SARAJEVO JOURNAL OF MATHEMATICS
Vol.6 (18) (2010), 109–115

DUAL FEUERBACH THEOREM IN AN ISOTROPIC PLANE

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Abstract. The dual Feuerbach theorem for an allowable triangle in an isotropic plane is proved analytically by means of the so–called standard triangle. A number of statements about relationships between some concepts of the triangle and their dual concepts are also proved.

In an isotropic plane (see e.g. [4] and [5]) the *distance* between the two points $T_i = (x_i, y_i)$ $(i = 1, 2)$ is defined by $T_1T_2 = x_2 - x_1$ and two lines with the equations $y = k_i x + l_i$ $(i = 1, 2)$ form the *angle* $k_2 - k_1$. Two points T_1, T_2 with $x_1 = x_2$ are said to be *parallel*; we shall also say they lie on the same *isotropic* line. Two lines with $k_1 = k_2$ are parallel.

A triangle is said to be *allowable* if none of its sides is isotropic. Each allowable triangle ABC can be realized by a suitable choice of a coordinate system in the standard position, in which its circumscribed circle has the equation $y = x^2$, its vertices are the points

$$A = (a, a^2), \quad B = (b, b^2), \quad C = (c, c^2), \tag{1}$$

and its sides BC, CA, AB have the equations

$$y = -ax - bc, \quad y = -bx - ca, \quad y = -cx - ab, \tag{2}$$

where

$$a + b + c = 0 \tag{3}$$

is fulfilled.

We shall say then that ABC is the *standard triangle*. To prove the geometric facts for each allowable triangle it is sufficient to provide a proof for the standard triangle (see [3]).

With the labels

$$p = abc, \quad q = bc + ca + ab \tag{4}$$

a number of useful equalities are proved in [3], for example $a^2 = bc - q$.

2000 *Mathematics Subject Classification.* 51N25.

In [2], (page 129-130) the dual Feurbach theorem for each allowable triangle in an isotropic plane is stated without proof. In this paper we shall give a short analytical proof of this theorem by means of the method of the standard triangle which is formed in [3] and [1].

In an isotropic plane the principe of projective duality is valid.

In [1] it is proved that the midpoints A_m, B_m, C_m of sides of the triangle ABC and feet A_h, B_h, C_h of its altitudes lie on the same circle, the *Euler circle* of that triangle. The dual objects of the midpoints A_m, B_m, C_m of the sides BC, CA, AB are the isotropic bisectors of the angles A, B, C. The dual objects of the points A_h, B_h, C_h on the lines BC, CA, AB, successively parallel to the points A, B, C are lines through the points A, B, C parallel to the lines BC, CA, AB. The last three lines are the lines B_nC_n, C_nA_n, A_nB_n, where $A_nB_nC_n$ is the anticomplementary triangle of the triangle ABC. Therefore the dual of the previous statement is the following theorem. (Some statements about anticomplementary triangle can be found in [3])

Theorem 1. *Angle bisectors of each allowable triangle and lines through its vertices which are parallel to its opposite sides, touch one circle (Figure 1), which in the case of the standard triangle ABC has the equation*

$$\mathcal{K}'_e \quad \ldots \quad y = -\frac{1}{8}x^2. \tag{5}$$

Proof. Because of [3] the line B_nC_n has the equation $y = -ax + 2a^2$. From this equation and the equation (5) we get the equation $\frac{1}{8}x^2 - ax + 2a^2 = 0$ with the double solution $x = 4a$ for abscissa of the common points of the line B_nC_n and the circle (5), therefore they are touching. We get $y = -2a^2$ for the ordinate of associated point of contact. According to [3] the lines AB and AC have the equations $y = -cx - ab$ and $y = -bx - ac$, and if we add them we get the equation $2y = -(b + c)x - a(b + c)$ of the bisector of these lines. Since $b + c = -a$ the bisector of the angle A has the equation $y = \frac{a}{2}x + \frac{a^2}{2}$. From this equation and the equation (5) we get the equation $\frac{1}{8}x^2 + \frac{a}{2}x + \frac{a^2}{2} = 0$ with the double solution $x = -2a$, so we get touching again. The associated ordinate of point of contact is $y = -\frac{1}{2}a^2$. In a completely analogous manner one shows the assertion of theorem for other lines. □

The circle $\mathcal{K}'_e$ from Theorem 1 will be called *dual Euler circle* of the triangle under consideration.

Corollary 1. *The dual Euler circle of the standard triangle ABC has the equation* (5) *and it touches the bisectors of the angles A, B, C at the points*

$$\Big(-2a,\ -\frac{1}{2}a^2\Big),\quad \Big(-2b,\ -\frac{1}{2}b^2\Big),\quad \Big(-2c,\ -\frac{1}{2}c^2\Big). \tag{6}$$

It touches the lines parallel to the lines BC, CA, AB through the points A, B, C at the points

$$\big(4a,\ -2a^2\big),\quad \big(4b,\ -2b^2\big),\quad \big(4c,\ -2c^2\big). \tag{7}$$

The points of contact (6) are successively parallel to the points A_n, B_n, C_n, whose abscissas are, owing to [3], $-2a$, $-2b$, $-2c$.
The following theorem is dual to the Feuerbach theorem from [1].

Theorem 2. *The circumscribed circle and the dual Euler circle of an allowable triangle touch each other at one point. In the case of the standard triangle ABC, it is at the point $\Phi' = (0,0)$, and the common tangent $\mathcal{F}'$ of these two circles at the point Φ' has the equation $y = 0$ (Figure 1).* (In [2] the first statement of this theorem appears.)

Proof. It is obvious that the circles with the equations $y = x^2$ and (5) touch each other at the point $(0,0)$ and their tangent at this point has the equation $y = 0$. □

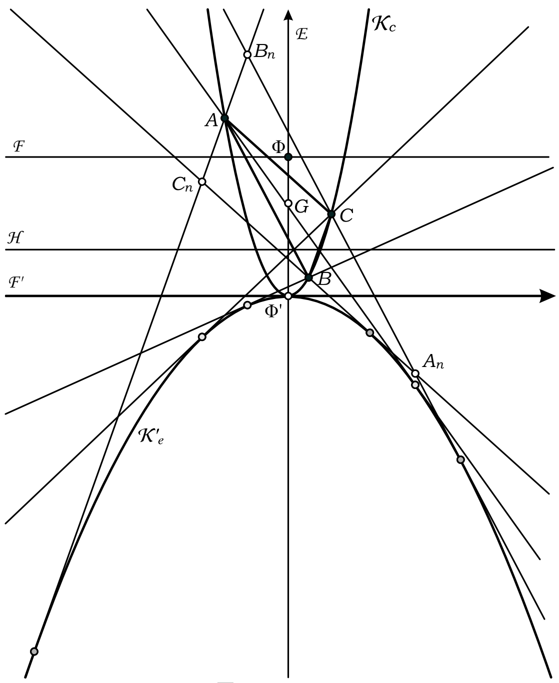

Figure 1.

The point Φ' and the line $\mathcal{F}'$ from Theorem 6 will be called the *dual Feuerbach point* and the *dual Feuerbach line* of the triangle ABC. Thus, with results from [1] we get:

Corollary 2. *The dual Feuerbach point and the dual Feuerbach line of an allowable triangle coincide with the Feurbach point and Feuerbach line of its anticomplementary and its tangential triangle. In the case of the standard triangle ABC this point and this line are the origin and abscissa–axis of coordinate system.*

Corollary 3. *The dual Feuerbach point of an allowable triangle lies on its Euler line, and the dual Feuerbach line of this triangle is parallel to its orthic line (Figure 1).*

Corollary 4. *In an allowable triangle the Feuerbach point and the dual Feuerbach point are parallel as well as the Feuerbach line and the dual Feuerbach line (Figure 1).*

Theorem 3. *The inscribed circle $\mathcal{K}_{ii}$ of the contact triangle $A_iB_iC_i$ of the standard triangle ABC has the equation*

$$\mathcal{K}_{ii} \quad \ldots \quad y = \frac{1}{16}x^2 - 2q. \tag{8}$$

Proof. According to [1] the line B_iC_i has the equation $y = \frac{a}{2}x - bc - q$. From this equation and the equation (8), setting to $bc - q = a^2$, the equation $\frac{1}{16}x^2 - \frac{a}{2}x + a^2 = 0$ with double solution $x = 4a$ follows. The associated ordinate is $a^2 - 2q = bc - 3q$. □

Corollary 5. *The points of contact of the circle from Theorem 3 with the lines B_iC_i, C_iA_i, A_iB_i are the points*

$$A_{ii} = (4a, bc - 3q), \qquad B_{ii} = (4b, ca - 3q), \qquad C_{ii} = (4c, ab - 3q). \tag{9}$$

The points of contact (9) are parallel successively to the points of contact (7) from Corollary 1.

According to [1] the inscribed circle $\mathcal{K}_i$ and the polar circle $\mathcal{K}_p$ of the standard triangle have the equations

$$\mathcal{K}_i \quad \ldots \quad y = \frac{1}{4}x^2 - q, \tag{10}$$

$$\mathcal{K}_p \quad \ldots \quad y = -\frac{1}{2}x^2 - \frac{q}{2}. \tag{11}$$

In [3] it is proved that by the substitution of coordinates $x \longrightarrow -\frac{1}{2}x$, $y \longrightarrow -\frac{1}{2}y - q$ into the equation of some curve we get the equation of its anticomplementary curve in the triangle ABC. It is easy to see that by these substitution of coordinates the equation (11) transforms to the equation (10), equation (10) to the equation (5), and finally the equation (5) to the equation (8), i.e. we get:

Theorem 4. *The inscribed circle is anticomplementary to polar circle, dual Euler circle is anticomplementary to the inscribed circle, and the inscribed circle of the contact triangle is anticomplementary to the dual Euler circle of the considered triangle (Figure 2).*

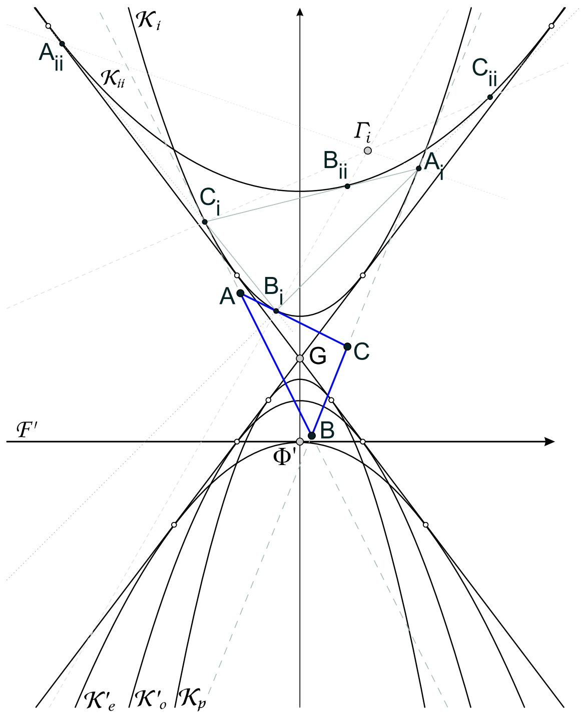

Figure 2.

Anticomplementarity with regard to the triangle ABC is in fact the application of homothecy $(G, -2)$, where G is centroid of this triangle. Therefore, from Theorem 4 we get that the four mentioned circles have common tangents at the point G.

Theorem 5. *In the case of the standard triangle ABC the lines with the equation*

$$y = kx + 2k^2, \tag{12}$$

where

$$k^2 = -\frac{q}{3}, \tag{13}$$

touch all four circles from Theorem 4.

Proof. It is sufficient to prove it for one of four circles. For example from the equation (11) and (12) we get, because of (13), the equation $\frac{1}{2}x^2+kx+\frac{1}{2}k^2=0$ with the double solution $x=-k$. □

From Theorem 4 it follows.

Corollary 6. *Homothecy with the center at the centroid of an allowable triangle and with the coefficient 4 maps its polar and its inscribed circle into its dual Euler circle and into the inscribed circle of its contact triangle respectively.*

There is one more interesting circle which touches two lines from Theorem 5.

Theorem 6. *The circle with the equation*

$$\mathcal{K}_o' \quad \ldots \quad y=-\frac{1}{4}x^2-\frac{q}{3} \tag{14}$$

touches the lines (12) *under the condition* (13)*, and the orthic line of the triangle* ABC *too, and it is symmetric to the inscribed circle of the triangle* ABC *with respect to its centroid* G*.*

Proof. From (14), with $y=-\frac{q}{3}$, the equation $-\frac{1}{4}x^2=0$ with double solution $x=0$ follows, so the orthic line touches the circle (14). Symmetry with respect to the centroid $G=\left(0,-\frac{2}{3}q\right)$ is the transformation $x\to -x,\quad y\to -y-\frac{4}{3}q$, and by this transformation the equation (10) transforms into the equation (14). □

We can prove thefollowing theorem by means of results from Corollary 5.

Theorem 7. *The contact triangle* $A_iB_iC_i$ *of the standard triangle* ABC *has Gergonne point*

$$\Gamma_i=\left(\frac{6p}{q},-\frac{7}{3}q\right). \tag{15}$$

Proof. According to [1] we get $A_i=(-2a,bc-2q)$. The points A_i, A_{ii} from (9) and the point Γ_i from (15) lie on a common line with the equation

$$y=-\frac{q}{6a}x+bc-\frac{7}{3}q.$$

The point Γ_i also lies on the analogous lines B_iB_{ii} and C_iC_{ii}. □

Acknowledgment. The authors are grateful to the referee for his/her very useful suggestions.

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(Received: January 20, 2009)
(Revised: September 14, 2009)

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