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B.-Y. CHEN INEQUALITIES FOR SLANT SUBMANIFOLDS IN KENMOTSU SPACE FORMS II

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ABSTRACT. In this article, we investigate sharp inequalities involving Chen invariants for a slant submanifold M of a Kenmotsu space form $\widetilde{M}(c)$, tangent to the structure vector field of the ambient space.

1. PRELIMINARIES

Let $(\widetilde{M}, g)$ be an odd-dimensional Riemannnian manifold. Then $\widetilde{M}$ is said to be an almost contact metric manifold if it admits an endomorphism φ of its tangent bundle $T\widetilde{M}$, a vector field ξ (structure vector field) and a 1-form η, which satisfy:

$$\varphi^2 = -I + \eta \otimes \xi, \eta(\xi) = 1, \varphi\xi = 0, \eta \circ \varphi = 0,$$
$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \eta(X) = g(X, \xi),$$

for any vector fields X, Y on $\widetilde{M}$.

An almost contact metric manifold is called a Kenmotsu manifold if

$$(\widetilde{\nabla}_X \varphi)(Y) = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \quad \widetilde{\nabla}_X \xi = -\varphi^2 X = X - \eta(X)\xi,$$

where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to g.

A plane section π in $T_p\widetilde{M}$ is called a φ -section if it is spanned by X and φX, where X is a unit tangent vector field orthogonal to ξ. The sectional curvature $\widetilde{K}(\pi)$ of a φ-section π is called φ-sectional curvature. A Kenmotsu manifold with constant φ-sectional curvature c is called a Kenmotsu space form and it is denoted by $\widetilde{M}(c)$. Then its curvature tensor $\widetilde{R}$ is expressed by

$$4\widetilde{R}(X, Y)Z = (c-3)[g(Y, Z)X - g(X, Z)Y] + (c+1)[g(\varphi Y, Z)\varphi X$$
$$-g(\varphi X, Z)\varphi Y - 2g(\varphi X, Y)\varphi Z + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi$$
$$+\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X].$$

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Let M be an n-dimensional Riemannian manifold with induced metric g isometrically immersed in $\widetilde{M}$. We denote by TM and $T^{\perp}M$ the tangent and the normal bundles of M respectively.

For any $X \in TM$, we write $\varphi X = PX + FX$, where PX (respectively FX) denotes the tangential (respectively normal) component of φX.

The equation of Gauss is given by

$$\widetilde{R}(X,Y,Z,W) = R(X,Y,Z,W) - g(h(X,Z),h(Y,W)) + g(h(X,W),h(Y,Z)),$$

for any vectors X, Y, Z, W tangent to M.

We denote by

$$\tau(p) = \sum_{1 \le i < j \le n} K(e_i \wedge e_j),$$

where $p \in M$ and $\{e_1, \ldots, e_n\} \subset T_pM$ is an orthonormal basis, the scalar curvature of M at $p \in M$.

The mean curvature vector H is defined by $H = \frac{1}{\dim M}$ trace h.

From now on, let n (respectively $2m+1$) be the dimension of M (respectively $\widetilde{M}$). We denote by

$$h_{ij}^r = g(h(e_i, e_j), e_r), i, j \in \{1, \ldots, n\}, r \in \{n+1, \ldots, 2m+1\};$$

then we have

$$\|H\|^2 = \frac{1}{n^2} \sum_{r=n+1}^{2m+1} (\sum_{i=1}^{n} h_{ii}^r)^2, \quad \|h\|^2 = \sum_{r=n+1}^{2m+1} \sum_{i,j=1}^{n} (h_{ij}^r)^2.$$

Also we put

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j),$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis of T_pM and $\{e_{n+1}, \ldots e_{2m+1}\}$ is an orthonormal basis of $T_p^{\perp}M$.

A Chen invariant is defined by

$$\delta'_M(p) = \tau(p) - \inf \big\{ K(\pi) | \pi \subset T_pM \text{ a plane section invariant by } P \big\}.$$

If the structure vector field ξ is tangent to M, we denote by D the orthogonal distribution to ξ in TM and we can consider the orthogonal direct decomposition $TM = D \oplus \langle \xi \rangle$.

Let $\pi \subset D_p$ a plane section at $p \in M$, orthogonal to ξ_p. Then, $\Phi^2(\pi) = g^2(Pe_1, e_2)$ is a real number which is independent of the choice of the orthonormal basis $\{e_1, e_2\}$ of π.

Let L be a subspace of T_pM of dimension $r \geq 2$ and $\{e_1, \ldots, e_r\}$ an orthonormal basis of L.The scalar curvature $\tau(L)$ of the r-plane section L

is given by:

$$\tau(L) = \sum_{1\leq\alpha<\beta\leq r} K(e_\alpha \wedge e_\beta)$$

and we denote by

$$\Psi(L) = \sum_{1\leq i<j<\leq r} g^2(Pe_i, e_j).$$

For an integer $k \geq 0$, we denote by $S(n,k)$ the finite set consisting of k-tuples $(n_1,\ldots,n_k)$ of integers ≥ 2 satisfying $n_1 < n, n_1+\cdots+n_k \leq n$. Denote by $S(n)$ the set of k-tuples with $k \geq 0$ for a fixed n.

For each k-tuples $(n_1,\ldots,n_k) \in S(n)$, Chen introduced a Riemannian invariant defined by:

$$\delta(n_1,\ldots,n_k)(p) = \tau(p) - S(n_1,\ldots,n_k)(p),$$

where $S(n_1,\ldots,n_k)(p) = \inf\{\tau(L_1)+\cdots+\tau(L_k)\}$ and at each point $p \in M$, $L_1,\ldots,L_k$ run over all k mutually orthogonal subspaces of T_pM such that $\dim L_j = n_j, j = 1,\ldots,k$.

We will consider the Chen invariant

$$\delta'(n_1,\ldots,n_k)(p) = \tau(p) - \inf\{\tau(L_1)+\cdots+\tau(L_k)\},$$

where $L_1,\ldots,L_k$ run over all k mutually orthogonal subspaces of T_pM, invariant by P, such that $\dim L_j = n_j, j = 1,\ldots,k$.

For each $(n_1,\ldots,n_k) \in S(n)$, let:

$$d(n_1,\ldots,n_k) = \frac{n^2\left(n+k-1-\sum_{j=1}^k n_j\right)}{2\left(n+k-\sum_{j=1}^k n_j\right)},$$

$$b(n_1,\ldots,n_k) = \frac{1}{2}\left[n(n-1) - \sum_{j=1}^k n_j(n_j-1)\right].$$

According to Lotta's definition (see [10]), a submanifold M immersed into an almost contact metric manifold $\widetilde{M}$ is called slant if the angle $\theta(X)$ between φX and T_pM is a constant θ, which is independent of the choice of $p \in M$ and $X \in T_pM - \langle\xi_p\rangle$. The angle θ of a slant immersion is called the slant angle of the immersion.

Invariant and anti-invariant immersions are slant immersions with slant angle θ equal to 0 and $\frac{\pi}{2}$, respectively. A slant immersions which is neither invariant nor anti-invariant is called a proper slant immersion.

2. Inequalities

We recall the following lemma due to Chen.

Lemma 1. *Let $a_1, \dots a_n, b$ be $n+1$ $(n \geq 2)$ real numbers such that:*

$$\left(\sum_{i=1}^{n} a_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} a_i^2 + b\right).$$

Then, $2a_1a_2 \geq b$, with the equality holding if and only if $a_1 + a_2 = a_3 = \dots = a_n$.

Using the above lemma, we proved a Chen first inequality (see [7]).

Theorem 2. *Let $\widetilde{M}(c)$ be a $(2m+1)$-dimensional Kenmotsu space form and M an $(n = 2k+1)$-dimensional non anti-invariant θ-slant submanifold, tangent to ξ. Then, for any point $p \in M$ and any plane section $\pi \subset D_p$, we have:*

$$\tau - K(\pi) \leq \frac{n-2}{2}\left\{\frac{n^2}{n-1}\|H\|^2 + \frac{(c-3)(n+1)}{4}\right\} + \frac{(c+1)(n-1)}{8}(3\cos^2\theta - 2) - 3\frac{c+1}{4}\Phi^2(\pi).$$

The equality case of the inequality holds at $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \dots, e_n\}$of T_pM and an orthonormal basis $\{e_{n+1}, \dots, e_{2m+1}\}$ of $T_p^{\perp}M$ such that $e_n = \xi$, π is spanned by e_1, e_2 and the shape operators of M in $\widetilde{M}(c)$ at p take the following forms:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 0 & b & 0 & \dots & 0 \\ 0 & 0 & \mu & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mu \end{pmatrix},\ a + b = \mu,$$

$$A_r = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \dots & 0 \\ h_{12}^r & -h_{11}^r & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},\ r \in \{n+2, \dots, 2m+1\}.$$

From Theorem 2 we derive the following.

Corollary 3. *Let $\widetilde{M}(c)$ be a $(2m+1)$-dimensional Kenmotsu space form and M an $(n = 2k+1)$-dimensional invariant submanifold, tangent to ξ. Then, for any point $p \in M$ and any plane section $\pi \subset D_p$, we have:*

$$\tau - K(\pi) \leq \frac{(c-3)(n-2)(n+1)}{8} + \frac{c+1}{4}[\frac{n-1}{2} - 3\Phi^2(\pi)].$$

Theorem 4. *Let $\widetilde{M}(c)$ be a $(2m+1)$-dimensional Kenmotsu space form and M an $(n = 2k+1)$-dimensional non anti-invariant θ-slant submanifold, tangent to ξ. Then:*

$$\delta'_M \leq \frac{n-2}{2}\left\{\frac{n^2}{n-1}\|H\|^2 + \frac{(c-3)(n+1)}{4}\right\} + \frac{c+1}{8}[3(n-3)\cos^2\theta - 2(n-1)].$$

The equality case of the inequality holds at $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \dots, e_n\}$of T_pM and an orthonormal basis $\{e_{n+1}, \dots, e_{2m+1}\}$ of $T_p^{\perp}M$ such that the shape operators of M in $\widetilde{M}(c)$ at p have the following forms:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 0 & b & 0 & \dots & 0 \\ 0 & 0 & \mu & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mu \end{pmatrix},\ a+b=\mu,$$

$$A_r = \begin{pmatrix} h^r_{11} & h^r_{12} & 0 & \dots & 0 \\ h^r_{12} & -h^r_{11} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix},\ r \in \{n+2, \dots, 2m+1\}.$$

Proof. The proof of Theorem 4 is similar with the proof of Theorem 2 considering $\pi = Sp\{e_1, e_2\}$, with $e_2 = \frac{1}{\cos\theta}Pe_1$, because π is invariant by P and so $\Phi^2(\pi) = \cos^2\theta$. □

Lemma 5. *Let M be an $(n = 2k+1)$-dimensional submanifold, tangent to ξ of a $(2m+1)$-dimensional Kenmotsu space form $\widetilde{M}(c)$. Let $n_1, \dots, n_k$ be integers ≥ 2 satisfying $n_1 < n, n_1 + \dots + n_k \leq n$. For $p \in M$, let $L_j \subset T_pM$ be subspaces of T_pM, orthogonal to ξ such that $\dim L_j = n_j, \forall j \in \{1, \dots, k\}$. Then, we have:*

$$\tau(p) - \sum_{j=1}^{k}\tau(L_j) \leq d(n_1, \dots, n_k)\|H\|^2 + b(n_1, \dots, n_k)\frac{c-3}{4} + \frac{c+1}{8}\left[3\|P\|^2 - 2n + 2 - \sum_{j=1}^{k}6\Psi(L_j)\right].$$

Proof. Let $p \in M$ and $\{e_1, \dots, e_n = \xi\}$ be an orthonormal basis of T_pM. From the Gauss equation we get

$$2\tau = n^2 \|H\|^2 - \|h\|^2 + \frac{c-3}{4} n(n-1) + \frac{c+1}{4}\left[3\|P\|^2 - 2(n-1)\right].$$

Denoting by

$$\eta = 2\tau - 2d(n_1, \dots, n_k)\|H\|^2 - \frac{c-3}{4} n(n-1) - \frac{c+1}{4}\left[3\|P\|^2 - 2(n-1)\right],$$

it follows that

$$n^2 \|H\|^2 = \left(\eta + \|h\|^2\right)\gamma, \tag{1}$$

where $\gamma = n + k - \sum_{j=1}^{k} n_j$.

Let e_{n+1} be a unit normal vector at p parallel to $H(p)$ and $\{e_{n+1}, \dots, e_{2m+1}\}$ an orthonormal basis of $T_p^\perp M$.

We denote by $a_i = h_{ii}^{n+1} = g(h(e_i, e_i), e_{n+1})$.

The relation (1) becomes

$$\left(\sum_{i=1}^{n} a_i\right)^2 = \gamma\left[\eta + \sum_{i \neq j}(h_{ij}^{n+1})^2 + \sum_{i=1}^{n}(h_{ii}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^{r})^2\right]. \tag{2}$$

Let $L_1, \dots, L_k$ be k mutually orthogonal subspaces of T_pM, $\dim L_j = n_j$, defined by:

$L_1 = Sp\{e_1, \dots, e_{n_1}\}$,
$L_2 = Sp\{e_{n_1+1}, \dots, e_{n_1+n_2}\}$,
$\vdots$
$L_k = Sp\left\{e_{n_1+\dots+n_{k-1}+1}, \dots, e_{n_1+\dots+n_k}\right\}$.

We denote by $D_j, j = 1, \dots, k$ the sets:
$D_1 = \{1, \dots, n_1\}$,
$D_2 = \{n_1 + 1, \dots, n_1 + n_2\}$,
$\vdots$
$D_k = \{n_1 + \dots + n_{k-1} + 1, \dots, n_1 + \dots + n_k\}$.

Also we put:
$b_1 = a_1$,
$b_2 = a_2 + \dots + a_{n_1}$,
$b_3 = a_{n_1+1} + \dots + a_{n_1+n_2}$,
$\vdots$
$b_{k+1} = a_{n_1+\dots+n_{k-1}+1} + \dots + a_{n_1+\dots+n_k}$,
$b_{k+2} = a_{n_1+\dots+n_k+1}$,
$\vdots$
$b_{\gamma+1} = a_n$.

Then the relation (2) is equivalent to

$$\left(\sum_{i=1}^{\gamma+1} b_i\right)^2 = \gamma\Bigg[\eta + \sum_{i=1}^{\gamma+1} b_i^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^r)^2$$

$$- 2\sum_{2\leq\alpha_1<\beta_1\leq n_1} a_{\alpha_1}a_{\beta_1} - 2\sum_{\substack{\alpha_2<\beta_2\\ \alpha_2,\beta_2\in D_2}} a_{\alpha_2}a_{\beta_2} - \cdots - 2\sum_{\substack{\alpha_k<\beta_k\\ \alpha_k,\beta_k\in D_k}} a_{\alpha_k}a_{\beta_k}\Bigg].$$

Applying the algebraic lemma we have:

$$2b_1b_2 \geq \eta + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^r)^2$$

$$- 2\Bigg(\sum_{2\leq\alpha_1<\beta_1\leq n_1} a_{\alpha_1}a_{\beta_1} + \sum_{\substack{\alpha_2<\beta_2\\ \alpha_2,\beta_2\in D_2}} a_{\alpha_2}a_{\beta_2} + \cdots + \sum_{\substack{\alpha_k<\beta_k\\ \alpha_k,\beta_k\in D_k}} a_{\alpha_k}a_{\beta_k}\Bigg),$$

which is equivalent to:

$$\sum_{\alpha_1<\beta_1} a_{\alpha_1}a_{\beta_1} + \cdots + \sum_{\alpha_k<\beta_k} a_{\alpha_k}a_{\beta_k} \geq \frac{1}{2}\Bigg[\eta + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^r)^2\Bigg],$$

with $\alpha_i, \beta_i \in D_i, \forall i = 1, \ldots, k$.

From the Gauss equation we obtain:

$$\tau(L_j) = \frac{n_j(n_j-1)(c-3)}{8} + \frac{3(c+1)}{4}\Psi(L_j) + \sum_{r=n+1}^{2m+1}\sum_{\alpha_j<\beta_j}\left[h_{\alpha_j\alpha_j}^r h_{\beta_j\beta_j}^r - \left(h_{\alpha_j\beta_j}^r\right)^2\right].$$

It follows that

$$\sum_{j=1}^{k}\sum_{r=n+1}^{2m+1}\sum_{\alpha_j<\beta_j}\left[h_{\alpha_j\alpha_j}^r h_{\beta_j\beta_j}^r - \left(h_{\alpha_j\beta_j}^r\right)^2\right] \geq \frac{\eta}{2} + \frac{1}{2}\sum_{r=n+1}^{2m+1}\sum_{(\alpha,\beta)\notin D^2}(h_{\alpha\beta}^r)^2 + \frac{1}{2}\sum_{r=n+2}^{2m+1}\sum_{j=1}^{k}\Bigg(\sum_{\alpha_j\in D_j} h_{\alpha_j\alpha_j}^r\Bigg)^2 \geq \frac{\eta}{2},$$

where $D^2 = (D_1\times D_1)\cup\cdots\cup(D_k\times D_k)$.

Thus

$$\sum_{j=1}^{k}\tau(L_j) \geq \frac{\eta}{2} + \sum_{j=1}^{k}\left[\frac{n_j(n_j-1)(c-3)}{8} + \frac{3(c+1)}{4}\Psi(L_j)\right]$$
$$= \tau - d(n_1,\dots,n_k)\left\|H\right\|^2 - \frac{c-3}{8}n(n-1) - \frac{c+1}{8}\left[3\left\|P\right\|^2 - 2(n-1)\right]$$
$$+ \sum_{j=1}^{k}\left[\frac{n_j(n_j-1)(c-3)}{8} + \frac{3(c+1)}{4}\Psi(L_j)\right],$$

which is equivalent with the relation that we want to prove. □

In particular, for slant submanifolds we derive:

Theorem 6. *Let M be an $(n = 2k+1)$-dimensional non anti-invariant θ-slant submanifold, tangent to ξ of a $(2m+1)$-dimensional Kenmotsu space form $\widetilde{M}(c)$. Let $n_1,\dots,n_k$ be integers ≥ 2 satisfying $n_1 < n, n_1+\dots+n_k \leq n$. For $p \in M$, let $L_j \subset T_pM$ be subspaces of T_pM, orthogonal to ξ such that $\dim L_j = n_j, \forall j \in \{1,\dots,k\}$. Then, we have:*

$$\tau(p) - \sum_{j=1}^{k}\tau(L_j) \leq d(n_1,\dots,n_k)\left\|H\right\|^2 + b(n_1,\dots,n_k)\frac{c-3}{4}$$
$$+ \frac{c+1}{8}\left[(n-1)(3\cos^2\theta - 2) - \sum_{j=1}^{k}6\Psi(L_j)\right].$$

The equality case of the inequality holds at $p \in M$ if and only if there exists an orthonormal basis $\{e_1,\dots,e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1},\dots,e_{2m+1}\}$ of $T_p^{\perp}M$ such that the shape operators of M in $\widetilde{M}(c)$ at p have the following forms:

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix},\ a_1+\dots+a_{n_1} = a_{n_1+1}+\dots+a_{n_1+n_2} = \cdots$$
$$= a_{n_1+\dots+n_{k-1}+1}+\dots+a_{n_1+\dots+n_k} = a_{n_1+\dots+n_k+1} = \dots = a_n,$$

$$A_r = \begin{pmatrix} A_1^r & 0 & 0 & 0 & \dots & 0 \\ 0 & \ddots & 0 & 0 & \dots & 0 \\ 0 & 0 & A_k^r & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \ A_j^r \in M_{n_j}(\mathbb{R}), {}^t A_j^r = A_j^r, \, TrA_j^r = 0,$$

$$\forall j = \overline{1, k}, \ \forall r \in \{n+2, \dots, 2m+1\}.$$

Proof. For a θ-slant submanifold of a Kenmotsu space form we have $\|P\|^2 = (n-1)\cos^2\theta$.

Equality at a point $p \in M$ holds if and only if the equality holds in all the previous inequalities and we have the equality in the algebraic lemma:

$h^r_{\alpha\beta} = 0, \forall r = \overline{n+1, 2m+1}, \forall(\alpha, \beta) \notin D^2$,

$\sum\limits_{\alpha_j \in D_j} h^r_{\alpha_j\alpha_j} = 0, \forall r = \overline{n+2, 2m+1}, \forall j = \overline{1, k}$,

$b_1 + b_2 = b_3 = \cdots = b_{\gamma+1}$. □

For invariant submanifolds we have the following.

Corollary 7. *Let M be an $(n = 2k+1)$-dimensional invariant submanifold, tangent to ξ of a $(2m+1)$-dimensional Kenmotsu space form $\widetilde{M}(c)$. Let $n_1, \dots, n_k$ be integers ≥ 2 satisfying $n_1 < n, n_1 + \cdots + n_k \leq n$. For $p \in M$, let $L_j \subset T_pM$ be subspaces of T_pM, orthogonal to ξ such that $\dim L_j = n_j, \forall j \in \{1, \dots, k\}$. Then, we have:*

$$\tau(p) - \sum_{j=1}^{k} \tau(L_j) \leq b(n_1, \dots, n_k)\frac{c-3}{4} + \frac{c+1}{8}\left[(n-1) - \sum_{j=1}^{k} 6\Psi(L_j)\right].$$

Proof. It is known that every invariant submanifold of a Kenmotsu space form is minimal. □

We obtain the following Chen inequality.

Theorem 8. *Let M be an $(n = 2k+1)$-dimensional non anti-invariant θ-slant submanifold, tangent to ξ of a $(2m+1)$-dimensional Kenmotsu space form $\widetilde{M}(c)$. Let $n_1, \dots, n_k$ be integers ≥ 2 satisfying $n_1 < n, n_1 + \cdots + n_k \leq n$. Then, we have:*

$$\delta'(n_1, \dots, n_k) \leq d(n_1, \dots, n_k)\|H\|^2 + b(n_1, \dots, n_k)\frac{c-3}{4}$$
$$+ \frac{c+1}{8}\left[3(n - 1 - \sum_{j=1}^{k} n_j)\cos^2\theta - 2(n-1)\right].$$

The equality case of the inequality holds at $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, \ldots, e_{2m+1}\}$ of $T_p^{\perp}M$ such that the shape operators of M in $\widetilde{M}(c)$ at p have the following forms:

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_n \end{pmatrix}, \; a_1 + \cdots + a_{n_1} = a_{n_1+1} + \cdots + a_{n_1+n_2} = \cdots$$

$$= a_{n_1+\cdots+n_{k-1}+1} + \cdots + a_{n_1+\cdots+n_k} = a_{n_1+\cdots+n_k+1} = \cdots = a_n,$$

$$A_r = \begin{pmatrix} A_1^r & 0 & 0 & 0 & \ldots & 0 \\ 0 & \ddots & 0 & 0 & \ldots & 0 \\ 0 & 0 & A_k^r & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \; A_j^r \in M_{n_j}(\mathbb{R}), {}^t A_j^r = A_j^r, \, TrA_j^r = 0,$$

$$\forall j = \overline{1,k}, \, \forall r \in \{n+2, \ldots, 2m+1\}.$$

Proof. Let $p \in M$ and $\{e_1, \ldots, e_n = \xi\}$ be an orthonormal basis of T_pM. Since we use subspaces invariant by P, we may choose $e_2 = \frac{1}{\cos\theta} Pe_1, \ldots, e_{2k} = \frac{1}{\cos\theta} Pe_{2k-1}$.

Let $L_1, \ldots, L_k$ be k mutually orthogonal subspaces of T_pM, $\dim L_j = n_j$, defined by:

$L_1 = Sp\{e_1, \ldots, e_{n_1}\}$,

$L_2 = Sp\{e_{n_1+1}, \ldots, e_{n_1+n_2}\}$,

$\vdots$

$L_k = Sp\left\{e_{n_1+\cdots+n_{k-1}+1}, \ldots, e_{n_1+\cdots+n_k}\right\}$.

Thus

$$\Psi(L_j) = \frac{n_j}{2}\cos^2\theta, \forall j = 1, \ldots, k.$$

□

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