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CURVATURE OF THE FOCAL CONIC IN THE ISOTROPIC PLANE

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Abstract. It is shown in [1] that every focal conic $\mathcal{C}$ in the isotropic plane can be represented by the equation $y^2 = \epsilon x^2 + x$, $\epsilon \in \{-1, 0, 1\}$ and a parametrization. This paper gives the equation of the circle of curvature at the point T of the focal conic $\mathcal{C}$. The radius of curvature ρ at the point T of the focal conic $\mathcal{C}$ is given as well as its relation to the span δ from the center of $\mathcal{C}$ to the tangent $\mathcal{T}$ at the point T and to the length of the half diameter of $\mathcal{C}$ on the diameter parallel to the tangent $\mathcal{T}$.

By choosing a suitable affine coordinate system (see [1]) every conic $\mathcal{C}$ with foci in the isotropic plane can be represented by the equation

$$y^2 = \epsilon x^2 + x \tag{1}$$

where $\epsilon = -1$, $\epsilon = 0$ or $\epsilon = 1$ depending on whether $\mathcal{C}$ is an ellipse, a parabola or a hyperbola. This conic has the x-axis as its axis and one focus has the form $O = (0, 0)$. In the case of an ellipse or a hyperbola the second focus is of the form $O = (-\epsilon, 0)$. Such a conic can be parametrized by the equations

$$x = \frac{1}{t^2 - \epsilon}, \quad y = \frac{t}{t^2 - \epsilon}, \tag{2}$$

i.e. its point T with the parameter t, denoted by $T = (t)$ and a tangent to the conic $\mathcal{C}$ at the point T are of the form

$$T = \Big(\frac{1}{t^2 - \epsilon}, \frac{t}{t^2 - \epsilon}\Big),$$

$$y = \frac{t^2 + \epsilon}{2t} x + \frac{1}{2t}, \tag{3}$$

respectively. All the notions related to the geometry of the isotropic plane can be found for example in Sachs [3] and Strubecker [4].

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Computation yields the equation of the straight line spanned by the two points $T_1 = (t_1)$ and $T_2 = (t_2)$ of (2):

$$y = \frac{t_1 t_2 + \epsilon}{t_1 + t_2} x + \frac{1}{t_1 + t_2}. \tag{4}$$

Under $t_1 = t_2 = t$ (4) turns into (3). The first theorem deals with the condition for the points $T_i = (t_i)\,(i = 1, 2, 3, 4)$ of the conic $\mathcal{C}$, given in (2), to be concyclic. So we have:

Theorem 1. *The points* $T_1 = (t_1)$, $T_2 = (t_2)$, $T_3 = (t_3)$, $T_4 = (t_4)$ *of the conic* $\mathcal{C}$*, given with the parametric equations* (2)*, are concyclic if and only if the equality*

$$\epsilon s_1 + s_3 = 0 \tag{5}$$

is valid, where

$$s_1 = t_1 + t_2 + t_3 + t_4, \quad s_3 = t_1 t_2 t_3 + t_1 t_2 t_4 + t_1 t_3 t_4 + t_2 t_3 t_4. \tag{6}$$

Proof. Inserting (2) in the equation $2\rho y = x^2 + ux + v$ of any circle, after multiplying the equation by $(t^2 - \epsilon)^2$ we get the equation

$$-2\rho t(t^2 - \epsilon) + 1 + u(t^2 - \epsilon) + v(t^2 - \epsilon)^2 = 0,$$

that is,

$$vt^4 - 2\rho t^3 + t^2(u - 2v\epsilon) + 2\rho\epsilon t + 1 - \epsilon u + v\epsilon^2 = 0.$$

If t_1, t_2, t_3, t_4 are the solutions of this equation, referring to Viete's formulae we have that

$$s_1 = \frac{2\rho}{v}, \quad s_3 = -\frac{2\rho\epsilon}{v},$$

out of which the equality (5) follows. □

What is a geometrical meaning of the equality (5)?

First, let us introduce the following definition: two lines are *antiparallel* with respect to the third line if they form two opposite angles with this line.

Theorem 2. *The points* T_1, T_2, T_3, T_4 *of the focal conic* $\mathcal{C}$ *are concyclic if and only if the lines* T_1T_2 *and* T_3T_4 *are antiparallel with respect to the axis of the conic* $\mathcal{C}$.

Proof. Let $T_i = (t_i)$ $(i = 1, 2, 3, 4)$ be the points of the conic $\mathcal{C}$. According to (4) the lines T_1T_2 and T_3T_4 have the slopes $\frac{t_1t_2+\epsilon}{t_1+t_2}$, $\frac{t_3t_4+\epsilon}{t_3+t_4}$, respectively and they are antiparallel with respect to the axis of the conic $\mathcal{C}$ (with the slope equal to zero) iff

$$\frac{t_1 t_2 + \epsilon}{t_1 + t_2} + \frac{t_3 t_4 + \epsilon}{t_3 + t_4} = 0.$$

Being written in the form of $t_1t_2(t_3+t_4)+t_3t_4(t_1+t_2)+\epsilon(t_1+t_2+t_3+t_4) = 0$, this is actually the equality (5). □

Letting $T_1 = T_2 = T_3 = T$ and $T_4 = T'$, from Theorem 2, we get:

Corollary 1. *If a circle of curvature of the focal conic $\mathcal{C}$ at its point T intersects this conic again in a point T', then the line TT' is antiparallel to the tangent $\mathcal{T}$ of the conic at the point T with respect to the axis of the conic $\mathcal{C}$.*

Under the assumptions that $t_1 = t_2 = t_3 = t$ and $t_4 = t'$, (6) gives $s_1 = 3t + t'$ and $s_3 = t^3 + 3t^2t'$. In this case the equality (5) yields

$$t' = -t\frac{t^2 + 3\epsilon}{3t^2 + \epsilon}. \tag{7}$$

As a consequence, we have

Corollary 2. *The circle of curvature at the point $T = (t)$ of the conic $\mathcal{C}$ given with parametric equations (2) intersects this conic again in the point $T' = (t')$ given by the parameter t' (7).*

Theorem 3. *The circle of curvature at the point $T = (t)$ of the conic $\mathcal{C}$ given with parametric equations (2) has the equation*

$$(t^2 - \epsilon)^3 x^2 - 2(3t^4 + \epsilon^2)x + 8t^3 y - 3t^2 - \epsilon = 0. \tag{8}$$

Proof. The line $\mathcal{T}'$ with the equation

$$y = -\frac{t^2 + \epsilon}{2t}x + \frac{3t^2 + \epsilon}{2t(t^2 - \epsilon)} \tag{9}$$

passes through the point given in (3) because of

$$-\frac{t^2 + \epsilon}{2t} \cdot \frac{1}{t^2 - \epsilon} + \frac{3t^2 + \epsilon}{2t(t^2 - \epsilon)} = \frac{t}{t^2 - \epsilon}.$$

The lines $\mathcal{T}$ and $\mathcal{T}'$ with the equations (3) and (9) are antiparallel with respect to the axis of the conic $\mathcal{C}$. According to Corollary 1, the line $\mathcal{T}'$ is the joint line TT' where

$$\mathcal{T} = (t^2 + \epsilon)x - 2ty + 1, \quad \mathcal{T}' = (t^2 + \epsilon)x + 2ty - \frac{3t^2 + \epsilon}{t^2 - \epsilon},$$

$\mathcal{T} = 0$ and $\mathcal{T}' = 0$ are the equations of the lines $\mathcal{T}$ and $\mathcal{T}'$. Under $\mathcal{C} = \epsilon x^2 - y^2 + x$ the equation $\mathcal{T}\mathcal{T}' + k\mathcal{C} = 0$ for a given $k \in \mathbb{R}$ associates a conic that passes through the point T' and at the point T osculates the conic $\mathcal{C}$. For $k = -4t^2$ we gain the osculating circle of $\mathcal{C}$ at T. So, the circle of curvature from the theorem has the equation

$$[(t^2 + \epsilon)x - 2ty + 1]\left[(t^2 + \epsilon)x + 2ty - \frac{3t^2 + \epsilon}{t^2 - \epsilon}\right] - 4t^2(\epsilon x^2 - y^2 + x) = 0.$$

Rearrangement of the upper equation yields (8). $\square$

It is known that any circle with the radius ρ has the equation of the form $2\rho y = x^2 + ux + v$. Based on this, using (8), we have:

Corollary 3. *The radius of curvature ρ of the conic $\mathcal{C}$ with the parametric equations* (2) *at the point $T = (t)$ is given by*

$$\rho = -\frac{4t^3}{(t^2-\epsilon)^3}. \tag{10}$$

The span from the point (x_0, y_0) to the line with the equation $y = kx+l$ is $y_0 - kx_0 - l$. The conic $\mathcal{C}$ given with the equation (1) where $\epsilon \in \{-1, 1\}$ has center at $(-\frac{\epsilon}{2}, 0)$, so the span from the center to the line $\mathcal{T}$ with equation given in (3) is equal to

$$\delta = -\frac{t^2+\epsilon}{2t} \cdot \Big(-\frac{\epsilon}{2}\Big) - \frac{1}{2t} = \frac{t^2-\epsilon}{4\epsilon t}. \tag{11}$$

We continue our work by investigating relations between the above mentioned span δ, the radius of curvature ρ, the half axes and the half diameter of the conic $\mathcal{C}$.

Let us denote by α and β the half axes of the conic $\mathcal{C}$ with the equation (1). Obviously, $\alpha = -\frac{\epsilon}{2}$, i.e. $\alpha^2 = \frac{1}{4}$. An isotropic line passing through the center of the conic has the equation $x = -\frac{\epsilon}{2}$. For this abscissae, using (1), an ordinate y is of the form

$$y^2 = -\frac{\epsilon}{4}.$$

Hence, it is formally $\beta^2 = -\frac{\epsilon}{4}$, having geometrical meaning in the case of an ellipse, and in the case of a hyperbola it is taken as a formal equality. In both cases we have

$$\alpha^2\beta^2 = -\frac{\epsilon}{16}.$$

On the other hand, (10) and (11) give

$$\rho\delta^3 = -\frac{4t^3}{(t^2-\epsilon)^3} \cdot \frac{(t^2-\epsilon)^3}{4^3\epsilon t^3} = -\frac{\epsilon}{16} = \alpha^2\beta^2,$$

and we get the following:

Theorem 4. *If α, β are the half axes of an ellipse or of a focal hyperbola, then the radius of curvature ρ of the conic at its point T has the form*

$$\rho = \frac{\alpha^2\beta^2}{\delta^3},$$

where δ is the span from the center of the conic to its tangent at the point$T$.

The line with the equation

$$y = \frac{t^2+\epsilon}{2t}\Big(x + \frac{\epsilon}{2}\Big)$$

is parallel to the line in (3) and is incident to the center $(-\frac{\epsilon}{2}, 0)$ of the conic (1). Inserting this expression for y in (1) for the abscissae of the points of intersection we get the equation

$$\frac{(t^2+\epsilon)^2}{4t^2}\Big(x^2+\epsilon x+\frac{1}{4}\Big)-\epsilon x^2-x=0$$

which, after the multiplication by $4t^2$ and after being rearranged, we get

$$x^2(t^2-\epsilon)^2+x\cdot\epsilon(t^2-\epsilon)^2+\frac{1}{4}(t^2+\epsilon)^2=0.$$

For the solutions of the latter equation we have that

$$x_1+x_2=-\epsilon, \quad x_1x_2=\frac{(t^2+\epsilon)^2}{4(t^2-\epsilon)^2}, \quad \text{and}$$

$$(x_1-x_2)^2=(x_1+x_2)^2-4x_1x_2=1-\frac{(t^2+\epsilon)^2}{(t^2-\epsilon)^2}=-\frac{4t^2\epsilon}{(t^2-\epsilon)^2}.$$

Denoting by τ the length of the half diameter on the given diameter, it follows that $4\tau^2=(x_1-x_2)^2$ i.e.

$$\tau^2=-\frac{\epsilon t^2}{(t^2-\epsilon)^2}. \tag{12}$$

Thus we have:

Theorem 5. *The half diameter τ of the ellipse or the focal hyperbola $\mathcal{C}$ given with* (2) *on its diameter parallel to the tangent at its point $T=(t)$ is given by the formula* (12).

Furthermore, (10) and (11) immediately give

$$\rho\delta=-\frac{\epsilon t^2}{(t^2-\epsilon)^2},$$

i.e. $\rho\delta=\tau^2$, because of (12). Thus we have the following theorem:

Theorem 6. *The radius of curvature ρ of an ellipse or of a focal hyperbola $\mathcal{C}$ at its point T equals*

$$\rho=\frac{\tau^2}{\delta},$$

where δ is the span from the center of the conic $\mathcal{C}$ to its tangent $\mathcal{T}$, and τ is the half diameter of the conic on the diameter parallel to the tangent $\mathcal{T}$.

In [1] it is shown that the point of intersection of the tangents to the conic (2) at its points $T_1=(t_1)$ and $T_2=(t_2)$ is of the form

$$T_{12}=\Big(\frac{1}{t_1t_2-\epsilon}, \frac{t_1+t_2}{2(t_1t_2-\epsilon)}\Big). \tag{13}$$

With (3), $t = t_1$, and (13) for the distances $\tau_1 = T_1T_{12}$ and $\tau_2 = T_2T_{12}$ we get

$$\tau_1 = T_1T_{12} = \frac{1}{t_1t_2 - \epsilon} - \frac{1}{{t_1}^2 - \epsilon} = \frac{t_1(t_1 - t_2)}{({t_1}^2 - \epsilon)(t_1t_2 - \epsilon)},$$
$$\tau_2 = T_2T_{12} = \frac{t_2(t_2 - t_1)}{({t_2}^2 - \epsilon)(t_1t_2 - \epsilon)},$$

respectively, whence

$$\frac{\tau_1}{\tau_2} = -\frac{t_1({t_2}^2 - \epsilon)}{t_2({t_1}^2 - \epsilon)}. \tag{14}$$

According to Corollary 3, for the radii of curvature ρ_1 and ρ_2 of the conic $\mathcal{C}$ at the points T_1 and T_2 we get the equality

$$\frac{\rho_1}{\rho_2} = \frac{{t_1}^3({t_2}^2 - \epsilon)^3}{{t_2}^3({t_1}^2 - \epsilon)^3}. \tag{15}$$

Finally, from (14) and (15) we get

$$\frac{\rho_1}{\rho_2} = -\frac{{\tau_1}^3}{{\tau_2}^3}. \tag{16}$$

This is summarized in

Theorem 7. *The radii of curvature of the focal conic at its two points are related as cubes of the segments of its tangents at these points measured from their points of contact to their point of intersection.*

In the case of a parabola where $\epsilon = 0$, the equation (8) can be transformed into

$$t^4x^2 - 6t^2x + 8ty - 3 = 0. \tag{17}$$

Thus,

Corollary 4. *A parabola with the equation* $y^2 = x$ *has at its point* $T = (t) = (\frac{1}{t^2}, \frac{1}{t})$ *the circle of curvature given in* (17) *and the radius of curvature* $\rho = -\frac{4}{t^3}$.

The diameter of a parabola at the point T has the equation $y = \frac{1}{t}$. This and (17) yields the equation $t^4x^2 - 6t^2x + 5 = 0$. The latter equation has two solutions in x, the one $x = \frac{1}{t^2}$ corresponds to the point T and the second $x = \frac{5}{t^2}$ corresponds to the point $T' = (\frac{5}{t^2}, \frac{1}{t})$ that is the second point of intersection of this diameter of the parabola with the circle given in (17). Now

$$OT = \frac{1}{t^2}, \quad TT' = \frac{5}{t^2} - \frac{1}{t^2} = \frac{4}{t^2} = 4 \cdot OT$$

holds, as well as

Theorem 8. *If the circle of curvature at the point T of the parabola with focus $O = (0, 0)$ intersects its diameter at the point T residually at the point T', then the equality $TT' = 4 \cdot OT$ is valid.*

The same statement is valid in Euclidean geometry as well (see e. g. [2], p. 561).

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