg(A_j)x,x\right\rangle}-1\right|\leq\frac{1}{4}\cdot\frac{(\Gamma-\gamma)(\Delta-\delta)}{\sqrt{\Gamma\gamma\Delta\delta}} \tag{5.20}$$

for each $x\in H$, with $\|x\|^2=1$, while the second inequality has the following equivalent form

$$\begin{aligned}&\left|\frac{\left\langle\sum_{j=1}^{n}p_jf(A_j)g(A_j)x,x\right\rangle}{\left[\left\langle\sum_{j=1}^{n}p_jf(A_j)x,x\right\rangle\cdot\left\langle\sum_{j=1}^{n}p_jg(A_j)x,x\right\rangle\right]^{1/2}}\right.\\&\qquad\left.-\left[\left\langle\sum_{j=1}^{n}p_jf(A_j)x,x\right\rangle\cdot\left\langle\sum_{j=1}^{n}p_jg(A_j)x,x\right\rangle\right]^{1/2}\right|\\&\qquad\qquad\leq\left(\sqrt{\Gamma}-\sqrt{\gamma}\right)\left(\sqrt{\Delta}-\sqrt{\delta}\right)\end{aligned} \tag{5.21}$$

for each $x\in H$, with $\|x\|^2=1$.

We know, from [10] that if f,g are synchronous (asynchronous) functions on the interval $[m,M]$, then we have the inequality

$$\left\langle\sum_{j=1}^{n}p_jf(A_j)g(A_j)x,x\right\rangle\geq(\leq)\left\langle\sum_{j=1}^{n}p_jf(A_j)x,x\right\rangle\cdot\left\langle\sum_{j=1}^{n}p_jg(A_j)x,x\right\rangle \tag{5.22}$$

for each $x\in H$, with $\|x\|^2=1$, provided f,g are continuous on $[m,M]$ and A_j are selfadjoint operators with $Sp(A_j)\subseteq[m,M]$, $j\in\{1,\dots,n\}$.

Therefore, if f,g are synchronous then we have from (5.20) and from (5.21) the following results:

$$\begin{aligned}0&\leq\frac{\left\langle\sum_{j=1}^{n}p_jf(A_j)g(A_j)x,x\right\rangle}{\left\langle\sum_{j=1}^{n}p_jf(A_j)x,x\right\rangle\cdot\left\langle\sum_{j=1}^{n}p_jg(A_j)x,x\right\rangle}-1\\&\qquad\leq\frac{1}{4}\cdot\frac{(\Gamma-\gamma)(\Delta-\delta)}{\sqrt{\Gamma\gamma\Delta\delta}}\end{aligned} \tag{5.23}$$

and

$$0\leq\frac{\left\langle\sum_{j=1}^{n}p_jf(A_j)g(A_j)x,x\right\rangle}{\left[\left\langle\sum_{j=1}^{n}p_jf(A_j)x,x\right\rangle\cdot\left\langle\sum_{j=1}^{n}p_jg(A_j)x,x\right\rangle\right]^{1/2}}$$

$$-\left[\left\langle\sum_{j=1}^{n}p_jf(A_j)x,x\right\rangle\cdot\left\langle\sum_{j=1}^{n}p_jg(A_j)x,x\right\rangle\right]^{1/2}$$
$$\leq\left(\sqrt{\Gamma}-\sqrt{\gamma}\right)\left(\sqrt{\Delta}-\sqrt{\delta}\right)\quad(5.24)$$

for each $x\in H$, with $\|x\|^2=1$, respectively.

If f,g are asynchronous then

$$0\leq 1-\frac{\left\langle\sum_{j=1}^{n}p_jf(A_j)g(A_j)x,x\right\rangle}{\left\langle\sum_{j=1}^{n}p_jf(A_j)x,x\right\rangle\cdot\left\langle\sum_{j=1}^{n}p_jg(A_j)x,x\right\rangle}$$
$$\leq\frac{1}{4}\cdot\frac{(\Gamma-\gamma)(\Delta-\delta)}{\sqrt{\Gamma\gamma\Delta\delta}}\quad(5.25)$$

and

$$0\leq\left[\left\langle\sum_{j=1}^{n}p_jf(A_j)x,x\right\rangle\cdot\left\langle\sum_{j=1}^{n}p_jg(A_j)x,x\right\rangle\right]^{1/2}$$
$$-\frac{\left\langle\sum_{j=1}^{n}p_jf(A_j)g(A_j)x,x\right\rangle}{\left[\left\langle\sum_{j=1}^{n}p_jf(A_j)x,x\right\rangle\cdot\left\langle\sum_{j=1}^{n}p_jg(A_j)x,x\right\rangle\right]^{1/2}}$$
$$\leq\left(\sqrt{\Gamma}-\sqrt{\gamma}\right)\left(\sqrt{\Delta}-\sqrt{\delta}\right)\quad(5.26)$$

for each $x\in H$, with $\|x\|^2=1$, respectively.

The above inequalities (5.23) - (5.26) can be used to state various particular inequalities as in the previous examples, however the details are left to the interested reader.

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Research Group in Math. Inequalities & Appl.
School of Engineering & Science
Victoria University, PO Box 14428
Melbourne City, MC 8001, Australia
E–mail: sever.dragomir@vu.edu.au
Url: http://www.staff.vu.edu.au/rgmia/dragomir/